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Subclasses of Yamakawa-Type Bi-Starlike functions subordinate to Gegenbaur polynomials associated with quantum calculus

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In this paper, we present a novel class of Yamakawa-type bi-starlike functions. These functions are defined using Gegenbauer polynomials associated with q-calculus. We have derived estimates for the Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the Yamakawa-type bi-starlike function subclass. Additionally, we have solved the Fekete-Szegö problems for functions in this new subclass. By specializing the parameters in our main results, we have obtained several new findings.

Keywords: q–Gegenbauer polynomials, starlike functions, subordination, bi-univalent functions, q–calculus, Fekete-Szegö problem

Mathematics Subject Classification: 30C45

1. Definitions and preliminaries

Legendre polynomials [27] are commonly used to solve ordinary differential equations with specific model constraints. Orthogonal polynomials, including Legendre polynomials, also play a significant role in approximation theory [17]. One kind of orthogonal polynomials are Gegenbauer polynomials

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that have a special relationship with the generating function and integral representation of real functions. This connection, as stated in [26], has resulted in various valuable inequalities within the field of Gegenbauer polynomials.

Quantum calculus, commonly referred to as q-calculus, has drawn the attention of numerous scholars because of its extensive applicability in physics and mathematics. The q-calculus increases the effectiveness of the conventional complement approach for various orthogonal polynomials modules and functions. This connection between the equilibrium states of differential formulas and their solutions provides an extremely efficient and well-designed technique for examining the characteristics of individual functions. The pioneers of q-calculus were Euler and Jacobi in the 18th century, while Jackson [24, 25] further developed and systematically applied q-calculus. Quantum calculus is necessary for Aral and Gupta [14, 15] to construct q-analogue of the Baskakov and Durrmeyer operator. Check out [4, 6, 7, 8] for some recent uses of the q-operator.

Yamakawa-type bi-starlike functions have attracted considerable interest in the field of complex analysis. These functions possess intricate geometric properties, particularly in relation to starlikeness. The study of Yamakawa-type bi-starlike functions entails examining their behavior and properties under different transformations and mappings. This investigation provides valuable insights into their analytical structure and applications in various mathematical contexts. Extensive research has been conducted on these functions, exploring integral representations, coefficient estimates, and other fundamental aspects. This research contributes to the broader understanding of complex function theory and its practical applications.

Let Δ represent the set of analytic functions Θ that are defined in the open unit disk $\Lambda = \{\beth \in \mathbb{C} : |\beth| < 1\}$ such that $\Theta(0) = 0$ and $\Theta'(0) = 1$. Therefore, each $\Theta \in \Delta$ has a Maclaurin series of the form:

$$\Theta(\beth) = \beth + \sum_{n=2}^{\infty} a_n \beth^n, \ (\beth \in \Lambda). \tag{1}$$

Also, the set of univalent functions $\Theta \in \Delta$ is indicated by $\mathcal{S}$ (for the set $\mathcal{S}$, refer to [20]).

The steady instruments provided by differential subordination of analytic functions are highly advantageous to the subject of geometric function theory. Miller and Mocanu [28] presented the first differential subordination problem; [29] offers more references. Miller and Mocanu's [30], which includes publication dates, provides a thorough history of the advances in this field.

Every function $\Theta \in \mathcal{S}$ has an inverse Θ^{-1} that is defined by

$$\Theta^{-1}(\Theta(\beth)) = \beth \qquad (\beth \in \Lambda), \tag{2}$$

and

$$\Theta^{-1}(\Theta(\varpi)) = \varpi \qquad (|\varpi| < r_0(\Theta);\ r_0(\Theta) \geq \frac{1}{4}), \tag{3}$$

where

$$\Theta^{-1}(\varpi) = \varpi - a_2\varpi^2 - (a_3 - 2a_2^2)\varpi^3 + (5a_2a_3 - 5a_2^3 - a_4)\varpi^4 + \cdots. \tag{4}$$

A function is considered bi-univalent in Λ if $\Theta(\beth)$ and $\Theta^{-1}(\beth)$ are univalent in Λ.

Let $\sum$ be the set of bi-univalent functions in Λ defined by (1). For example, the following functions in the set $\sum$

$$\Theta(\beth) = \frac{\beth}{1-\beth} \text{ and } \Theta(\beth) = \log\sqrt{\frac{1+\beth}{1-\beth}}.$$

However, the Koebe function, which is familiar to many, is not included in the set $\sum$. There are also other commonly used functions in Λ, such as:

$$\Theta(\beth) = \frac{2\beth - \beth^2}{2} \text{ and } \Theta(\beth) = \frac{\beth}{1-\beth^2}$$

They are also not members of $\sum$.
Askey and Ismail [16] identified a class of polynomials that correspond to the Gegenbauer polynomials in terms of q, where $\mathfrak{B}_q^{(\aleph)}(\varkappa,\sqsupset)$ is the notation used to represent these polynomials

$$\mathfrak{G}_q^{(\iota)}(\varkappa,\sqsupset)=\sum_{n=0}^{\infty}\mathcal{C}_n^{(\iota)}(\varkappa;q)\sqsupset^n. \tag{5}$$

Chakrabarti et al. [19] identified polynomials based on the following recurrence relations:

$$\begin{aligned}&\mathcal{C}_0^{(\iota)}(\varkappa;q)=1,\\&\mathcal{C}_1^{(\iota)}(\varkappa;q)=2[\iota]_q\varkappa,\\&\mathcal{C}_2^{(\iota)}(\varkappa;q)=2\left([\iota]_{q^2}+[\iota]_q^2\right)\varkappa^2-[\iota]_{q^2}.\end{aligned} \tag{6}$$

Amourah et al. [13] presented new subclasses using q-Gegenbauer polynomials. In contrast, Alsoboh et al. [5] presented a new subclass by combining the q-Gegenbauer polynomials with a generalized neutrosophic Poisson distribution. Fekete-Szegö inequalities can be established for these subclasses, along with the initial coefficient bounds $|a_2|$ and $|a_3|$.

Recently, researchers have begun investigating subclasses related to orthogonal polynomials. In their studies, they have found estimates for coefficients of these functions. However, the issue of determining sharp bounds for the coefficients $|a_n|$, $(n=3,4,5,\cdots)$ remains unsolved, as indicated by a number of sources (see [1]-[3], [9]-[12], [18], [21]-[23], [31]-[39]).

2. Identify and study the class $\mathcal{GY}_\Sigma^\iota(\wp,\mathfrak{G}_q^{(\iota)}(\varkappa,\sqsupset))$

Definition 2.1. *A function* $\Theta\in\sum$ *given by* (1) *be in the class* $\mathcal{GY}_\Sigma^\iota(\wp,\mathfrak{G}_q^{(\iota)}(\varkappa,\sqsupset))$ *if satisfied*

$$\frac{\Theta(\sqsupset)}{(1-\wp)\sqsupset+\wp\sqsupset\partial_q\Theta(\sqsupset)}\prec\mathfrak{G}_q^{(\iota)}(\varkappa,\sqsupset) \tag{7}$$

and

$$\frac{g(\varpi)}{(1-\wp)\varpi+\wp\varpi\partial_q g(\varpi)}\prec\mathfrak{G}_q^{(\iota)}(\varkappa,\varpi), \tag{8}$$

where $\varkappa\in\left(\frac{1}{2},1\right],0\le\wp\le1,\mathfrak{G}_q^{(\iota)}$ is given by (5) and $g(\varpi)=\Theta^{-1}(\varpi)$ is defined by (4).

We define a new subclass of Yamakawa type by specializing the parameter $\wp$ as follows:

Example 2.1. *For* $q\to1^-$, *we get* $\mathcal{GY}_\Sigma^\iota(\wp,\mathfrak{G}_\varkappa^\iota)=:\lim_{q\to1^-}\mathcal{GY}_\Sigma^\iota(\wp,\mathfrak{G}_q^{(\iota)}(\varkappa,\sqsupset))$.

$$\frac{\Theta(\sqsupset)}{(1-\wp)\sqsupset+\wp\sqsupset\Theta'(\sqsupset)}\prec\mathfrak{G}_\varkappa^\iota(\sqsupset)\quad\text{and}\quad\frac{g(\varpi)}{(1-\wp)\varpi+\wp\varpi g'(\varpi)}\prec\mathfrak{G}_\varkappa^\iota(\varpi),$$

where $z,\varpi\in\Lambda$ and $g(\varpi)=\Theta^{-1}(\varpi)$ *is defined by* (4).

Example 2.2. *For* $q\to1^-$ and $\wp=1$, *we get* $\mathcal{YS}_\Sigma^*(\mathfrak{G}_\varkappa^\iota)=:\lim_{q\to1^-}\mathcal{GY}_\Sigma^\iota(1,\mathfrak{G}_q^{(\iota)}(\varkappa,\sqsupset))$. *Thus* $f\in\mathcal{YS}_\Sigma^*(\mathfrak{G}_\varkappa^\iota)$ *if* $f\in\sum$ *and the following subordinations hold:*

$$\frac{\Theta(\sqsupset)}{\sqsupset\Theta'(\sqsupset)}\prec\mathfrak{G}_\varkappa^\iota(\sqsupset)\quad\text{and}\quad\frac{g(\varpi)}{\varpi g'(\varpi)}\prec\mathfrak{G}_\varkappa^\iota(\varpi),$$

where $\sqsupset,\varpi\in\Lambda$ *and* $g(\varpi)=\Theta^{-1}(\varpi)$ *is defined by* (4).

Example 2.3. *For* $q \to 1^-$ *and* $\wp = 0$, *we get* $\mathcal{N}_\Sigma(\mathfrak{G}^{\iota}_{\varkappa}) =: \lim\limits_{q\to 1^-} \mathcal{GY}_\Sigma(0, \mathfrak{G}^{(\iota)}_q(\varkappa, \beth))$. *Thus* $\Theta \in \mathcal{N}_\Sigma(\mathfrak{G}^{\iota}_{x})$ if $\Theta \in \sum$ *and the following subordinations hold:*

$$\frac{\Theta(\beth)}{\beth} \prec \mathfrak{G}^{\iota}_{\varkappa}(\beth) \text{ and } \frac{g(\varpi)}{\varpi} \prec \mathfrak{G}^{\iota}_{\varkappa}(\varpi),$$

where $z, \varpi \in \Lambda$ *and* $g(\varpi) = \Theta^{-1}(\varpi)$ *is defined by* (4).

First, let's provide the coefficient estimates for the class $\mathcal{GY}_\Sigma(\wp, \mathfrak{G}^{(\iota)}_q(\varkappa, \beth))$ as defined in Definition 2.1.

Theorem 2.2. *Let* $f \in \sum$ *given by* (1) *be in the class* $\mathcal{GY}_\Sigma(\wp, \mathfrak{G}^{(\iota)}_q(\varkappa, \beth))$. *Then*

$$|a_2| \le \frac{2\,|[\iota]_q|\,\varkappa\sqrt{2[\iota]_q\varkappa}}{\sqrt{\left(4[\iota]_q^2(1-([3]_q+[2]_q)\vartheta+[2]_q^2\vartheta^2) - 2\left(1-[2]_q\vartheta\right)^2([\iota]_{q^2}+[\iota]_q^2)\right)\varkappa^2 + \left(1-[2]_q\vartheta\right)^2[\iota]_{q^2}}}$$

and

$$|a_3| \le \frac{4[\iota]_q^2\varkappa^2}{(1-[2]_q\wp)^2} + \frac{2\,|[\iota]_q|\,\varkappa}{|1-[3]_q\wp|},$$

where $\wp \ne \frac{1}{[2]_q}$ *and* $\wp \ne \frac{1}{[3]_q}$.

Proof. Let $\Theta \in \mathcal{GY}_\Sigma(\wp, \mathfrak{G}^{(\iota)}_q(\varkappa, \beth))$. Then, from Definition 2.1, there exists two functions υ_1, υ_2 such that $\upsilon_1(0) = \upsilon_2(0) = 0$ and $|\upsilon_1(z)| < 1$, $|\upsilon_2(\varpi)| < 1$ for all $\beth, \varpi \in \Lambda$, after which we may write

$$\frac{\Theta(\beth)}{(1-\wp)\beth + \wp\beth\partial_q\Theta(\beth)} = \mathfrak{G}^{(\iota)}_q(\varkappa, \upsilon_1(\beth)) \tag{9}$$

and

$$\frac{g(\varpi)}{(1-\wp)\varpi + \wp\varpi\partial_q g(\varpi)} = \mathfrak{G}^{(\iota)}_q(\varkappa, \upsilon_2(\varpi)). \tag{10}$$

From (9) and (10), we have

$$\frac{\Theta(\beth)}{(1-\wp)\beth + \wp\beth\partial_q\Theta(\beth)} = 1 + C_1^{(\iota)}(\varkappa;q)c_1\beth + \left[C_1^{(\iota)}(\varkappa;q)c_2 + C_2^{(\iota)}(\varkappa;q)c_1^2\right]\beth^2 + \cdots \tag{11}$$

and

$$\frac{g(\varpi)}{(1-\wp)\varpi + \wp\varpi\partial_q g(\varpi)} = 1 + C_1^{(\iota)}(\varkappa;q)d_1\varpi + \left[C_1^{(\iota)}(\varkappa;q)d_2 + C_2^{(\iota)}(\varkappa;q)d_1^2\right]\varpi^2 + \cdots. \tag{12}$$

Pretty known if

$$|\upsilon_1(z)| = |c_1\beth + c_2\beth^2 + c_3\beth^3 + \cdots| < 1, \ (\beth \in \Lambda)$$

and

$$|\upsilon_2(\varpi)| = |d_1\varpi + d_2\varpi^2 + d_3\varpi^3 + \cdots| < 1, \ (\varpi \in \Lambda),$$

then

$$|c_j| \le 1 \text{ and } |d_j| \le 1 \text{ for all } j \in \mathbb{N}. \tag{13}$$

In view of (1), (4), from (11) and (12), we obtain

$$\begin{aligned}&1-([2]_q\wp-1)a_2-(([3]_q\wp-1)a_3+[2]_q\wp([2]_q\wp-1)a_2^2)\beth^2+\cdots\\&=1+C_1^{(\iota)}(\varkappa;q)c_1\beth+\left[C_1^{(\iota)}(\varkappa;q)c_2+C_2^{(\iota)}(\varkappa;q)c_1^2\right]\beth^2+\cdots\end{aligned}$$

and

$$\begin{aligned}&1+([2]_q\wp+1)a_2+\{([3]_q\wp-1)a_3+(2-(2[3]_q+[2]_q)\wp+[2]_q^2\wp^2)a_2^2\}\varpi^2+\cdots\\&=1+C_1^{(\iota)}(\varkappa;q)d_1\varpi+\left[C_1^{(\iota)}(\varkappa;q)d_2+C_2^{(\iota)}(\varkappa;q)d_1^2\right]\varpi^2+\cdots.\end{aligned}$$

Thus, from equations (11) and (12), we have

$$(1-[2]_q\wp)a_2=C_1^{(\iota)}(\varkappa;q)c_1, \tag{14}$$

$$(1-[3]_q\wp)a_3+[2]_q\wp([2]_q\wp-1)a_2^2=C_1^{(\iota)}(\varkappa;q)c_2+C_2^{(\iota)}(\varkappa;q)c_1^2, \tag{15}$$

and

$$-(1-[2]_q\wp)a_2=C_1^{(\iota)}(\varkappa;q)d_1, \tag{16}$$

$$([3]_q\wp-1)a_3+(2-(2[3]_q+[2]_q)\wp+[2]_q^2\wp^2)a_2^2=C_1^{(\iota)}(\varkappa;q)d_2+C_2^{(\iota)}(\varkappa;q)d_1^2. \tag{17}$$

It follows from (14) and (16) that

$$c_1=-d_1 \tag{18}$$

and

$$2(1-[2]_q\vartheta)^2a_2^2=\left[C_1^{(\iota)}(\varkappa;q)\right]^2\left(c_1^2+d_1^2\right)$$

$$a_2^2=\frac{\left[C_1^{(\iota)}(\varkappa;q)\right]^2}{2(1-[2]_q\vartheta)^2}\left(c_1^2+d_1^2\right) \tag{19}$$

By adding (15) and (16), we get

$$2(1-([3]_q+[2]_q)\wp+[2]_q^2\wp^2)a_2^2=C_1^{(\iota)}(\varkappa;q)(c_2+d_2)+C_2^{(\iota)}(\varkappa;q)(c_1^2+d_1^2).$$

Substituting the value of $(c_1^2+d_1^2)$ from (19) gives that

$$a_2^2=\frac{\left[C_1^{(\iota)}(\varkappa;q)\right]^3\left(c_2+d_2\right)}{2(1-([3]_q+[2]_q)\vartheta+[2]_q^2\vartheta^2)\left[C_1^{(\iota)}(\varkappa;q)\right]^2-2\left(1-[2]_q\vartheta\right)^2C_2^{(\iota)}(\varkappa;q)} \tag{20}$$

By applying (13) for the coefficients c_2 and d_2 and using (6), we obtain

$$|a_2|\leq\frac{2|[\iota]_q|\varkappa\sqrt{2[\iota]_q\varkappa}}{\sqrt{\left|(4[\iota]_q^2(1-([3]_q+[2]_q)\vartheta+[2]_q^2\vartheta^2)-2\left(1-[2]_q\vartheta\right)^2([\iota]_{q^2}+[\iota]_q^2))\varkappa^2+\left(1-[2]_q\vartheta\right)^2[\iota]_{q^2}\right|}}$$

By subtracting (17) from (15), we get

$$2(1-[3]_q\wp)(a_3-a_2^2)=C_1^{(\iota)}(\varkappa;q)\left(c_2-d_2\right)+C_2^{(\iota)}(\varkappa;q)\left(c_1^2-d_1^2\right). \tag{21}$$

Then, in view of (18) and (19), Eq. (21) becomes

$$a_3=\frac{\left[C_1^{(\iota)}(\varkappa;q)\right]^2}{2(1-[2]_q\wp)^2}\left(c_1^2+d_1^2\right)+\frac{C_1^{(\iota)}(\varkappa;q)}{2(1-[3]_q\wp)}\left(c_2-d_2\right) \tag{22}$$

Thus by applying (6), we conclude that

$$|a_3| \leq \frac{4[\iota]_q^2 \varkappa^2}{(1-[2]_q \wp)^2} + \frac{2\,|[\iota]_q|\,\varkappa}{|1-[3]_q \wp|}.$$

By taking $q \to 1^-$, $\wp = 0$ or $\wp = 1$, and $\varkappa \in (0,1)$, we can find the upper bounds for $|a_2|$ and $|a_3|$ for the function classes $\mathcal{GY}_\Sigma(\wp, \mathfrak{G}_\varkappa^\iota)$, $\mathcal{YS}_\Sigma^*(\mathfrak{G}_\varkappa^\iota)$ and $\mathcal{N}_\Sigma(\mathfrak{G}_\varkappa^\iota)$.

Corollary 2.3. *If* $\Theta \in \mathcal{GY}_\Sigma(\wp, \mathfrak{G}_\varkappa^\iota)$, *then*

$$|a_2| \leq \frac{2|\iota|\varkappa\sqrt{2\iota\varkappa}}{\sqrt{4\iota^2\varkappa^2(1-5\wp+4\wp^2)+(1-2\wp)^2\left(\iota-2(\iota+\iota^2)\varkappa^2\right)}}$$

and

$$|a_3| \leq \frac{4(\iota x)^2}{(1-2\wp)^2} + \frac{2|\iota|\varkappa}{|1-3\wp|},$$

where $\wp \neq \frac{1}{2}$ *and* $\wp \neq \frac{1}{3}$.

Corollary 2.4. *If* $\Theta \in \mathcal{YS}_\Sigma^*(\mathfrak{G}_\varkappa^\iota)$, *then*

$$|a_2| \leq \frac{2|\iota|\varkappa\sqrt{2\iota\varkappa}}{\sqrt{\iota-2(\iota+\iota^2)\varkappa^2}}$$

and

$$|a_3| \leq 4(\iota x)^2 + |\iota|\varkappa.$$

In the above Corollaries, by fixing $\iota = 1$ and $\iota = \frac{1}{2}$, we acquire the new approximations of $|a_2|$ and $|a_2|$ for the function classes $\mathcal{GY}_\Sigma(\wp, \mathfrak{G}_\varkappa^\iota)$ and $\mathcal{GY}_\Sigma(\wp, \mathfrak{G}_\varkappa^\iota)$ related with Chebyshev polynomials and Legendre polynomials, respectively.

Corollary 2.5. *If* $\Theta \in \mathcal{N}_\Sigma(\mathfrak{G}_\varkappa^\iota)$, *then*

$$|a_2| \leq \frac{2|\iota|\varkappa\sqrt{2\iota\varkappa}}{\sqrt{\left(4\iota^2-2(\iota+\iota^2)\right)\varkappa^2+\iota}}$$

and

$$|a_3| \leq 4(\iota x)^2 + 2|\iota|\varkappa.$$

Based on the findings of Zaprawa [40], we will now explore the Fekete–Szegö inequality for functions in $\mathcal{GY}_\Sigma(\wp, \mathfrak{G}_q^{(\iota)}(\varkappa, z))$.

Theorem 2.6. *Let* $\Theta \in \sum$ *given by* (1) *be in the class* $\mathcal{GY}_\Sigma(\wp, \mathfrak{G}_q^{(\iota)}(\varkappa, z))$. *and* $\mu \in \mathbb{R}$. *Then, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{2\,|[\iota]_q|\,\varkappa}{1-[3]_q\wp}, & |\mu-1| \leq \mathcal{Q}(\wp,\iota,\varkappa;q), \\ 4\,|[\iota]_q|\,\varkappa\,|\mathcal{H}(\mu)|, & |\mu-1| \geq \mathcal{Q}(\wp,\iota,\varkappa;q), \end{cases}$$

where

$$\mathcal{H}(\mu) = \frac{(1-\mu)\left[C_1^{(\iota)}(\varkappa;q)\right]^2}{2(1-([3]_q+[2]_q)\wp+[2]_q^2\wp^2)\left[C_1^{(\iota)}(\varkappa;q)\right]^2 - 2\left(1-[2]_q\wp\right)^2 C_2^{(\iota)}(\varkappa;q)},$$

$$\mathcal{Q}(\wp,\iota,\varkappa;q)=\left|1+\frac{([2]_q\wp-1)\left(4[\iota]_q^2\varkappa^2[2]_q\wp+\left(1-[2]_q\wp\right)(2\left([\iota]_{q^2}+[\iota]_q^2\right)\varkappa^2-[\iota]_{q^2})\right)}{4[\iota]_q^2\varkappa^2(1-[3]_q\wp)}\right|,$$

and $\wp\neq\frac{1}{[2]_q}$, $\wp\neq\frac{1}{[3]_q}$.

Proof. If $\Theta\in\mathcal{GY}_\Sigma^{\iota}(\wp,\mathfrak{G}_q^{(\iota)}(\varkappa,\beth))$ is given by (1), from (20) and (22) we have

$$a_3-\mu a_2^2=\frac{C_1^{(\iota)}(\varkappa;q)}{2(1-[3]_q\wp)}(c_2-d_2)+(1-\mu)a_2^2=\frac{C_1^{(\iota)}(\varkappa;q)}{2(1-[3]_q\wp)}(c_2-d_2)$$
$$+\frac{(1-\mu)\left[C_1^{(\iota)}(\varkappa;q)\right]^3}{2(1-([3]_q+[2]_q)\wp+[2]_q^2\wp^2)\left[C_1^{(\iota)}(\varkappa;q)\right]^2-2\left(1-[2]_q\wp\right)^2C_2^{(\iota)}(\varkappa;q)}(c_2+d_2)$$
$$=C_1^{(\iota)}(\varkappa;q)\left[\left(\mathcal{H}(\mu)+\frac{1}{2(1-[3]_q\wp)}\right)c_2+\left(\mathcal{H}(\mu)-\frac{1}{2(1-[3]_q\wp)}\right)d_2\right],$$

where

$$\mathcal{H}(\mu)=\frac{(1-\mu)\left[C_1^{(\iota)}(\varkappa;q)\right]^2}{2(1-([3]_q+[2]_q)\wp+[2]_q^2\wp^2)\left[C_1^{(\iota)}(\varkappa;q)\right]^2-2\left(1-[2]_q\wp\right)^2C_2^{(\iota)}(\varkappa;q)}.$$

Then, in view of (6), we get

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\dfrac{\left|C_1^{(\iota)}(\varkappa;q)\right|}{1-[3]_q\wp}, & \left|\mathcal{H}(\mu)\right|\leq\dfrac{1}{2(1-[3]_q\wp)},\\[2mm] 2\left|C_1^{(\iota)}(\varkappa;q)\right|\left|\mathcal{H}(\mu)\right|, & \left|\mathcal{H}(\mu)\right|\geq\dfrac{1}{2(1-[3]_q\wp)}.\end{cases}$$

Theorems 2.2 and 2.6 lead to the following corollaries, which are similar to Examples 2.2 and 2.3.

Corollary 2.7. *If* $\Theta\in\mathcal{GY}_\Sigma^{\iota}(\wp,\mathfrak{G}_\varkappa^{\iota})$ *and* $\mu\in\mathbb{R}$, *then we have*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\dfrac{2|\iota|\varkappa}{1-3\wp}, & \left|\mu-1\right|\leq\mathcal{Q}(\wp,\iota,\varkappa),\\[2mm] 4|\iota|\varkappa\left|\mathcal{H}(\mu)\right|, & \left|\mu-1\right|\geq\mathcal{Q}(\wp,\iota,\varkappa),\end{cases}$$

where

$$\mathcal{H}(\mu)=\frac{(1-\mu)\left[C_1^{(\iota)}(\varkappa;1)\right]^2}{2(1-5\wp+4\wp^2)\left[C_1^{(\iota)}(\varkappa;1)\right]^2-2\left(1-2\wp\right)^2C_2^{(\iota)}(\varkappa;1)},$$
$$\mathcal{Q}(\wp,\iota,\varkappa)=\left|1+\frac{(2\wp-1)\left(8\iota^2\varkappa^2\wp+\left(1-2\wp\right)(2\left(\iota+\iota^2\right)\varkappa^2-\iota)\right)}{4\iota^2\varkappa^2(1-3\wp)}\right|,$$

and $\wp\neq\frac{1}{2}$, $\wp\neq\frac{1}{3}$.

Corollary 2.8. *If* $\Theta \in \mathcal{YS}^{*}_{\Sigma}(\mathfrak{G}^{\iota}_{\varkappa})$ *and* $\mu \in \mathbb{R}$, *then we have*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} -|\iota|\varkappa, & \left|\mu - 1\right| \le \mathcal{Q}(1,\iota,\varkappa), \\ 4|\iota|\varkappa\left|\mathcal{H}(\mu)\right|, & \left|\mu - 1\right| \ge \mathcal{Q}(1,\iota,\varkappa), \end{cases}$$

where

$$\mathcal{H}(\mu) = \frac{(1-\mu)\left[C_1^{(\iota)}(\varkappa;1)\right]^2}{-2C_2^{(\iota)}(\varkappa;1)},$$

$$\mathcal{Q}(1,\iota,\varkappa) = \left|1 - \frac{\left(8\iota^2\varkappa^2 - \left(2\left(\iota+\iota^2\right)\varkappa^2 - \iota\right)\right)}{8\iota^2\varkappa^2}\right|.$$

Corollary 2.9. *If* $\Theta \in \mathcal{N}_{\Sigma}(\mathfrak{G}^{\iota}_{\varkappa})$ *and* $\mu \in \mathbb{R}$, *then we have*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 2|\iota|\varkappa, & \left|\mu - 1\right| \le \mathcal{Q}(0,\iota,\varkappa), \\ 4|\iota|\varkappa\left|\mathcal{H}(\mu)\right|, & \left|\mu - 1\right| \ge \mathcal{Q}(0,\iota,\varkappa), \end{cases}$$

where

$$\mathcal{H}(\mu) = \frac{(1-\mu)\left[C_1^{(\iota)}(\varkappa;1)\right]^2}{2\left[C_1^{(\iota)}(\varkappa;1)\right]^2 - 2C_2^{(\iota)}(\varkappa;1)},$$

$$\mathcal{Q}(0,\iota,\varkappa) = \left|1 + \frac{\left(2\left(\iota+\iota^2\right)\varkappa^2 - \iota\right)}{4\iota^2\varkappa^2}\right|.$$

Concluding Remark: We have introduced and studied the coefficients bounds associated with a new subclass $\mathcal{GY}^{\lambda}_{\Sigma}(\wp,\mathfrak{G}^{(\iota)}_{q}(\varkappa,\beth))$ of functions in Λ. These specific classes of bi-univalent functions are defined in Definitions 2.1. In this class, we have derived estimates for the Maclaurin coefficients $|a_2|$ and $|a_3|$, as well as solved Fekete-Szegö issues for functions in the subclass $\mathcal{GY}^{\lambda}_{\Sigma}(\wp,\mathfrak{G}^{(\iota)}_{q}(\varkappa,\beth))$. By specializing the parameters, we have obtained several new results. However, obtaining the upper bound of $\left|a_n\right|$ for $n \ge 4;\ n \in \mathbb{N}$ for the subclass $\mathcal{GY}^{\lambda}_{\Sigma}(\wp,\mathfrak{G}^{(\iota)}_{q}(\varkappa,\beth))$ remains an open problem.

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