Results in Nonlinear Analysis 7 (2024) No. 3, 55–64
https://doi.org/10.31838/rna/2024.07.03.004
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556
Peer Reviewed Scientific Journal

On generalized Weyl conformal curvature tensor in para-kenmotsu manifolds

M. Bilal[1], T. Raghuwanshi[2], M. K. Pandey[3], A. Haseeb[4], A. Goyal[5]

[1]Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Al Qura University, Makkah 21955, Saudi Arabia; [2,3,5]Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal, Madhya Pradesh, India, 462033; [4]Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan 45142, Kingdom of Saudi Arabia.

In the present study, we consider a generalized Weyl conformal curvature tensor on para-Kenmotsu manifolds (briefly, PK-manifolds). First we describe certain vanishing properties of generalized Weyl conformal curvature tensor (brifly, GWC-curvature tensor) on a PK-manifold. Later, we study generalized Weyl conformally semi-symmetric PK-manifold that turns out to an Einstein manifold. Among others, it has been shown that the generalized Weyl conformally ϕ-symmetric PK-manifold is of constant curvature or $dr(\psi) = 0$.

Keywords and phrases: Weyl conformal curvature tensor, para-Kenmotsu manifolds, Einstein manifolds

Mathematics Subject Classification (2010): 53C15, 53C25.

1. Introduction

If a pseudo-Riemannian metric g on a manifold M is conformally related with a flat pseudo-Euclidean metric, then g is called conformally flat. A pseudo-Riemannian manifold with a conformally flat pseudo-Riemannian metric is called a conformally flat manifold. Using the tools of conformal transformation, H. Weyl (see [5, 6]) introduced a generalized curvature tensor which vanishes whenever the metric is conformally flat (for this reason named the confomal curvature tensor). Later, the properties of Weyl conformal curvature tensor is extensively studied by many author such as [4, 8, 16, 17].

Email addresses: mohd7bilal@gmail.com (Mohd. Bilal); teerathramsgs@gmail.com (T. Raghuwanshi); mkp_apsu@rediffmail.com (M. K. Pandey); malikhaseeb80@gmail.com, haseeb@jazanu.edu.sa (A. Haseeb)*; anil_goyal03@rediffmail.com (A. Goyal)

Received May 25, 2024; Accepted June 17, 2024; Online July 3, 2024

A (0,3)-type Weyl conformal curvature tensor C is given by

$$C(l_1,m_1,n_1)=R(l_1,m_1,n_1)-\frac{1}{(\Upsilon-2)}[S(m_1,n_1)l_1-S(l_1,n_1)m_1 \\ -g(l_1,n_1)Qm_1+g(m_1,n_1)Ql_1] \\ -\frac{r}{(\Upsilon-1)(\Upsilon-2)}[g(l_1,n_1)m_1-g(m_1,n_1)l_1], \quad (1)$$

for all vector fields l_1,m_1 and $n_1\in\mathfrak{X}(M)$, here $\mathfrak{X}(M)$ is the set of all differentiable vector fields on a manifold M. The symbol R refers to the Riemannian curvature tensor of type (0,3) and S denotes the Ricci tensor, such that

$$S(l_1,m_1)=g(Ql_1,m_1),$$

where Q being the Ricci operator of type (1,1).

On the other hand, the structure of an almost para-contact manifold was first developed and studied by I. Sato [19]. Recently, many authors published the papers by adopting the work of S. Zamkovoy [25]. At this point, we refer the papers [2, 9–11, 14, 20, 21] and references therein for the detailed overview of the results on *PK*-manifolds.

In this paper, we undertake the study of GWC-curvature tensor of para-Kenmotsu manifolds. This paper is comprised of 6 sections. Section 2 presents preliminaries on *PK*-manifolds. In Section 3, we briefly describe the GWC-curvature tensor on a *PK*-manifold and study some of its basic properties. In Section 4, we prove that a generalized Weyl conformally semi-symmetric *PK*-manifold is an Einstein manifold. Further in Section 5, we prove that a generalized Weyl conformally *PK*-manifold satisfying the condition $\tilde{C}(l_1,m_1)\cdot S=0$ is either Einstein manifold or $\psi=\frac{r+(\Upsilon-1)^2-(\Upsilon-1)(\Upsilon-2)}{2(\Upsilon-1)}$, here ψ is an arbitrary scalar function. In section 6, we prove that a *PK*-manifold satisfying $\phi^2((\nabla_G\tilde{C})(l_1,m_1,n_1))=0$ is of constant curvature.

2. Preliminaries

A differentiable manifold $M^\Upsilon(\Upsilon=\text{dimension } M)$ equipped with the structure (ϕ,ζ,τ) with a tensor field ϕ of type (1, 1), a vector field ζ and a 1-form τ satisfying the following relations

$$\phi^2(l_1)=l_1-\tau(l_1)\zeta, \quad (2)$$

$$\tau(\phi l_1)=0, \quad (3)$$

$$\phi(\zeta)=0, \quad (4)$$

and

$$\tau(\zeta)=1 \quad (5)$$

is called an almost para-contact manifold. An M^Υ admitting a pseudo-Riemannian metric g with the conditions

$$\tau(l_1)=g(l_1,\zeta) \quad (6)$$

and

$$g(\phi l_1,\phi m_1)=-g(l_1,m_1)+\tau(l_1)\tau(m_1) \quad (7)$$

is known as an almost para-contact Riemannian manifold [1, 19].

Definition 2.1. *An almost para-contact metric structure* (ϕ,ζ,τ,g) *is called para-Kenmotsu* (*PK-manifold*) [7, 13] *if the Levi-Civita connection* ∇ *of g holds* $(\nabla_{l_1}\phi)m_1=g(\phi l_1,m_1)\zeta-\tau(m_1)\phi l_1$ *for any* $l_1,m_1\in\mathfrak{X}(M)$.

A. M. Blaga [7] has given the following example of *PK*-manifold:

Example 2.2. *Let* $M = \{(l,m,n) \in \mathbb{R}^3, n \neq 0\}$ *, where* (l,m,n) *are the standard co-ordinates in* $\mathbb{R}^3$*. We set* $\phi := \frac{\partial}{\partial m} \otimes dl + \frac{\partial}{\partial l} \otimes dm, \zeta := -\frac{\partial}{\partial n}, \tau := -dn,\ g := dl \otimes dl - dm \otimes dm + dn \otimes dn.$ *Then* (ϕ, ζ, τ, g) *defines a para-Kenmotsu structure on* $\mathbb{R}^3$.

On a *PK*-manifold, the following relations hold [7, 18]:

$$(\nabla_{l_1}\phi)m_1 = g(\phi l_1, m_1)\zeta - \tau(m_1)\phi l_1, \tag{8}$$

$$\nabla_{l_1}\zeta = l_1 - \tau(l_1)\zeta, \tag{9}$$

$$(\nabla_{l_1}\tau)m_1 = g(l_1, m_1) - \tau(l_1)\tau(m_1), \tag{10}$$

$$\tau(R(l_1, m_1, q_1)) = g(l_1, q_1)\tau(m_1) - g(m_1, q_1)\tau(l_1), \tag{11}$$

$$R(l_1, m_1, \zeta) = \tau(l_1)m_1 - \tau(m_1)l_1, \tag{12}$$

$$R(l_1, \zeta, m_1) = -R(\zeta, l_1, m_1) = g(l_1, m_1)\zeta - \tau(m_1)l_1, \tag{13}$$

$$S(\phi l_1, \phi m_1) = -(\Upsilon - 1)g(\phi l_1, \phi m_1), \tag{14}$$

$$S(l_1, \zeta) = (1 - \Upsilon)\tau(l_1), \tag{15}$$

$$Q\zeta = (1 - \Upsilon)\zeta, \tag{16}$$

$$r = \Upsilon(1 - \Upsilon), \tag{17}$$

for all l_1 and m_1 on M.

3. *PK*-manifolds admitting some vanishing properties of GWC-curvature tensor

In this section, we consider Weyl conformal curvature tensor of a *PK*-manifold and generalize it with the help of $\mathcal{Z}$ tensor and state some of its properties.

Taking the inner product of (1) with g, we obtain a (0,4) type tensor field $'C$ given below

$$\begin{aligned} 'C(l_1, m_1, n_1, p_1) = {}& 'R(l_1, m_1, n_1, p_1) - \frac{1}{(\Upsilon - 2)}[S(m_1, n_1)g(l_1, p_1) \\ & - S(l_1, n_1)g(m_1, p_1) + g(m_1, n_1)S(l_1, p_1) - g(l_1, n_1)S(m_1, p_1)] \\ & - \frac{r}{(\Upsilon - 1)(\Upsilon - 2)}[g(l_1, n_1)g(m_1, p_1) - g(m_1, n_1)g(l_1, p_1)], \end{aligned} \tag{18}$$

where

$$'C(l_1, m_1, n_1, p_1) = g(C(l_1, m_1, n_1), p_1)$$

and

$$'R(l_1, m_1, n_1, p_1) = g(R(l_1, m_1, n_1), p_1)$$

where l_1, m_1, n_1 and p_1 are the vector fields of M^Υ.

Furthermore, covariant differentiation of the equation (1) respecting to G, yields

$$\begin{aligned} (\nabla_G C)(l_1, m_1)n_1 = {}& (\nabla_G R)(l_1, m_1)n_1 - \frac{1}{(\Upsilon - 2)}[(\nabla_G S)(m_1, n_1)l_1 \\ & - (\nabla_G S)(l_1, n_1)m_1 + g(m_1, n_1)(\nabla_G Q)l_1 - g(l_1, n_1)(\nabla_G Q)m_1] \\ & - \frac{dr(G)}{(\Upsilon - 1)(\Upsilon - 2)}[g(l_1, n_1)m_1 - g(m_1, n_1)l_1]. \end{aligned} \tag{19}$$

A symmetric (0,2) type tensor $\mathcal{Z}$ is a generalized $\mathcal{Z}$ tensor if [3, 12]

$$\mathcal{Z}(l_1, m_1) = S(l_1, m_1) + \psi g(l_1, m_1) \tag{20}$$

with ψ as an arbitrary scalar function. This tensor $\mathcal{Z}$ has been used by [15, 18] to obtain a new tensor field out of a given tensor field. We use it to generalize the Weyl conformal curvature tensor.

Using (20) in (18), we get

$$'C(l_1,m_1,n_1,p_1) = {}'R(l_1,m_1,n_1,p_1) - \frac{1}{(\Upsilon-2)}[\mathcal{Z}(m_1,n_1)g(l_1,p_1) - \mathcal{Z}(l_1,n_1)g(m_1,p_1) + g(m_1,n_1)\mathcal{Z}(l_1,p_1) - g(l_1,n_1)\mathcal{Z}(m_1,p_1)] + \frac{r}{(\Upsilon-1)(\Upsilon-2)}[g(m_1,n_1)g(l_1,p_1) - g(l_1,n_1)g(m_1,p_1)] - \frac{2\psi}{(\Upsilon-2)}[g(m_1,p_1)g(l_1,n_1) - g(m_1,n_1)g(l_1,p_1)],$$

which can be rewritten as

$$'\tilde{C}(l_1,m_1,n_1,p_1) = {}'C(l_1,m_1,n_1,p_1) + \frac{2\psi}{(\Upsilon-2)}[g(m_1,p_1)g(l_1,n_1) - g(l_1,p_1)g(m_1,n_1)], \tag{21}$$

where

$$'\tilde{C}(l_1,m_1,n_1,p_1) = {}'R(l_1,m_1,n_1,p_1) - \frac{1}{(\Upsilon-2)}[\mathcal{Z}(m_1,n_1)g(l_1,p_1) - \mathcal{Z}(l_1,n_1)g(m_1,p_1) + g(m_1,n_1)\mathcal{Z}(l_1,p_1) - g(l_1,n_1)\mathcal{Z}(m_1,p_1)] + \frac{r}{(\Upsilon-1)(\Upsilon-2)}[g(m_1,n_1)g(l_1,p_1) - g(l_1,n_1)g(m_1,p_1)]. \tag{22}$$

The new tensor field $'\tilde{C}$ defined by the equation (22) is termed as GWC-curvature tensor of a *PK*-manifold.

Obviously, if we set $\psi = 0$, then the equation (21) turns to

$$'C(l_1,m_1,n_1,p_1) = {}'\tilde{C}(l_1,m_1,n_1,p_1). \tag{23}$$

Thus, vanishing of the scalar function ψ implies that the two tensor fields $'C$ and $'\tilde{C}$ are identical.

Lemma 3.1. *The GWC-curvature tensor $'\tilde{C}$ of a PK-manifold satisfies Bianchi's first identity.*

Remark 3.2. *The GWC-curvature tensor $'\tilde{C}$ of a PK-manifold is*

(a) skew-symmetric in the first two slots,
(b) skew-symmetric in the last two slots,
(c) symmetric in pair of slots.

Theorem 3.3. *The GWC-curvature tensor $'\tilde{C}$ of a PK-manifold satisfies the following identities:*

(a) $$\tilde{C}(\zeta,m_1,n_1) = -\tilde{C}(m_1,\zeta,n_1) = \frac{1}{(\Upsilon-2)}[\tau(n_1)Qm_1 - S(m_1,n_1)\zeta]\times + \left[1 - \frac{r}{(\Upsilon-1)(\Upsilon-2)} + \frac{2\psi}{(\Upsilon-2)} - \frac{(\Upsilon-1)}{(\Upsilon-2)}\right]\times\ [\tau(n_1)m_1 - g(m_1,n_1)\zeta],$$

(b) $$\tilde{C}(l_1,m_1,\zeta) = \left[1 - \frac{r}{(\Upsilon-1)(\Upsilon-2)} + \frac{2\psi}{(\Upsilon-2)} - \frac{(\Upsilon-1)}{(\Upsilon-2)}\right]\times[\tau(l_1)m_1 - \tau(m_1)l_1] - \frac{1}{(\Upsilon-2)}[\tau(m_1)Ql_1 - \tau(l_1)Qm_1], \tag{24}$$

(c) $$\tau(\tilde{C}(n_1,p_1,m_1)) = \left[1 - \frac{r}{(\Upsilon-1)(\Upsilon-2)} + \frac{2\psi}{(\Upsilon-2)} - \frac{(\Upsilon-1)}{(\Upsilon-2)}\right]\times[g(n_1,m_1)\tau(p_1) - g(p_1,m_1)\tau(n_1)] - \frac{1}{(\Upsilon-2)}[S(p_1,m_1)\tau(n_1) - S(n_1,m_1)\tau(p_1)]. \tag{25}$$

4. Generalized Weyl conformally semi-symmetric *PK*-manifolds satisfying $(R(\zeta, l_1)\cdot\tilde{C})(n_1, p_1, m_1)=0$

In this section, we study implication of a condition of semi-symmetry type. We first present the following definitions.

Definition 4.1. *A PK-manifold is called as semi-symmetric manifold* [22] *if*

$$R(l_1, m_1)\cdot R=0, \tag{26}$$

here the curvature operator $R(l_1, m_1)$ *is assumed to be the derivation of the tensor algebra at each point of the PK-manifold.*

On similar lines, we propose the following definition:

Definition 4.2. *A PK-manifold is called generalized Weyl conformally semi-symmetric if*

$$R(l_1, m_1)\cdot\tilde{C}=0, \tag{27}$$

where $\tilde{C}$ *is the GWC-curvature tensor of a PK-manifold.*

Also, we iterate:

Definition 4.3. [24] *A PK-manifold is an Einstein manifold if its S of type* (0,2) *assumes the form* $S(l_1, m_1)=ag(l_1, m_1)$ *for some smooth function a, where* l_1 *and* m_1 *are the vector fields.*

We now state the following theorem:

Theorem 4.4. *A generalized Weyl conformally semi-symmetric PK-manifold is an Einstein manifold.*

Proof. We consider

$$R(l_1, m_1)\cdot\tilde{C}=0,$$

Now, we put $l_1=\zeta$ in the above expression to get

$$(R(\zeta, l_1)\cdot\tilde{C})(n_1, p_1, m_1)=0$$

for all $l_1, m_1, n_1, p_1\in\chi(M)$, which gives

$$R(\zeta, l_1, \tilde{C}(n_1, p_1, m_1))-\tilde{C}(R(\zeta, l_1, n_1), p_1, m_1) - \tilde{C}(n_1, R(\zeta, l_1, p_1), m_1)-\tilde{C}(n_1, p_1, R(\zeta, l_1, m_1)=0. \tag{28}$$

Due to the relation (13), the above equation reduces to

$$\begin{aligned}0=&\tau(\tilde{C}(n_1, p_1, m_1))l_1-{}'\tilde{C}(n_1, p_1, m_1, l_1)\zeta-\tau(n_1)\tilde{C}(l_1, p_1, m_1)\\&+g(l_1, n_1)\tilde{C}(\zeta, p_1, m_1)-\tau(p_1)\tilde{C}(n_1, l_1, m_1)+g(l_1, p_1)\tilde{C}(n_1, \zeta, m_1)\\&-\tau(m_1)\tilde{C}(n_1, p_1, l_1)+g(l_1, m_1)\tilde{C}(n_1, p_1, \zeta).\end{aligned}$$

We take inner product of the above expression with ζ and use (5), (6), (15), (21), (24), (24) and (25) to get

$$\begin{aligned}C(n_1, p_1, m_1, l_1)=&-\frac{2\psi}{(\Upsilon-2)}[g(l_1, p_1)g(m_1, n_1)-g(l_1, n_1)g(m_1, p_1)]+\left[1-\frac{r}{(\Upsilon-2)(\Upsilon-1)}+\frac{2\psi}{(\Upsilon-2)}-\frac{(\Upsilon-1)}{(\Upsilon-2)}\right]\\&\times[g(l_1, p_1)g(n_1, m_1)-g(l_1, n_1)g(p_1, m_1)]+\frac{1}{(\Upsilon-2)}[S(l_1, p_1)\tau(n_1)\tau(m_1)-S(l_1, n_1)\tau(p_1)\tau(m_1)]\\&+\frac{1}{(\Upsilon-2)}[S(m_1, n_1)g(l_1, p_1)-S(m_1, p_1)g(l_1, n_1)]+\left[\frac{1-\Upsilon}{2-\Upsilon}\right][g(l_1, p_1)\tau(m_1)\tau(n_1)\\&-g(l_1, n_1)\tau(m_1)\tau(p_1)],\end{aligned}$$

which by virtue of (18) reduces to

$$'R(n_1,p_1,m_1,l_1)=\frac{1}{(\Upsilon-2)}[S(l_1,p_1)\tau(n_1)\tau(m_1)-S(l_1,n_1)\tau(p_1)\tau(m_1)]+\left[1-\frac{(1-\Upsilon)}{(2-\Upsilon)}\right][g(l_1,p_1)g(m_1,n_1)$$
$$-g(l_1,n_1)g(m_1,p_1)]+\left[\frac{1-\Upsilon}{2-\Upsilon}\right][g(l_1,p_1)\tau(m_1)\tau(n_1)-g(l_1,n_1)\tau(m_1)\tau(p_1)]$$
$$-\frac{1}{(2-\Upsilon)}[S(l_1,n_1)g(m_1,p_1)-S(l_1,p_1)g(m_1,n_1)]. \tag{29}$$

We take $\{e_i\}_{i=1}^n$ to be an orthonormal basis and put $l_1=U=e_i$ in (29), then taking summation over i, we obtain

$$S(m_1,p_1)=-(\Upsilon-1)g(m_1,p_1).$$

which proves the theorem.

5. *PK*-manifolds satisfying $\tilde{C}(l_1,m_1)\cdot S=0$

We now consider a *PK*-manifold which satisfies the condition

$$\tilde{C}(l_1,m_1)\cdot S=0 \tag{30}$$

for all vector fields l_1 and m_1 and $\tilde{C}$ is the GWC-curvature tensor field of a *PK*-manifold.

Theorem 5.1. *A PK-manifold satisfying the condition* $\tilde{C}(l_1,m_1)\cdot S=0$ *is either an Einstein manifold or* $\psi=\dfrac{r+(\Upsilon-1)^2-(\Upsilon-1)(\Upsilon-2)}{2(\Upsilon-1)}$.

Proof. Suppose the *PK*-manifold satisfies the condition

$$(\tilde{C}(\zeta,l_1)\cdot S)(n_1,p_1)=0.$$

This implies that

$$S(\tilde{C}(\zeta,l_1,n_1),p_1)+S(n_1,\tilde{C}(\zeta,l_1,p_1))=0.$$

We use the equations (15), (16) and (24) in the above equation to get

$$0=\left[1-\frac{r}{(\Upsilon-2)(\Upsilon-1)}+\frac{2\psi}{(\Upsilon-2)}-\frac{(\Upsilon-1)}{(\Upsilon-2)}\right][S(l_1,p_1)\tau(n_1)$$
$$+S(l_1,n_1)\tau(p_1)+(\Upsilon-1)g(l_1,n_1)\tau(p_1)+(\Upsilon-1)g(l_1,p_1)\tau(n_1)].$$

Replacing $n_1=\zeta$ in the above expression and using (5), (6) and (15), we obtain

$$\left[1-\frac{r}{(\Upsilon-2)(\Upsilon-1)}+\frac{2\psi}{(\Upsilon-2)}-\frac{(\Upsilon-1)}{(\Upsilon-2)}\right][S(l_1,p_1)+(\Upsilon-1)g(l_1,p_1)]=0,$$

which implies that either

$$\psi=\frac{r+(\Upsilon-1)^2-(\Upsilon-1)(\Upsilon-2)}{2(\Upsilon-1)},$$

or

$$S(l_1,p_1)=-(\Upsilon-1)g(l_1,p_1).$$

This proves the theorem.

6. *PK*-manifolds satisfying $\phi^2((\nabla_G R)(l_1,m_1,n_1))=0$

In this section, we study a *PK*-manifold satisfying ϕ-symmetric condition. Below, we present the two definitions given by Takahashi [23].

Definition 6.1. *A PK-manifold is known to be*
(*i*) *locally Weyl conformally ϕ-symmetric if*

$$\phi^2((\nabla_G\tilde{C})(l_1,m_1,n_1))=0,\ \textit{for vector fields } l_1,m_1,n_1 \textit{ and } G \textit{ orthogonal to } \zeta,$$

(*ii*) *Weyl conformally ϕ-symmetric if*

$$\phi^2((\nabla_G\tilde{C})(l_1,m_1,n_1))=0,\ \textit{for arbitrary vector fields } l_1,m_1,n_1 \textit{ and } G.$$

Theorem 6.2. *A PK-manifold satisfying* $\phi^2((\nabla_G\tilde{C})(l_1,m_1,n_1))=0,$ *is of constant curvature or* $dr(\psi)=0$.

Proof. The covariant differentiation of (21) respecting to *G* gives us

$$(\nabla_G\tilde{C})(l_1,m_1,n_1)=(\nabla_G C)(l_1,m_1,n_1)+\frac{2dr(\psi)}{(\Upsilon-2)}[g(l_1,n_1)m_1-g(m_1,n_1)l_1]. \tag{31}$$

Using (19) in (31), we get

$$\begin{aligned}(\nabla_G\tilde{C})(l_1,m_1,n_1)=&(\nabla_G R)(l_1,m_1,n_1)-\frac{2dr(\psi)}{(\Upsilon-2)}[g(m_1,n_1)l_1-g(l_1,n_1)m_1]\\&-\frac{1}{(\Upsilon-2)}[(\nabla_G S)(m_1,n_1)l_1-(\nabla_G S)(l_1,n_1)m_1\\&+g(m_1,n_1)(\nabla_G Q)l_1-g(l_1,n_1)(\nabla_G Q)m_1]\\&-\frac{dr(G)}{(\Upsilon-1)(\Upsilon-2)}[g(l_1,n_1)m_1-g(m_1,n_1)l_1].\end{aligned} \tag{32}$$

We now assume that the *PK*-manifold satisfies

$$\phi^2((\nabla_G\tilde{C})(l_1,m_1,n_1))=0,$$

which, due to (2), gives

$$(\nabla_G\tilde{C})(l_1,m_1,n_1)=\tau((\nabla_G\tilde{C})(l_1,m_1,n_1))\zeta. \tag{33}$$

Now, by using (32) in (33), we have

$$\begin{aligned}(\nabla_G R)(l_1,m_1,n_1)=&-\frac{2dr(\psi)}{(\Upsilon-2)}[g(l_1,n_1)m_1-g(m_1,n_1)l_1]\\&+\frac{1}{(\Upsilon-2)}[(\nabla_G S)(m_1,n_1)l_1-(\nabla_G S)(l_1,n_1)m_1\\&+g(m_1,n_1)(\nabla_G Q)l_1-g(l_1,n_1)(\nabla_G Q)m_1]\\&-\frac{dr(G)}{(\Upsilon-1)(\Upsilon-2)}[g(m_1,n_1)l_1-g(l_1,n_1)m_1]+\tau((\nabla_G R)(l_1,m_1,n_1))\zeta\\&+\frac{2dr(\psi)}{(\Upsilon-2)}[g(l_1,n_1)\tau(m_1)-g(m_1,n_1)\tau(l_1)]\zeta\\&-\frac{1}{(\Upsilon-2)}[(\nabla_G S)(m_1,n_1)\tau(l_1)-(\nabla_G S)(l_1,n_1)\tau(m_1)\\&+g(m_1,n_1)\tau((\nabla_G Q)l_1)-g(l_1,n_1)\tau((\nabla_G Q)m_1)]\zeta\\&-\frac{dr(G)}{(\Upsilon-1)(\Upsilon-2)}[g(l_1,n_1)\tau(m_1)-g(m_1,n_1)\tau(l_1)]\zeta.\end{aligned} \tag{34}$$

The inner product of (34) with p_1 yields

$$\begin{aligned} g((\nabla_G R)(l_1,m_1,n_1),p_1) = &-\frac{2dr(\psi)}{(\Upsilon-2)}[g(l_1,n_1)g(m_1,p_1)-g(m_1,n_1)g(l_1,p_1)] \\ &+\frac{1}{(\Upsilon-2)}[(\nabla_G S)(m_1,n_1)g(l_1,p_1)-(\nabla_G S)(l_1,n_1)g(m_1,p_1) \\ &+g(m_1,n_1)g((\nabla_G Q)l_1,p_1)-g(l_1,n_1)g((\nabla_G Q)m_1,p_1)] \\ &-\frac{dr(G)}{(\Upsilon-1)(\Upsilon-2)}[g(m_1,n_1)g(l_1,p_1)-g(l_1,n_1)g(m_1,p_1)] \\ &+\tau((\nabla_G R)(l_1,m_1,n_1))\tau(p_1) \\ &+\frac{2dr(\psi)}{(\Upsilon-2)}[g(l_1,n_1)\tau(m_1)-g(m_1,n_1)\tau(l_1)]\tau(p_1) \\ &-\frac{1}{(\Upsilon-2)}[(\nabla_G S)(m_1,n_1)\tau(l_1)-(\nabla_G S)(l_1,n_1)\tau(m_1) \\ &+g(m_1,n_1)\tau((\nabla_G Q)l_1)-g(l_1,n_1)\tau((\nabla_G Q)m_1)]\tau(p_1) \\ &+\frac{dr(G)}{(\Upsilon-1)(\Upsilon-2)}[g(m_1,n_1)\tau(l_1)-g(l_1,n_1)\tau(m_1)]\tau(p_1). \end{aligned} \tag{35}$$

We replace $l_1 = p_1 = e_i$ in (35) and take summation over i to obtain

$$\begin{aligned} \frac{1}{(\Upsilon-2)}(\nabla_G S)(m_1,n_1) = &-\frac{1}{(\Upsilon-2)}[g(m_1,n_1)g((\nabla_G Q)e_i,e_i)-g((\nabla_G Q)m_1,n_1)] \\ &+\frac{dr(G)}{(\Upsilon-2)}g(m_1,n_1)-\frac{2(1-\Upsilon)dr(\psi)}{(2-\Upsilon)}g(m_1,n_1) \\ &-\tau((\nabla_G R)(e_i,m_1,n_1))\tau(e_i) \\ &+\frac{1}{(\Upsilon-2)}[(\nabla_G S)(m_1,n_1)-(\nabla_G S)(e_i,n_1)\tau(m_1)\tau(e_i) \\ &+g(m_1,n_1)\tau((\nabla_G Q)e_i)\tau(e_i)-\tau((\nabla_G Q)m_1)\tau(n_1)] \\ &-\frac{dr(G)}{(\Upsilon-2)(\Upsilon-1)}[g(m_1,n_1)-\tau(m_1)\tau(n_1)] \\ &-\frac{2dr(\psi)}{(\Upsilon-2)}[\tau(m_1)\tau(n_1)-g(m_1,n_1)] \\ =& 0. \end{aligned} \tag{36}$$

Taking $n_1 = \zeta$ in the above equation, we have

$$\begin{aligned} \tau((\nabla_G R)(e_i,m_1,\zeta))\tau(e_i) = &-\frac{2(\Upsilon-1)}{(\Upsilon-2)}dr(\psi)\tau(m_1)+\frac{1}{(\Upsilon-2)}[\tau((\nabla_G Q)e_i)\tau(e_i)\tau(m_1) \\ &-(\nabla_G S)(e_i,\zeta)\tau(e_i)\tau(m_1)] \\ =& 0. \end{aligned} \tag{37}$$

Now, since

$$\tau((\nabla_G R)(e_i,m_1,\zeta)\tau(e_i) = g((\nabla_G R)(e_i,m_1,\zeta),\zeta)g(e_i,\zeta). \tag{38}$$

Also

$$\begin{aligned} g((\nabla_G R)(e_i,m_1,\zeta),\zeta) = & g(\nabla_G R(e_i,m_1,\zeta),\zeta)-g(R(\nabla_G e_i,m_1,\zeta),\zeta) \\ &-g(R(e_i,\nabla_G m_1,\zeta),\zeta)-g(R(e_i,m_1,\nabla_G\zeta),\zeta). \end{aligned} \tag{39}$$

We note that $\nabla_{l_1} e_i = 0,$ as $\{e_i\}$ is an orthonormal basis. Therefore, using (12) to get

$$g(R(e_i,\nabla_G m_1,\zeta),\zeta)=0.$$

Since

$$g(R(e_i,m_1,\zeta),\zeta)+g(R(\zeta,\zeta,m_1),e_i)=0,$$

therefore, we have

$$g(\nabla_G R(e_i,m_1,\zeta),\zeta)+g(R(e_i,m_1,\zeta),\nabla_G\zeta)=0.$$

Using the above equation in (39), we obtain

$$g((\nabla_G R)(e_i,m_1,\zeta),\zeta)=0. \tag{40}$$

Also, we have

$$\tau((\nabla_G Q)e_i)\tau(e_i)=g((\nabla_G Q)e_i,\zeta)g(e_i,\zeta)=g((\nabla_G Q)\zeta,\zeta).$$

By using (9) and (15), we get

$$\tau((\nabla_G Q)e_i)\tau(e_i)=0. \tag{41}$$

In view of (40) and (41), (37) yields

$$dr(\psi)(\Upsilon-1)\tau(m_1)=0. \tag{42}$$

Now, taking $m_1=\zeta$ in the above expression and using the equation (5), we get

$$dr(\psi)=0 \tag{43}$$

which completes the proof.

7. Example

Let us assume a 3 dimensional manifold $\mathcal{M}^3=\{(l_1,l_2,l_3)\in\mathbb{R}^3 : l_3\neq 0\}$, where (l_1,l_2,l_3) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1=e^{l_3}\frac{\partial}{\partial l_1},\quad e_2=e^{l_3}\frac{\partial}{\partial l_2},\quad e_3=-\frac{\partial}{\partial l_3}=\zeta,$$

which are linearly independent at each point of $\mathcal{M}^3$. Let the semi-Riemannian metric tensor g is defined as

$$g(e_1,e_1)=-1,\quad g(e_2,e_2)=1,\quad g(e_3,e_3)=1.$$

Let τ be the 1-form such that $\tau(l_1)=g(l_1,e_3),\ \forall\ l_1\in“(T\mathcal{M}^3)$. Now, we define the tensor field ϕ of (1,1) type such that

$$\phi e_1=e_2,\quad \phi e_2=e_1,\quad \phi e_3=0.$$

Then, we can easily see that

$$\tau(e_3)=1,\quad \text{for}\quad e_3=\zeta,$$
$$\phi^2 l_1=l_1-\eta(l_1)\zeta,$$

and

$$g(\phi l_1,\phi l_2)=-g(l_1,l_2)+\tau(l_1)\tau(l_2)$$

$\forall\ l_1,l_2\in“(T\mathcal{M}^3).$.

Thus, $\mathcal{M}^3(\phi,\zeta,\tau,g)$ defines an almost paracontact metric manifold. Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1,e_2]=0,\quad [e_1,e_3]=e_1,\quad [e_2,e_3]=e_2.$$

By using Koszul's formula, we have

$$\nabla_{e_1} e_3 = e_1, \quad \nabla_{e_2} e_3 = e_2, \quad \nabla_{e_3} e_3 = 0,$$
$$\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_3} e_2 = 0,$$
$$\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0.$$

Thus, we have $\nabla_{l_1} e_3 = l_1 - \tau(l_1)e_3$. Hence, $\mathcal{M}^3(\phi, \zeta, \tau, g)$ is a para Kenmotsu manifold of dimension 3.

References

[1] T. Adati and T. Miyazava, On para-contact Riemannian manifolds, *Tru Math.* **13(2)** (1977), 27–39.

[2] A. Haseeb, R. Prasad, Certain results on Lorentzian para-Kenmotsu manifolds, *Bol. Soc. Parana. Mat.* **39(3)** (2021), 201–220.

[3] M. Ali, A. Haseeb, F. Mofarreh, M. Vasiulla, Z-symmetric manifolds admitting Schouten tensor, *Mathematics* **10(22)** (2022), 1–10.

[4] A. Haseeb, Some new results on para-Sasakian manifold with a quarter-symmetric metric connection, Facta Universitatis (NIS), *Ser. Math. Inform.* **30(5)** (2015), 765–776.

[5] H. Weyl, Reine Infinitesimalgeometrie, *Math. Z.,* **2(3–4)** (1918), 384–411.

[6] H. Weyl, Zur Infinitesimalgeometrie, Einordnung der projektiven und der konformen Auffassung, *Gottingen Nachrichten*, (1921), 99–112.

[7] A. M. Blaga, η-Ricci solitons on para-Kenmotsu manifolds, *Balkan J. Geom. Appl.* **20** (2015), 1–13.

[8] Venkatesha, D. M Naik, H. A. Kumara, Conformal curvature tensor on para-contact metric manifolds, *Matematicki Vesnik* **72(3)** (2020), 215–225.

[9] B. Cappelletti-Montano, I. Kupeli Erken, C. Murathan, Nullity conditions in paracontact geometry, *Differ. Geom. Appl.* **30** (2012), 665–693.

[10] I. Kupeli Erken, Yamabe solitons on three-dimensional normal almost paracontact metric manifolds, *Periodica Math. Hung.* **80** (2020), 172–184.

[11] I. Kupeli Erken and C. Murathan, A study of three-dimensional paracontact (k,μ,v)-spaces, *Int. J. Geom. Methods Mod. Phys.* **14(7)** (2017), 1750106.

[12] C. A. Mantica, Y. J. Suh, Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors, *Int. J. Geom. Meth. Mod. Phys.* **9(1)** (2012), 1250004, 21 pages.

[13] Z. Olszak, The Schouten-van Kampen affine connection adapted to an almost (para) contact metric structure, *Publ. Inst. Math. nouv. sér.* **94(108)** (2013), 31–42.

[14] D. G. Prakasha, M. R. Amruthalakshmi, F. Mofarreh, A. Haseeb, Generalized Lorentzian Sasakian-space-forms with M-projective curvature tensor, *Mathematics* **10(16)** (2022), 1–14.

[15] T. Raghuwanshi, S. K. Pandey, M. K. Pandey, A. Goyal, On generalized W2-curvature tensor of para-Kenmotsu manifolds, *Filomat* **36(3)** (2022), 741–752.

[16] S. S. Shukla, D. D. Singh, On conformal curvature tensor of (ε)-para Sasakian manifolds, *SUT J. Math.* **47(2)** (2011), 119–128.

[17] M. Z. Petrovic, N. O. Vesic, M. Lj. Zlatanovic, Curvature properties of metric and semi-symmetric linear connections, *Quaestiones Mathematicae* **45(10)** (2022), 1603–1627.

[18] T. Raghuwanshi, G. Pandey, M. K. Pandey, A. Goyal, On generalized pseudo-projective curvature tensor of para-Kenmotsu manifolds, Bulletin of the *Transilvania* University of Barşov, *Series III: Mathematics and Computer Science* **1(63)**, no. 1 (2021), 143–160.

[19] I. Sato, On a structure similar to the almost contact structure, *Tensor*, (*N.S.*) **30** (1976), 219–224.

[20] R. Prasad, L. Rathor, P. Gupta, A. Haseeb, A note on transversal hypersurfaces of para-Kenmotsu manifolds, *Proceedings of the Jangjeon Mathematical Society* **26(1)** (2023), 97–106.

[21] A. Sardar, U. C. De, η-Ricci solitons on para-Kenmotsu manifolds, *Differential Geometry Dynamical Systems* **22** (2020), 218–228.

[22] Z. I. Szabo, Structure theorem on Riemannian space satisfying $R(X,Y).R = 0$. I. The local version, *J. Diff.Geom.* **17** (1982), 531–582.

[23] T. Takahashi, Sasakian manifold with pseudo-Riemannian metric, *Tohoku Math. J.* **2(21)** (1969), no. 2, 271–290.

[24] K. Yano and M. Kon, Structures on manifolds, *Series in Pure Math.* Vol. 3 (1984), World Scientific.

[25] S. Zamkovoy, Canonical connections on paracontact manifolds, *Ann. Glob. Anal. Geom.* **36** (2009), 37–60.