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Mitigating Gibbs phenomenon: A localized Padé-Chebyshev approach and its conservation law applications

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Approximating non-smooth functions presents a significant challenge due to the emergence of unwanted oscillations near discontinuities, commonly known as Gibbs' phenomena. Traditional methods like finite Fourier or Chebyshev representations only achieve convergence on the order of $O(1)$. A promising avenue in addressing this issue lies in nonlinear and essentially non-oscillatory approximation techniques, such as rational or Padé approximation. A recent and notable endeavor to mitigate Gibbs' oscillations is through singular Padé-Chebyshev approximation. However, a drawback of this approach is the requirement to specify the discontinuity location within the algorithm, which is often unknown in practical applications. To tackle this obstacle, we propose a localized Padé-Chebyshev approximation method. Fortunately, our efforts yield success; the proposed localized variant effectively captures jump locations in non-smooth functions while maintaining an essentially non-oscillatory character. Furthermore, we employ Padé-Chebyshev approximation within a finite volume framework to address scalar hyperbolic conservation laws. Remarkably, the resulting rational numerical scheme demonstrates stability regardless of wave propagation direction. Consequently, we introduce a central rational numerical scheme for scalar hyperbolic conservation laws, offering robust and accurate computation of solutions.

Keywords: Chebyshev expansion, Local Padé approximation, Gibbs phenomenon, Numerical scheme for scalar conservation laws

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1. Introduction

In practical scenarios, the need often arises to approximate complex functions with simpler ones, such as polynomials or splines. This necessity stems from the inherent challenge of explicitly knowing the function, particularly when it arises as a solution to differential equations. While approximating smooth functions is a thoroughly explored topic, as evidenced by existing literature [31, 32, 36, 14], the task becomes more intricate when dealing with piecewise smooth functions. Despite its complexity, this task holds significant importance across various applications. For example, in weather forecasting, considerations must be made for factors like the earth's atmosphere, air pressure, and terrain gradients, especially in hilly regions. Similarly, in space engineering, understanding and predicting shock waves in fluid flow are crucial aspects.

Numerous approximation techniques exist for general functions, with linear approximation being a classical approach [1, 35]. Examples include truncated Taylor and Chebyshev expansions, Legendre polynomial approximation, among others. These methods are renowned for their optimal performance and exponential convergence when applied to sufficiently smooth functions [11, 14, 31, 33]. However, linear approximations encounter limitations when applied to piecewise smooth functions, resulting in accuracy limited to $O(1)$ [8, 2]. Moreover, these approximations often fail to accurately represent the function due to the emergence of oscillations, known as Gibbs' phenomenon.

In a study examining Gibbs' phenomenon, Shim and Park [37] observed that series representations or linear approximations, such as Fourier series, wavelet series, or sampling series using piecewise linear splines, are insufficient in fully eliminating Gibbs oscillations. Gottlieb and Shu [17] reviewed various techniques aimed at mitigating Gibbs' oscillations. Tadmor [42] proposed the use of mollifiers and filters to address Gibbs' oscillations, along with concentrated kernels to detect sharp edges. Filtering involves multiplying the coefficients of a series by a rapidly decreasing factor to expedite convergence. Kaber and Maday [25] discussed a specific case where filtering is employed with rational approximation to enhance the convergence of the approximant towards the target function.

An alternative strategy to mitigate Gibbs oscillations and enhance accuracy involves employing nonlinear approximations, where the approximant cannot be expressed as a linear combination of a finite number of linearly independent functions. Nonlinear approximation methods play a crucial role in addressing challenges in digital image processing and nonlinear partial differential equations. These techniques are commonly utilized for post-processing noisy data and image segmentation [12]. There are three primary types of nonlinear approximations: (1) Essentially Non-Oscillatory (ENO) or Weighted Essentially Non-Oscillatory (WENO) methods [38, 18, 24], (2) rational approximation [16, 10, 9], and (3) approximations originating from a nonlinear space as defined by DeVore [12, 13, 15]. These approaches offer effective means to reduce oscillations and elevate the accuracy of approximations, catering to diverse application domains. Non linear approximations are popular in machine learning (ML) domain and specifically these approximation techniques can be used in spectral graph neural networks [7, 3, 5, 4, 28], an emerging area on ML.

In this article, we aim to employ specific nonlinear approximation techniques tailored for functions exhibiting singularities [10, 9, 12, 13]. Our focus primarily lies on rational approximations. Existing literature indicates that in nearly all methods for mitigating Gibbs' phenomenon, knowledge of the jump location is crucial. This information is typically required in the form of a smoothness indicator [18, 39] or weak truncated local error [27, 26], enabling adaptive or post-processing techniques. However, we propose a rational approximation-based scheme that circumvents the need for jump location information entirely. The numerical scheme based on Padé-Chebyshev approximation, which we advocate, eliminates these complexities. It effectively captures shocks while preserving a non-disturbing solution profile.

We aim to investigate the effectiveness of nonlinear approximations compared to linear approximations for discontinuous functions. Harten et al. [21, 22] proposed finite volume essentially nonoscillatory (ENO) schemes, which utilize adaptive stencils based on the local smoothness of the function. These schemes achieve higher-order accuracy in smooth regions while maintaining a less oscillatory profile near discontinuities. ENO schemes are essentially modified total variation diminishing (TVD) schemes [19, 20]. Shu and Osher [40, 41] introduced finite difference ENO schemes. Liu et al. [29] introduced an enhanced version of ENO schemes known as Weighted Essentially Non-Oscillatory (WENO) schemes. WENO schemes utilize a nonlinear convex combination of candidate stencils to increase accuracy in smooth regions and reduce Gibbs' phenomenon near discontinuities. Specifically, a WENO scheme constructed using an r-th order ENO scheme achieves (r+1)-st order accuracy. Shu and Jiang [23] proposed a new smoothness indicator to assess the smoothness of numerical solutions, leading to the construction of third (for $r = 2$) and fifth (for $r = 3$) order WENO schemes. Despite their computational expense, these schemes find widespread use across numerous applications.

We introduce a numerical scheme founded on piecewise rational approximation of the function. This innovative nonlinear approach is specifically designed to formulate a numerical scheme for scalar conservation laws. In this scheme, we employ cell averages for reconstructing the solution, departing from the conventional use of point values of the function.

2. Padé-Chebyshev Approximants

Padé approximation, introduced by Henri Padé in 1890, is a rational approximation method for a given function. It has been observed that rational approximations outperform typical truncated series expansions. By rearranging a series expansion into a ratio of two finite-degree polynomials, significant acceleration can be achieved. Padé-Chebyshev approximation is one such rearrangement of truncated Chebyshev series.

2.1. Chebyshev series expansion

The Chebyshev series of a function represents an expansion of the function in terms of Chebyshev polynomials, which are a significant family of orthogonal functions in numerical analysis. For $n \geq 0$, a Chebyshev polynomial of degree n is defined as [31]:

$$T_n(t) = cos(n\theta), \qquad t \in [-1,1], \tag{1}$$

where $\theta = \cos^{-1}(t)$.

For a given function $f : [-1,1] \to \mathbb{R}$, where $f \in L_2[-1,1]$, the Chebyshev series expansion of f is given by [30]

$$f(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n(t), \tag{2}$$

where

$$c_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(s)T_n(s)}{\sqrt{1-s^2}} ds. \tag{3}$$

For some functions like *Signum* function, see [30], one can exactly evaluate the Chebyshev coefficients c_n but for a general function evaluation of the integral (3) accurately is difficult. Hence we use the Gauss Chebyshev quadrature rule to approximate the integral numerically and evaluate the coefficients c_n approximately. The approximated Chebyshev coefficients we denoted by $c_{n,m}$

$$c_n \approx c_{n,m} = \frac{2}{m} \sum_{l=1}^{m} f(t_l) T_n(t_l), \tag{4}$$

where the points

$$t_l = \cos\left(\frac{(2l+1)\pi}{2m}\right), \qquad l = 1, 2, \ldots, m \tag{5}$$

are the m roots of a Chebyshev polynomial $T_m(t)$ and are called as Chebyshev points. Here we use the Gauss Chebyshev quadrature rule to approximate the integral numericallyand evaluate the coefficients c_n approximately.

Notations: Let us denote the truncated Chebyshev series expansion of degree d of a function f by

$$\mathsf{C}_{d,m}[f](t) = \sum_{n=0}^{d} c_{n,m}[f]T_n(t). \tag{6}$$

Using $T_n = \cos(n\theta)$ and $\cos\theta = \frac{e^{\iota\theta} - e^{-\iota\theta}}{2}$ we write the Chebyshev series representation of the function f, given in (2), as

$$f(t) = \frac{1}{2}(c_{0,m} + \sum_{r=1}^{\infty} c_{r,m} z^r + \sum_{r=1}^{\infty} c_{r,m} z^{-r}) \tag{7}$$

where $z = e^{\iota \cos^{-1}(t)}$. Define

$$\mathsf{C}_{d,m}[f](z) = \frac{c_{0,m}}{2} + \sum_{r=1}^{\infty} c_{r,m} z^r, \tag{8}$$

then (7) becomes

$$f(t) = \frac{1}{2}(\mathsf{C}_{d,m}[f](z) + \mathsf{C}_{d,m}[f](z^{-1})). \tag{9}$$

2.2. Padé Approximation

A rational function with numerator degree n_p and denominator degree n_q

$$\mathsf{R}(z) = \frac{P_{n_p}(z)}{Q_{n_q}(z)} = \frac{\sum_{i=0}^{n_p} p_i z^i}{\sum_{i=0}^{n_q} q_i z^i} \tag{10}$$

is called a Padé approximant of order $[n_p / n_q]$, where $n_p, n_q \in \mathbb{Z}^+$. The Padé-Chebyshev approximant of a function f can be calculated by rationalizing the truncated Chebyshev series expansion $\mathsf{C}_{d,m}[f](z)$ of f, *i.e.*,

$$\frac{c_{0,m}}{2} + c_{1,m} z + c_{2,m} z^2 + \cdots + c_{d,m} z^d = \frac{P_{n_p}(z)}{Q_{n_q}(z)} + O(z^{d+1}) \tag{11}$$

such that

$$Q_{n_q}(z)\mathsf{C}_{d,m}[f](z) - P_{n_p}(z) = O(z^{d+1}), \quad z \to 0, \tag{12}$$

where $d = n_p + n_q$. To find the Padé approximant $\mathsf{R}(z)$ we need to calculate the coefficients of P_{n_p} and Q_{n_q} in such a way that

$$\begin{aligned} &d(P_{n_p}) \le n_p \\ &d(Q_{n_q}) \le n_q \\ &O(Q_{n_q}\mathsf{C}_{d,m}[f] - P_{n_p}) \ge n_p + n_q + 1, \end{aligned} \tag{13}$$

where $d(P)$ denotes the degree of a polynomial P and $O(P)$ denotes the order of a polynomial P, *i.e.*, the degree of the first nonzero term in P. The last inequality in (13) indicates that the coefficients in the series $Q_{n_q}\mathsf{C}_{d,m}[f] - P_{n_p}$ with index $< n_p + nq + 1$ vanishes.

Conditions (13) leads to two linear systems (14) and (15) in polynomial coefficients. The coefficients of the denominator Q_{n_q} can be computed by solving the following system of linear equations of size $n_q \times (n_q + 1)$ [16]

$$\begin{bmatrix} c_{n_p+1} & c_{n_p} & \cdots & c_{n_p-n_q+1} \\ c_{n_p+2} & c_{n_p+1} & \cdots & c_{n_p-n_q+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n_p+n_q} & c_{n_p+n_q-1} & \cdots & c_{n_p} \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n_q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{14}$$

and the coefficients of P_{n_p} are computed by the following matrix vector multiplication

$$\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n_p} \end{bmatrix} = \begin{bmatrix} c_0/2 & 0 & \cdots & 0 \\ c_1 & c_0/2 & \cdots & 0 \\ \vdots & & & \\ c_{n_p} & c_{n_p-1} & \cdots & c_0/2 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n_q} \end{bmatrix}. \tag{15}$$

As $\mathrm{Re}(e^{1\theta}) = \cos(\theta)$ for any $\theta \in \mathbb{R}$, from (9) we observe that the Padé approximation of the function *f* is the sum of the real part of the approximation of the function $\mathsf{C}_{d,m}[f](z)$ and $\mathsf{C}_{d,m}[f](z^{-1})$. Since the real part of $\mathsf{C}_{d,m}[f](z)$ and $\mathsf{C}_{d,m}[f](z^{-1})$ are same, the Padé approximation of *f*(*t*) is the real part of the approximation of the function $\mathsf{C}_{d,m}[f](z)$.

Following is a result on the uniqueness of the Padé approximant to a function *f*.

Theorem 2.1. [25] *Let* $R_1(t)$ *and* $R_2(t)$ *be two Padé approximants to a function f, such that,*

$$\mathsf{R}_1(t) = \frac{P_1(t)}{Q_1(t)} \text{ and } \mathsf{R}_2(t) = \frac{P_2(t)}{Q_2(t)}.$$

If the polynomials $P_1(t), Q_1(t)$ *and* $P_2(t), Q_2(t)$ *satisfies the relation* (13) *then*

$$\mathsf{R}_1(t) = \mathsf{R}_2(t).$$

3. Local Padé-Chebyshev Reconstruction

The Local Padé-Chebyshev method (LocPCM) is a straightforward technique for accurately approximating non-smooth functions with reduced computational complexity. Unlike many recent approximation methods, LocPCM does not necessitate prior knowledge of the jump location, as highlighted in various literature [16, 43, 44]. In this study, we provide numerical examples and compare LocPCM to Singular Padé-Chebyshev approximations.

3.1. Local Chebyshev Approximation

Consider a bounded piecewise smooth function *f* defined on a bounded interval [*a*, *b*], which has isolated singularities. Let us first discretize the domain into *N* cells with cell boundaries $\{x_1, x_2, ..., x_N\}$. Assume that the function values are known at cell boundaries, denoted by $\{f_1, f_2, ..., f_N\}$. The aim is to approximate the function at the point

$$t_i = \frac{x_i + x_{i+1}}{2}$$

for $i = 1, 2, \ldots, N-1$, using truncated Chebyshev series expansion. We do this through the following steps:

1. Consider a stencil $S_i = \{x_{i-(\frac{n-1}{2})}, \ldots, x_{i+(\frac{n-1}{2})}\}$ of size *n*, where *n* is an odd integer.
2. Generate *n* Chebyshev points in the reference interval [–1,1], denoted by $s_1, s_2, ..., s_n$.
3. Consider the bijection map $\mathsf{G} : [-1,1] \to [x, y]$ given by

$$G(t) = x + (y - x)\frac{(t+1)}{2},$$

with $x = x_{i-(\frac{n-1}{2})}$ and $y = x_{i+(\frac{n-1}{2})}$.

4. To approximate the function f at a point $t_i = \dfrac{x_i + x_{i+1}}{2}$ in each stencil S_i, for $i = 1, 2, \ldots, N-1$, we need its Chebyshev series expansion in S_i. The Chebyshev series expansion of the function f is given by

$$f(x) = \frac{c_{0,n}}{2} + \sum_{l=1}^{\infty} c_{l,n} T_l(x), t \in [x_{i-\frac{n-1}{2}}, x_{i+\frac{n-1}{2}}], \tag{16}$$

where

$$c_{m,n} = \frac{2}{n}\sum_{l=1}^{n} f\,|_{s_l}\; T_m(s_l). \tag{17}$$

Note: For a given $n = 2m+1$, where $m = 1,2,3,4,5,6$, and equidistant points in an interval $[a,b]$ along with n Chebyshev points in $[-1,1]$, not a single scaled Chebyshev point falls between three consecutive equidistant points. Therefore, we can compute the function values at any scaled Chebyshev point using linear interpolation. The challenge arises when we have function values f at points $\{x_{i-\frac{n-1}{2}}, \ldots, x_{i+\frac{n-1}{2}}\}$, but for a local Chebyshev expansion, we require function values at Chebyshev points. To address this, we compute a linear approximation $l(f)$ of the function f within $[x_j, x_{j+1}]$ for $j = i - \dfrac{n-1}{2}, \ldots, i + \dfrac{n-3}{2}$. Once we have the linear approximation of f within $[x_j, x_{j+1}]$, we can calculate the values of f at Chebyshev points using this interpolation.

$$f\,|_{s_l} = l[f](s_l) \quad l = 1, 2, \ldots, n.$$

5. The truncated local series expansion of degree $n_p + n_q$ in each stencil S_i for $i = 1, 2, \ldots, N$ is

$$\mathsf{C}^i_{n_p+n_q,n}[f](z) := \frac{c^i_{0,n}}{2} + c^i_{1,n} z + \cdots + c^i_{n_p+n_q,n} z^{n_p+n_q}, \tag{18}$$

where $z = e^{i\cos^{-1}(t)}$ for all $t = \dfrac{x_i + x_{i+1}}{2} \in S_i$.

Hence $C^i_{d,n}[f](t)$ is the desired local Chebyshev series expansion which approximate the function f at t_i.

3.2. Local Padé-Chebyshev Approximation

For local Padé-Chebyshev approximation of a bounded piecewise smooth function f, we need to rationalize the local truncated Chebyshev series expansion in each stencil, which is explained in Section 3.1. Consider the Chebyshev series expansion (18) in a stencil S_i, for $i = 1, 2, \ldots, N$ and as explained in Section 2.2, we rationalize the truncated local series expansion (18) as

$$\frac{c^i_{0,n}}{2} + c^i_{1,n} z + \cdots + c^i_{n_p+n_q,n} z^{n_p+n_q} = \mathsf{R}^i_n(z) + O(z^{n_p+n_q+1}), \tag{19}$$

where $z = e^{i\cos^{-1}(t)}$, for all $t = \dfrac{x_i + x_{i+1}}{2} \in S_i$, and n_p, $n_q \geq 1$ are the degrees of numerator and denominator polynomials of the PC approximation

$$R_n^i(z) = \frac{\sum_{j=0}^{n_p} p_j z^j}{\sum_{j=0}^{n_q} q_j z^j}. \tag{20}$$

Recall that the coefficients p_j, $j = 1,2,...,n_p$ and q_j, $j = 1,2,...,n_q$, are obtained by solving the systems (14) and (15). We construct the PC approximation of the function f in each stencil S_i, for $i = 1,2,...,N$. Hence $R_n^i[f](t)$ is the required local Padé-Chebyshev approximation of the function f at t_i.

4. Numerical Scheme: LocPCM for Conservation Laws

In this section, we outline a reconstruction process for the numerical flux employing local Chebyshev and local Padé-Chebyshev approximation methods. Our approach utilizes a finite volume framework to construct a numerical scheme utilizing LocPCM. Consider the general form of conservation laws.

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, \qquad \text{in } \mathbb{R}\times(0,\infty) \tag{21}$$

with initial conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},$$

where $f(u) = f(u(x,t))$ is the flux function, $u_0 \in C^\infty(\mathbb{R})$ and $u = u(x,t)$ is the unknown function, $x \in \mathbb{R}$.

Let h be a function defined as

$$f(u(x_j,t)) = f(x_j,t) = \frac{1}{\Delta x_j}\int_{x_j-\Delta x_j/2}^{x_j+\Delta x_j/2} h(\xi)d\xi, \quad \text{for a fixed } t > 0, \tag{22}$$

where x_j are the Chebyshev points defined in (5), $\Delta x_j = x_j - x_{j-1}$ for $j = 1,\cdots,m$. Upon differentiating (22) with respect to x, we get

$$\frac{\partial}{\partial x}f(u) = \frac{1}{\Delta x_j}\left[h(x_j + \frac{\Delta x_j}{2}) - h(x_j - \frac{\Delta x_j}{2})\right]. \tag{23}$$

Thus (21) can be written as

$$\frac{\partial u}{\partial t} = -\frac{1}{\Delta x_j}\left[h(x_j + \frac{\Delta x_j}{2}) - h(x_j - \frac{\Delta x_j}{2})\right]. \tag{24}$$

Here the right hand side expression of (24) is an exact representation of the derivative of the flux function f in space variable and not the central difference approximation.

4.1. Chebyshev and Padé-Chebyshev reconstruction in FVM

We will use truncated Chebyshev and Padé-Chebyshev approximation to approximate the unknown function h and call the approximant H. By replacing h by H, we can reduce (24) to an ordinary differential equation of the form

$$\frac{du}{dt} = \frac{1}{\Delta x_j}\left[H(x_j + \frac{\Delta x_j}{2}) - H(x_j - \frac{\Delta x_j}{2})\right].$$

Note that, H can be taken as polynomial interpolation, spline approximation or any other approximation of h. In our study we will use truncated Chebyshev and Padé-Chebyshev approximation.

To approximate a function we need to know the function values at some points, for example in polynomial interpolation we need to know the function values at nodes. Similarly, in our case we need the values of h at Chebyshev points. But as we can see in (22), h is an implicit function and we do not know the value of h at any point. To overcome this difficulty we introduce a new function F, defined as

$$F(x) = \int_{-\infty}^{x} h(\xi)\, d\xi.$$

Then by using (22), we get

$$\begin{aligned} F(x_j + \Delta x_j / 2) &= \sum_{k=-\infty}^{j} \int_{x_k - \Delta x_k/2}^{x_k + \Delta x_k/2} h(\xi) d\xi \\ &= \Delta x_k \sum_{k=-\infty}^{j} f(x_k), \end{aligned} \tag{25}$$

for $j = 1, ..., m–1$. Since h is an unknown function, F is also unknown. However, the expression (25) shows that the value of F can be obtained at Chebyshev points in terms of the values of the flux function f at these points.

Now we approximate the derivative of F to compute the approximation of the function h, as F is the primitive function of h. We do this by applying the above discussed truncated Chebyshev and Padé technique on the differentiated Chebyshev series expansion of the function F. Finally we denote this approximation by $H \approx F'$, thus

$$h \approx H.$$

We can see that this is nothing but the approximation of the flux function f.

$$f(u(x_k, t)) = \frac{1}{\Delta x_k} \int_{x_k - \Delta x_k/2}^{x_k + \Delta x_k/2} h(\xi) d\xi,$$

differentiating the above equation, we get

$$\begin{aligned} \frac{\partial}{\partial x} f(u(x_k, t)) &= \frac{1}{\Delta x_k} \left[h(x_k + \Delta x_k / 2) - h(x_k - \Delta x_k / 2) \right] \\ &\approx \frac{1}{\Delta x_k} (H(x_k + \Delta x_k / 2) - H(x_k - \Delta x_k / 2)) \end{aligned} \tag{26}$$

Thus, we have

$$f(u(x_k, t)) \approx H(x_k),$$

for some fixed $t > 0$.

Using the approximation H, we get a semi-discrete scheme in space

$$\frac{du}{dt} = -\frac{1}{\Delta x_j} \left(H(x_j + \Delta x_j / 2) - H(x_j - \Delta x_j / 2) \right). \tag{27}$$

4.2. LocPC Reconstruction in FVM

Given a grid

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b,$$

define cells, cell center and size of a cell by

$$\begin{aligned} I_i &\equiv [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \qquad x_i \equiv \frac{x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}}{2}, \\ \Delta x_i &\equiv x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \qquad i = 1, 2, \ldots, N \end{aligned}$$

and the maximum cell size is denoted by

$$\Delta x \equiv \max_{1\le i\le N} \Delta x_i.$$

Given a function u to be approximated and let $\bar{u}_i$ denotes its cell average on the cell I_i

$$\bar{u}_i \equiv \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x)dx, \quad i=1,2,\ldots,N. \tag{28}$$

We assume that the cell averages $\bar{u}_i$ for all $i=1,2,\ldots,N$ are known. Our aim is to approximate the function u at any given point, say $x_{i+\frac{1}{2}}$, with an order of accuracy k.

Consider a stencil $S_i=\{I_{i-2},I_{i-1},I_i,I_{i+1},I_{i+2}\}$ and assume that $\mathsf{R}_i(x)$ is the rational approximation of order $[n_p/n_q]$, which approximates the function $u_i=u|_S$ on the stencil S.

$$d\bar{u}(x_i,t)dt=-\frac{1}{\Delta x_i}\Big(f(u(x_{i+\frac{1}{2}},t))-f(u(x_{i-\frac{1}{2}},t))\Big), \tag{29}$$

where

$$\bar{u}(x_i,t)=\frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(\xi,t)d\xi, \tag{30}$$

is the cell average. We approximate (29) by the following conservative scheme

$$\frac{d\bar{u}_i(t)}{dt}=-\frac{1}{\Delta x_i}(\hat{f}_{i+\frac{1}{2}}-\hat{f}_{i-\frac{1}{2}}), \tag{31}$$

where $\bar{u}_i(t)$ is the numerical approximation to the cell average $\bar{u}(x_i,t)$, and the numerical flux $\hat{f}_{i+\frac{1}{2}}$ is defined by

$$\hat{f}_{i+\frac{1}{2}}=h(u^-_{i+\frac{1}{2}},u^+_{i+\frac{1}{2}}) \tag{32}$$

with the values $u^{\pm}_{i+\frac{1}{2}}$ obtained by PPC reconstruction, *i.e.*

$$u^+_{i+\frac{1}{2}}=\mathsf{R}_1(x_{i+\frac{1}{2}})|_S, \tag{33}$$

$$u^-_{i+\frac{1}{2}}=\mathsf{R}'(x_{i+\frac{1}{2}})|'s \tag{34}$$

where $S'=\{I_{i-3},I_{i-2},I_{i-1},I_i,I_{i+1}\}$. The function h satisfies

1. h is a monotone flux.
2. $h(u,v)$ is non-increasing in second argument v and is non-decreasing in first argument u.
3. $h(u,v)$ is a Lipschitz continuous function in u as well as v.
4. $h(u,v)$ is consistent, that is, $h(u,u)=f(u)$, where f is the physical flux.

We will use Lax-Friedrichs flux,

$$h(u,v)=\frac{1}{2}[f(u)+f(v)-\alpha(v-u)], \tag{35}$$

where $\alpha=\max_u|f'(u)|$, with LocPC reconstruction of u.

By using a finite difference approximation for time we can get a fully discrete scheme in time as well as in space.

5. Numerical Results

We apply the aforementioned approximation techniques within the finite volume method to solve both linear and nonlinear scalar conservation laws. Specifically, we present results obtained for nonlinear Burgers' equations through numerical experiments, demonstrating the effectiveness of the scheme in accurately computing solutions.The rationale behind transitioning from polynomial to rational functions can be readily understood and observed.

In Figure 1(a), we observe that employing a polynomial reconstruction for the numerical flux function in the finite volume setup leads to rapid solution blow-up, even with a small CFL number. Conversely, in Figure 1(b), we note that utilizing Padé-Chebyshev approximation ensures stability, although it fails to mitigate the Gibbs phenomenon. To address this challenge, we explore a local scheme employing local Padé-Chebyshev approximation. Figure 1(c) presents the solution obtained using local Padé-Chebyshev approximation.

Example 5.1. *A well-known example of nonlinear scalar conservation law is the inviscid Burgers' equation with flux function* $f(u)=\frac{u^2}{2}$, *which is given by*

$$\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0, \tag{36}$$

with the initial condition

$$u(x,0)=e^{-x^2}, \quad x\in\mathbb{R}. \tag{37}$$

We employ Padé-Chebyshev approximation for approximating the space derivative, while the time derivative is approximated using Euler forward and fourth-order Runge-Kutta discretization. The respective degrees of the numerator and denominator, denoted as n_p *and* n_q *are specified in the caption. We utilize a total of* $N=500$ *nodes for approximating the Chebyshev coefficients.*

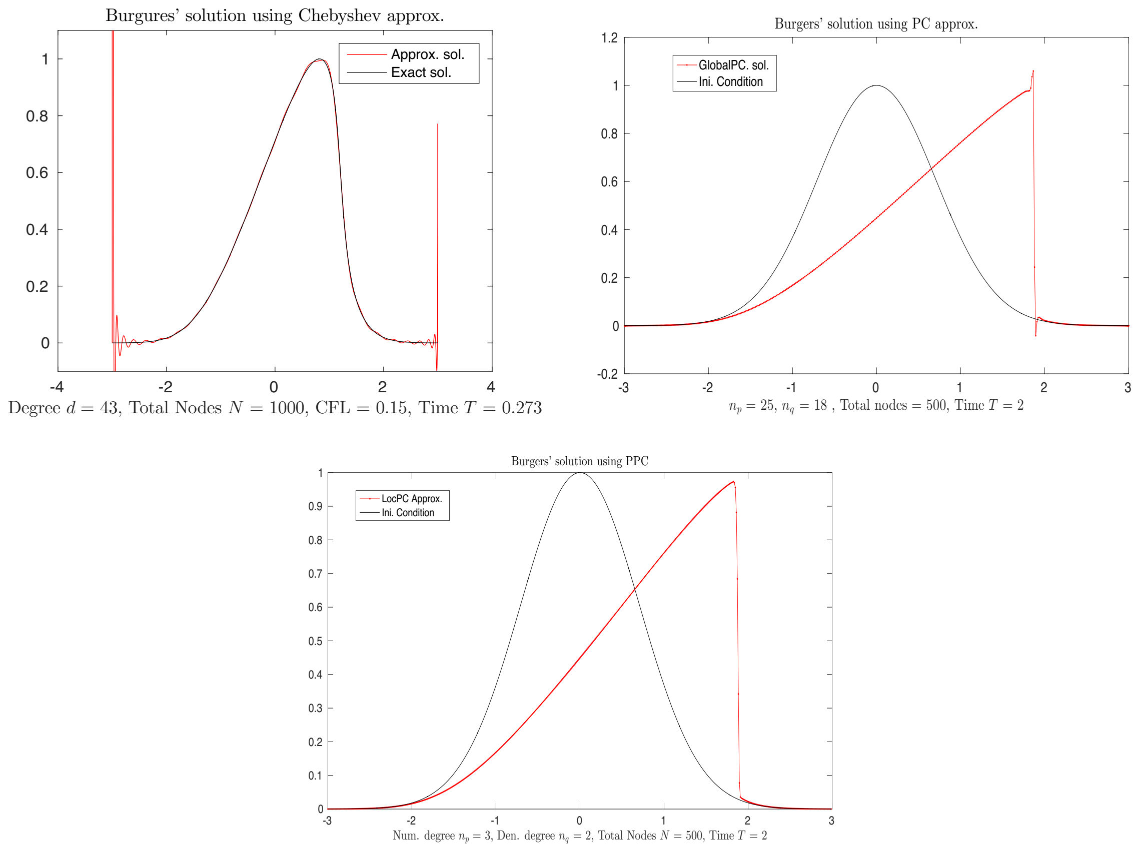

Figure 1: Comparison of all three methods on Burgers' equation.

In Figure 1, we compare all three proposed schemes for conservation laws and apply them to Burgers' equation. We depict the initial condition and the approximate solution for the Burgers' equation using Euler forward discretization in time.

In the first plot, we utilize truncated Chebyshev series approximation to reconstruct the numerical flux. It is evident from this plot that the solution becomes unbounded even after a few time steps.

In the second plot, we employ Padé-Chebyshev approximation to reconstruct the flux. In this case, we observe stability up to two shock lengths, indicating that the approximate solution does not blow up. However, numerically, we find that the proposed numerical scheme fails to eliminate oscillations.

In the third plot, we employ LocPC reconstruction of the solution u at each time level. In this scenario, we successfully mitigate oscillations from the solution.

6. Conclusion

We propose a non-linear local approximation algorithm, namely the Local Padé-Chebyshev (LocPC) algorithm, designed for functions with discontinuities. This algorithm effectively approximates the function with minimal Gibbs' oscillations. We provide numerical results to support our approach.

Furthermore, we successfully apply the LocPC algorithm to formulate a numerical scheme for scalar hyperbolic conservation laws. To the best of our knowledge, this marks the first attempt to utilize non-linear approximation in numerical schemes for solving hyperbolic partial difference equations (PDEs).

Considering the superior [34, 6] performance of non-linear functions over linear functions in approximating non-smooth functions, we anticipate promising outcomes for future numerical schemes for PDEs. Our future study direction involves the following analytical investigations:

1. Analysis of local Padé-Chebyshev approximation of non-smooth functions.
2. Exploration of the relationship between the order of accuracy and various parameters, such as numerator degree, denominator degree, number of cells, etc.

These endeavors aim to further elucidate the efficacy and potential applications of non-linear approximation techniques in the realm of numerical methods for PDEs.

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