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A fixed point technique for solving boundary value problems in Branciari Suprametric Spaces

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Abstract

The technique to broaden the scope of fixed point theory is to extend the class of spaces that have stronger conceptual frameworks than metric spaces. Therefore, this paper explores the introduction of novel metric spaces, namely, Branciari suprametric spaces, and investigates some of its fundamental topological properties. An illustration is provided to validate the newly defined idea of Branciari suprametric spaces. Further two intriguing, fixed point results are proved, and a corollary is presented as an implication of our main result. The following is a specification of the analogue of the rectangle inequality in Branciari suprametric spaces $d_B(\tau,\iota) \le d_B(\tau,\nu) + d_B(\nu,\sigma) + d_B(\sigma,\iota) + \mu d_B(\tau,\nu) d_B(\nu,\sigma) d_B(\sigma,\iota)$ for all $\tau \ne \nu$, $\nu \ne \sigma$ and $\sigma \ne \iota$. Furthermore, by employing the results obtained, the present study intends to provide an appropriate solution for the nonlinear fractional differential equations of the Riemann-Liouville type.

Keywords: Branciari suprametric space, contraction, fixed point, nonlinear fractional differential equations of the Riemann-Liouville type

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1. Introduction and Preliminaries

Fixed point approaches, especially are being applied in domains that include biology, engineering, chemistry, physics, game theory, and economics. In several disciplines of mathematics and computational science, the problems of the existence varied solutions to mathematical models relates to the existence of a fixed point problem for a particular map. Therefore, in order a variety of scientific fields across all areas rely substantially on the study of fixed points.

The development of fixed point theory was facilitated by the French mathematician Frechet's idea of metric spaces. In an attempt to improve the possibility of obtaining more generic fixed point results, the conception of a metric space has undergone numerous approaches to be extended. In analysis, the triangle inequality is among the most essential and effective inequalities. The triangle inequality has been extended to varying generalizations that are satisfied by different distance functions. These include the b-metric spaces by Czerwik [7], at which the triangle inequality equation is multiplied by the constant b on the right side; the extended b-metric spaces by Kamran [23], from which the triangle inequality equation is multiplied by the funtion $\theta(\tau,\iota)$ on the right side; and the generalized metric spaces by Branciari [5], wherein the triangular inequality of metric spaces has been supplanted by a new inequality that is referred to as rectangular inequality.

Furthermore, a significant number of new distance functions were developed by combining, easing, or expanding some of the tenets of the already-existing distance functions with the objective to address the growing uncertainty of practical applications. A number of articles (see [1, 11, 24–27, 30] with the cited works therein) have addressed fixed point theory for single-valued mappings throughout various abstract spaces.

The idea of b-metric spaces was established by Czerwik [7] as follows:

Definition 1.1. *[7] Let $Y \neq \emptyset$ and $b \geq 1$ be any real number. A function $d_b : Y \times Y \to [0,\infty)$ is a b-metric if and only if for each $\tau, \iota, \nu \in Y$, the preceding conditions are satisfied:*

1. $d_b(\tau,\iota) \geq 0$ *and* $d_b(\tau,\iota) = 0$ *if and only if* $\tau = \iota$;
2. $d_b(\tau,\iota) = d_b(\iota,\tau)$;
3. $d_b(\tau,\iota) \leq b[d_b(\tau,\nu) + d_b(\nu,\iota)]$.

Then d_b is said to be a b-metric on Y and (Y, d_b) is called a b-metric space.

Rectangular metric spaces were initially developed by Branciari [5] with the following definition:

Definition 1.2. *[5] Let $Y \neq \emptyset$ and the mapping $d_R : Y \times Y \to [0,\infty)$ is a rectangular metric if and only if the conditions listed below are fulfilled:*

1. $d_R(\tau,\iota) \geq 0$ *and* $d_R(\tau,\iota) = 0$ *if and only if* $\tau = \iota$;
2. $d_R(\tau, \iota) = d_R(\iota, \tau)$;
3. $d_R(\tau,\iota) \leq d_R(\tau,\nu) + d_R(\nu,\sigma) + d_R(\sigma,\iota)$, *for all* $\tau,\iota \in Y$ *and all distinct* $\nu,\sigma \in Y \setminus \{\tau,\iota\}$

Then (Y, d_R) is said to be a rectangular metric space.

Recently, a modified triangular inequality was employed in the new metric known as suprametric, originally presented by Maher Berzig [2], who additionally investigated several important aspects of its topology. Following that, the author demonstrated that specific contraction maps in suprametric spaces possess an unique fixed point if the space is complete or comprises a non-empty $\varkappa$-limit set. For more current works on suprametric spaces, researchers can refer to the articles [3, 4, 28, 31].

The following gives the definition of suprametric space:

Definition 1.3. *[2] Let Y be a nonempty set. A function is $d_s : Y \times Y \to [0,\infty)$ a suprametric if for all $\tau,\iota,\nu \in Y$, the following conditions hold:*

1. $d_s(\tau,\iota)=0$ *if and only if* $\tau=\iota$;

2. $d_s(\tau,\iota)=d_s(\iota,\tau)$ *for all* $\tau,\iota\in Y$;

3. $d_s(\tau,\iota)\le d_s(\tau,\nu)+d_s(\nu,\iota)+\mu d_s(\tau,\nu)d_b(\nu,\iota)$ *for some constant* $\mu\in\mathbb{R}^+$.

Then d_s *is called a suprametric on Y and (Y,* d_s*) is named a suprametric space.*

Definition 1.4. *[2] Let* γ *be a metric on Y and* α *be a positive real. Then* $d_s(\tau,\iota)=\alpha(e^{\gamma(\tau,\iota)}-1)$ *is a suprametric with constant* $\mu=\dfrac{1}{\alpha}$ *respectively.*

On the other hand, Liouville and Riemann established the very first definition of fractional derivative at the culmination of the 19th century, however Leibniz and L'Hospital initially proposed the idea of non-integer derivative and integral in 1695 as an interpretation of the standard integer order differential and integral calculus. In practical terms, derivatives with fractional values provides an ideal approach to articulate the memory and inherited qualities associated with various processes and techniques. In recent years, the research on fractional differential equations has increased significantly. In order to determine the existence of and distinctiveness of or the multiplicity of solutions to nonlinear fractional differential equation boundary value problems, as well as to deal with other problems involving nonlinear fractional differential equations, nonlinear analysis techniques, which serve as the primary approach for accomplishing this, play a significant part in the investigations within this field (see [8–10, 13–22, 27, 29, 32–38] and the sources listed therein).

Incited by all of the works listed above, in the current study, we extend the works of Maher Berzig [2] in Section 2 by setting up the idea of Branciari (or rectangular) suprametric spaces by presenting an example for the given metric. In Section 3, we establish several intriguing fixed point results under varying contractive conditions. The boundary value problem of a class of fractional differential equations consisting of the Riemann-Liouville fractional derivative is then examined in Section 4 by employing the established fixed point result, with the goal of determining if such solutions exist and whether they are unique.

2. Main Results

The present section explores the conception of Branciari suprametric spaces which is an appropriate extension that encompasses suprametric spaces [2] and rectangular metric spaces [5].

Definition 2.1. *Let Y be a nonempty set and* $\mu\in\mathbb{R}^+$*. A function* $d_{\mathcal{B}}:Y\times Y\to[0,\infty)$ *is called a Branciari suprametric if it satisfies:*

1. $d_{\mathcal{B}}(\tau,\iota)=0$ *if and only if* $\tau=\iota$ *for all* $\tau,\iota\in Y$;

2. $d_{\mathcal{B}}(\tau,\iota)=d_{\mathcal{B}}(\iota,\tau)$ *for all* $\tau,\iota\in Y$;

3. $d_{\mathcal{B}}(\tau,\iota)\le d_{\mathcal{B}}(\tau,\nu)+d_{\mathcal{B}}(\nu,\sigma)+d_{\mathcal{B}}(\sigma,\iota)+\mu d_{\mathcal{B}}(\tau,\nu)d_{\mathcal{B}}(\nu,\sigma)d_{\mathcal{B}}(\sigma,\iota)$.

for all $\tau,\iota\in Y$ *and for all* $\nu,\sigma\in Y$ *each distinct from* τ *and* ι *respectively. The pair* $(Y,d_{\mathcal{B}})$ *is named a Branciari suprametric space.*

Definition 2.2. *Let* $Y=\mathbb{N}$ *and* κ *is a rectangular metric on Y which is given by*

$$\kappa(\tau,\iota)=\begin{cases}0, & if\ \tau=\iota\\ 3, & if\ \tau,\iota\in\{5,6\}\ and\ \tau\neq\iota\\ 1, & if\ \tau \text{ or } \iota\notin\{5,6\}\ and\ \tau\neq\iota\end{cases}$$

Let $d_{\mathcal{B}} : Y \times Y \to [0,\infty)$ be defined by $d_{\mathcal{B}}(\tau,\iota) = \kappa(\tau,\iota)(\gamma + \kappa(\tau,\iota))$ where γ is a positive real number. It is apparent that $d_{\mathcal{B}}$ is a Branciari suprametric with the constant $\mu = \frac{6}{\gamma^3}$. Nevertheless, we observe that (i) $(Y,d_{\mathcal{B}})$ is not a rectangular metric space for $\gamma = 1$ because of the fact that $d_{\mathcal{B}}(5,6) = 12 > d_{\mathcal{B}}(5,1) + d_{\mathcal{B}}(1,2) + d_{\mathcal{B}}(2,6) = 6$ (ii) $(Y,d_{\mathcal{B}})$ is not a suprametric space for $\gamma = 3$ owing to the fact that $d_{\mathcal{B}}(5,6) = 18 > d_{\mathcal{B}}(5,2) + d_{\mathcal{B}}(2,6) + \frac{6}{3^3} d_{\mathcal{B}}(5,2) d_{\mathcal{B}}(2,6) = 11.56$.

Definition 2.3. *Let $(Y,d_{\mathcal{B}})$ be a Branciari suprametric space. The set*

$$B_{\mathcal{B}}(\tau_0,r_0) = \{\iota \in Y : d_{\mathcal{B}}(\tau_0,\iota) < r_0\},$$

where $r_0 > 0$ and $\tau_0 \in Y$ is called an open ball of radius r_0 and center τ_0.

Definition 2.4. *Let $(Y,d_{\mathcal{B}})$ be a Branciari suprametric space. A sequence $\{\tau_\mathfrak{y}\}$ in Y referred to as:*

1. *converges to $\tau \Leftrightarrow$ for every $\epsilon > 0$ there is $\mathsf{N} = \mathsf{N}(\epsilon) \in \mathbb{N}$ so that $d_{\mathcal{B}}(\tau_\mathfrak{y},\tau) < \epsilon$ for all $\mathfrak{y} \geq \mathsf{N}$.*
2. *Cauchy $\Leftrightarrow$ for every $\epsilon > 0$ there is $\mathsf{N} = \mathsf{N}(\epsilon) \in \mathbb{N}$ so that $d_{\mathcal{B}}(\tau_\mathfrak{y},\tau_\mathfrak{z}) < \epsilon$ for all $\mathfrak{y},\mathfrak{z} \geq \mathsf{N}$.*

Definition 2.5. *A Branciari suprametric space $(Y,d_{\mathcal{B}})$ is complete if and only if every Cauchy sequence in Y is convergent.*

3. Fixed Point Theorems on Branciari Suprametric Spaces

In this section, we establish two intriguing fixed point results depending on specific contractive conditions in the conceptual framework of Branciari suprametric spaces.

Theorem 3.1. *Let $(Y,d_{\mathcal{B}})$ be a complete Branciari suprametric space and $G : Y \to Y$ is a mapping satisfying*

$$d_{\mathcal{B}}(G\tau,G\iota) \leq g d_{\mathcal{B}}(\tau,\iota), \text{ for all } \tau,\iota \in Y \tag{3.1}$$

where $g \in [0,1)$. Then the fixed point of G is unique.

Prof. For every $\tau_0 \in Y$, the iterative sequence $\{\tau_\mathfrak{y}\}$ is specified by $\tau_\mathfrak{y} = G\tau_{\mathfrak{y}-1}, \mathfrak{y} \in \mathbb{N}$. Inequality (3.1) yields

$$\begin{aligned} d_{\mathcal{B}}(\tau_\mathfrak{y},\tau_{\mathfrak{y}+1}) &= d_{\mathcal{B}}(G\tau_{\mathfrak{y}-1},G\tau_\mathfrak{y}) \\ &\leq g d_{\mathcal{B}}(\tau_{\mathfrak{y}-1},\tau_\mathfrak{y}) \\ &< d_{\mathcal{B}}(\tau_{\mathfrak{y}-1},\tau_\mathfrak{y}) \end{aligned} \tag{3.2}$$

Similarly to that, we observe

$$\begin{aligned} d_{\mathcal{B}}(\tau_\mathfrak{y},\tau_{\mathfrak{y}+2}) &= d_{\mathcal{B}}(G\tau_{\mathfrak{y}-1},G\tau_{\mathfrak{y}+1}) \\ &\leq g d_{\mathcal{B}}(\tau_{\mathfrak{y}-1},\tau_{\mathfrak{y}+1}) \\ &< d_{\mathcal{B}}(\tau_{\mathfrak{y}-1},\tau_{\mathfrak{y}+1}) \end{aligned} \tag{3.3}$$

Thereby, the sequences $\{d_{\mathcal{B}}(\tau_\mathfrak{y},\tau_{\mathfrak{y}+1})\}$ and $\{d_{\mathcal{B}}(\tau_\mathfrak{y},\tau_{\mathfrak{y}+2})\}$ are decreasing and with regard to all fixed integer k and for all $\mathfrak{y} > \text{k}$, it fulfils

$$d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1}) \le g^{\mathfrak{y}-\mathrm{k}} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) \tag{3.4}$$

and

$$d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2}) \le g^{\mathfrak{y}-\mathrm{k}} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+2}). \tag{3.5}$$

This implies $\lim_{\mathfrak{y}\to\infty} d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1}) = \lim_{\mathfrak{y}\to\infty} d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2}) = 0$. Consequently there exists $\mathrm{k}\in\mathbb{N}$ so that $d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1}) \le 1$ and $d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2}) \le 1$ for all $\mathfrak{y} \ge \mathrm{k}$. In order to illustrate $\{\tau_{\mathfrak{y}}\}$ is Cauchy, we consider $d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+\mathfrak{z}})$ in a pair of distinct cases.

Case 1: We proceed by delving into an odd number represented as $\mathfrak{z} = 2\mathfrak{m}+1$, where $\mathfrak{m} \ge 1$, then the following is derived from inequality (3.4):

$$\begin{aligned} d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) &\le d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\quad + \mu d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\le g^{\mathfrak{y}-\mathrm{k}} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) + g^{\mathfrak{y}-\mathrm{k}+1} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\quad + \mu g^{\mathfrak{y}-\mathrm{k}} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) g^{\mathfrak{y}-\mathrm{k}+1} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+1} + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+1}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) &\le d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+3}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+4}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\quad + \mu d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+3}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+4}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\le g^{\mathfrak{y}-\mathrm{k}+2} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) + g^{\mathfrak{y}-\mathrm{k}+3} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\quad + \mu g^{\mathfrak{y}-\mathrm{k}+2} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) g^{\mathfrak{y}-\mathrm{k}+3} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\le g^{\mathfrak{y}-\mathrm{k}+2} + g^{\mathfrak{y}-\mathrm{k}+3} + (1+\mu g^{\mathfrak{y}-\mathrm{k}+2} g^{\mathfrak{y}-\mathrm{k}+3}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \end{aligned}$$

Accordingly, the inequality (3.6) yields

$$\begin{aligned} d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) &\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+1} + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+1}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+1} + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+1})[g^{\mathfrak{y}-\mathrm{k}+2} + g^{\mathfrak{y}-\mathrm{k}+3} \\ &\quad + (1+\mu g^{\mathfrak{y}-\mathrm{k}+2} g^{\mathfrak{y}-\mathrm{k}+3}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1})] \\ &\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+1} + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+1})[g^{\mathfrak{y}-\mathrm{k}+2} + g^{\mathfrak{y}-\mathrm{k}+3}] \\ &\quad + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+1})(1+\mu g^{\mathfrak{y}-\mathrm{k}+2} g^{\mathfrak{y}-\mathrm{k}+3}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+1} + (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+1})[g^{\mathfrak{y}-\mathrm{k}+2} + g^{\mathfrak{y}-\mathrm{k}+3}] \\ &\quad + (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+1})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+5})[g^{\mathfrak{y}-\mathrm{k}+4} + g^{\mathfrak{y}-\mathrm{k}+5}] \\ &\quad \vdots \\ &\quad + (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+1})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+5})\ldots(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathfrak{m}-7}) \\ &\quad [g^{\mathfrak{y}-\mathrm{k}+2\mathfrak{m}-2} + g^{\mathfrak{y}-\mathrm{k}+2\mathfrak{m}-1}] + (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+1})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+5})\ldots \\ &\quad (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathfrak{m}-7})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathfrak{m}-3}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2\mathfrak{m}},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \\ &= g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+1} + \sum_{i=1}^{\mathfrak{m}-1} [g^{\mathfrak{y}-\mathrm{k}+2i} + g^{\mathfrak{y}-\mathrm{k}+2i+1}] \prod_{j=1}^{i} (1+\mu g^{2(\mathfrak{y}-k)+4j-3}) \\ &\quad + \prod_{\mathrm{i}=1}^{\mathfrak{m}} (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathrm{i}-3}) g^{\mathfrak{y}-\mathrm{k}+2\mathfrak{m}} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) \end{aligned}$$

Considering the fact that $g \in [0,1)$, it implies

$$d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) \le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+1} + g^{\mathfrak{y}-\mathrm{k}}\sum_{i=1}^{\mathfrak{m}-1}[g^{2i}+g^{2i+1}]\prod_{j=1}^{i}(1+\mu g^{4j-3}) + g^{\mathfrak{y}-\mathrm{k}+2\mathfrak{m}}\sum_{i=1}^{\mathfrak{m}}(1+\mu g^{4i-3}) \tag{3.7}$$

Let $\mathcal{W}_{\mathrm{i}} = (g^{2\mathrm{i}}+g^{2\mathrm{i}+1})\prod_{\mathrm{j}=1}^{\mathrm{i}}(1+\mu g^{4\mathrm{j}-3})$. By ratio test, $\sum_{\mathrm{i}=1}^{\infty}\mathcal{W}_{\mathrm{i}}$ converges, as $\lim_{\mathrm{i}\to\infty}\left|\frac{\mathcal{W}_{\mathrm{i}+1}}{\mathcal{W}_{\mathrm{i}}}\right|<1$ and $g\in[0,1)$. Equation (3.4) thus leads to the conclusion that $d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}+1})$ tends to zero as $\mathfrak{y},\mathfrak{m}$ tend to infinity.

Case 2: Now, suppose an even number represented by $\mathfrak{z} = 2\mathfrak{m}$, where $\mathfrak{m} \ge 1$, then the following is obtained from inequalities (3.4) and (3.5):

$$\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}}) &\le d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+3}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\quad +\mu d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+3}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\le g^{\mathfrak{y}-\mathrm{k}} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+2}) + g^{\mathfrak{y}-\mathrm{k}+2} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\quad +\mu g^{\mathfrak{y}-\mathrm{k}} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+2}) g^{\mathfrak{y}-\mathrm{k}+2} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+2} + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+2}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+2\mathfrak{m}}),
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+2\mathfrak{m}}) &\le d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+4}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+5}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+5},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\quad +\mu d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+4}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+5}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+5},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\le g^{\mathfrak{y}-\mathrm{k}+3} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) + g^{\mathfrak{y}-\mathrm{k}+4} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+5},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\quad +\mu g^{\mathfrak{y}-\mathrm{k}+3} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) g^{\mathfrak{y}-\mathrm{k}+4} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+5},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\le g^{\mathfrak{y}-\mathrm{k}+3} + g^{\mathfrak{y}-\mathrm{k}+4} + (1+\mu g^{\mathfrak{y}-\mathrm{k}+3} g^{\mathfrak{y}-\mathrm{k}+4}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+5},\tau_{\mathfrak{y}+2\mathfrak{m}})
\end{aligned}$$

Accordingly, the inequality (3.8) yields

$$\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}}) &\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+2} + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+2}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+2} + (1+\mu g^{\mathfrak{y}-\mathrm{k}} g^{\mathfrak{y}-\mathrm{k}+2})[g^{\mathfrak{y}-\mathrm{k}+3} + g^{\mathfrak{y}-\mathrm{k}+4}\\
&\quad +(1+\mu g^{\mathfrak{y}-\mathrm{k}+3} g^{\mathfrak{y}-\mathrm{k}+4}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+5},\tau_{\mathfrak{y}+2\mathfrak{m}})]\\
&\;\;\vdots\\
&\le g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+2} + (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+2})[g^{\mathfrak{y}-\mathrm{k}+3} + g^{\mathfrak{y}-\mathrm{k}+4}]\\
&\quad +(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+2})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+7})[g^{\mathfrak{y}-\mathrm{k}+5} + g^{\mathfrak{y}-\mathrm{k}+6}]\\
&\;\;\vdots\\
&\quad +(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+2})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+7})\dots(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathfrak{m}-9})\\
&\quad [g^{\mathfrak{y}-\mathrm{k}+2\mathfrak{m}-3} + g^{\mathfrak{y}-\mathrm{k}+2\mathfrak{m}-2}] + (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+2})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+7})\dots\\
&\quad (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathfrak{m}-9})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathfrak{m}-5}) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2\mathfrak{m}-1},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&= g^{\mathfrak{y}-\mathrm{k}} + g^{\mathfrak{y}-\mathrm{k}+2} + (1+\mu g^{2(\mathfrak{y}-\mathrm{k})+2})[g^{\mathfrak{y}-\mathrm{k}+3} + g^{\mathfrak{y}-\mathrm{k}+4}]\\
&\quad +(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+2})\sum_{\mathrm{i}=1}^{\mathfrak{m}-3}\left[g^{\mathfrak{y}-\mathrm{k}+2\mathrm{i}+3} + g^{\mathfrak{y}-\mathrm{k}+2\mathrm{i}+4}\right]\prod_{\mathrm{j}=1}^{\mathrm{i}}(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathrm{j}+3})\\
&\quad +\prod_{\mathrm{i}=1}^{\mathfrak{m}-2}(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+4\mathrm{i}+3})(1+\mu g^{2(\mathfrak{y}-\mathrm{k})+2}) g^{\mathfrak{y}-\mathrm{k}+2\mathfrak{m}-1} d_{\mathcal{B}}(\tau_{\mathrm{k}},\tau_{\mathrm{k}+1})
\end{aligned}$$

Considering the fact that $g \in [0,1)$, it implies

$$\begin{aligned} d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+2\mathfrak{m}}) &\leq g^{\mathfrak{y}-k} + g^{\mathfrak{y}-k+2} + (1+\mu g^2)[g^{\mathfrak{y}-k+3} + g^{\mathfrak{y}-k+4}] \\ &\quad + (1+\mu g^2)\sum_{i=1}^{\mathfrak{m}-3} g^{\mathfrak{y}-k}[g^{2i+3} + g^{2i+4}]\prod_{j=1}^{i}(1+\mu g^{4j+3}) \\ &\quad + \prod^{\mathfrak{m}-2}(1+\mu g^{4i+3})(1+\mu g^2)g^{\mathfrak{y}-k+2\mathfrak{m}-1} \end{aligned} \tag{3.9}$$

Let $\mathcal{W}_i = (g^{2i+3} + g^{2i+4})\prod_{j=1}^{i}(1+\mu g^{4j+3})$. Through ratio test, we find that $\sum_{i=1}^{\infty}\mathcal{W}_i$ converges, as $\lim_{i\to\infty}\left|\frac{\mathcal{W}_{i+1}}{\mathcal{W}_i}\right| < 1$ and $g \in [0,1)$. Equation (3.9) thus results in the conclusion that $d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+2\mathfrak{m}})$ tends to zero as $\mathfrak{y}, \mathfrak{m}$ tend to infinity. Accordingly, from both the cases, we get

$$d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+\mathfrak{z}}) = 0, \forall \mathfrak{y}, \mathfrak{z} \in \mathbb{N}.$$

Therefore the sequence $\{\tau_{\mathfrak{y}}\}$ is Cauchy. The existence of some $\tau \in Y$ corresponding to $\tau_{\mathfrak{y}} \to \tau$ is ensured by the completeness of *Y*. Furthermore, we establish that τ is a fixed point of *G*. Consider

$$\begin{aligned} d_{\mathcal{B}}(G\tau, \tau) &\leq d_{\mathcal{B}}(G\tau, G\tau_{\mathfrak{y}}) + d_{\mathcal{B}}(G\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau) + \mu d_{\mathcal{B}}(G\tau, G\tau_{\mathfrak{y}})d_{\mathcal{B}}(G\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}})d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau) \\ &\leq g d_{\mathcal{B}}(\tau, \tau_{\mathfrak{y}}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}+1}, \tau_{\mathfrak{y}}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau) + \mu g d_{\mathcal{B}}(\tau, \tau_{\mathfrak{y}})d_{\mathcal{B}}(\tau_{\mathfrak{y}+1}, \tau_{\mathfrak{y}})d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau) \end{aligned}$$

Letting $\mathfrak{y} \to \infty$ in the previously given inequality, we find $d_{\mathcal{B}}(G\tau, \tau) = 0$ i.e., $G\tau = \tau$. Thereby τ is a fixed point of *G*. Through the use of inequality (3.1), we can easily show that τ is a unique fixed point of *G*. □

Theorem 3.2. *Let $(Y, d_{\mathcal{B}})$ be a complete Branciari suprametric space and $G: Y \to Y$ is a mapping. Suppose that there exist mappings $g_1, g_2 : Y \times Y \to [0,\infty)$ such that $g_1 + g_2 < 1$ and*

$$d_{\mathcal{B}}(G\tau, G\iota) \leq g_1(\tau, G\tau)d_{\mathcal{B}}(\tau, G\tau) + g_2(\iota, G\iota)d_{\mathcal{B}}(\iota, G\iota), \tag{3.10}$$

for all $\tau, \iota \in Y$. Then the fixed point of G is unique.

Prof. For every $\tau_0 \in Y$, the iterative sequence $\{\tau_{\mathfrak{y}}\}$ is specified by $\tau_{\mathfrak{y}} = G\tau_{\mathfrak{y}-1}, \mathfrak{y} \in \mathbb{N}$. Inequality (3.10) gives with the following

$$\begin{aligned} d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+1}) &= d_{\mathcal{B}}(G\tau_{\mathfrak{y}-1}, G\tau_{\mathfrak{y}}) \\ &\leq g_1(\tau_{\mathfrak{y}-1}, G\tau_{\mathfrak{y}-1})d_{\mathcal{B}}(\tau_{\mathfrak{y}-1}, G\tau_{\mathfrak{y}-1}) + g_2(\tau_{\mathfrak{y}}, G\tau_{\mathfrak{y}})d_{\mathcal{B}}(\tau_{\mathfrak{y}}, G\tau_{\mathfrak{y}}) \\ &= g_1(\tau_{\mathfrak{y}-1}, \tau_{\mathfrak{y}})d_{\mathcal{B}}(\tau_{\mathfrak{y}-1}, \tau_{\mathfrak{y}}) + g_2(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+1})d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+1}) \end{aligned}$$

Therefore, we get

$$\begin{aligned} d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+1}) &\leq \Big[\frac{g_1(\tau_{\mathfrak{y}-1}, \tau_{\mathfrak{y}})}{1 - g_2(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+1})}\Big]d_{\mathcal{B}}(\tau_{\mathfrak{y}-1}, \tau_{\mathfrak{y}}) \\ &= \kappa_{\mathfrak{y}-1}d_{\mathcal{B}}(\tau_{\mathfrak{y}-1}, \tau_{\mathfrak{y}}) \\ &< d_{\mathcal{B}}(\tau_{\mathfrak{y}-1}, \tau_{\mathfrak{y}}), \end{aligned} \tag{3.11}$$

where $\kappa_{\mathfrak{y}-1} = \dfrac{g_1(\tau_{\mathfrak{y}-1}, \tau_{\mathfrak{y}})}{1 - g_2(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+1})} < 1$. As a result, the sequence $\{d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+1})\}$ is decreasing and with regard to all fixed integer k and for all $\mathfrak{y} > k$, it fulfils

$$d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1})\leq\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}d_{\mathcal{B}}(\tau_{k},\tau_{k+1}) \tag{3.12}$$

This implies $\lim_{\mathfrak{y}\to\infty}d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1})=0$, subsequently there exists $k\in\mathbb{N}$ so that for all $\mathfrak{y}\geq k$, we have $d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1})\leq 1$. Consider

$$\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2}) &= d_{\mathcal{B}}(G\tau_{\mathfrak{y}-1},G\tau_{\mathfrak{y}+1})\\
&\leq g_1(\tau_{\mathfrak{y}-1},G\tau_{\mathfrak{y}-1})d_{\mathcal{B}}(\tau_{\mathfrak{y}-1},G\tau_{\mathfrak{y}-1})+g_2(\tau_{\mathfrak{y}+1},G\tau_{\mathfrak{y}+1})d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},G\tau_{\mathfrak{y}+1})\\
&= g_1(\tau_{\mathfrak{y}-1},\tau_{\mathfrak{y}})d_{\mathcal{B}}(\tau_{\mathfrak{y}-1},\tau_{\mathfrak{y}})+g_2(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})\\
&\leq g_1(\tau_{\mathfrak{y}-1},\tau_{\mathfrak{y}})\prod_{i=1}^{\mathfrak{y}-k-1}\kappa_{\mathfrak{y}-i-1}d_{\mathcal{B}}(\tau_{k},\tau_{k+1})+g_2(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}d_{\mathcal{B}}(\tau_{k},\tau_{k+1})\\
&= [g_1(\tau_{\mathfrak{y}-1},\tau_{\mathfrak{y}})+\kappa_{k}\kappa_{k+1}g_2(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})]\prod_{i=1}^{\mathfrak{y}-k-1}\kappa_{\mathfrak{y}-i-1}d_{\mathcal{B}}(\tau_{k},\tau_{k+1})\\
&= \xi_{\mathfrak{y}}\prod_{i=1}^{\mathfrak{y}-k-1}\kappa_{\mathfrak{y}-i-1}d_{\mathcal{B}}(\tau_{k},\tau_{k+1}),
\end{aligned} \tag{3.13}$$

where $\xi_{\mathfrak{y}}=g_1(\tau_{\mathfrak{y}-1},\tau_{\mathfrak{y}})+\kappa_{k}\kappa_{k+1}g_2(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})>0$. In order to illustrate $\{\tau_{\mathfrak{y}}\}$ is Cauchy, we take into $d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+\mathfrak{z}})$ in a pair of distinct cases.

Case 1: We proceed by delving into an odd number represented as $\mathfrak{z}=2\mathfrak{m}+1$, where $\mathfrak{m}\geq 1$, then the following is obtained from (3.12):

$$\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) &\leq d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1})+d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})+d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\quad+\mu d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1})d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\leq \prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}d_{\mathcal{B}}(\tau_{k},\tau_{k+1})+\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}d_{\mathcal{B}}(\tau_{k},\tau_{k+1})+d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\quad+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}d_{\mathcal{B}}(\tau_{k},\tau_{k+1})\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}d_{\mathcal{B}}(\tau_{k},\tau_{k+1})d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\leq \prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}+\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1})
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) &\leq d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+3})+d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+4})+d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\quad+\mu d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+3})d_{\mathcal{B}}(\tau_{\mathfrak{y}+3},\tau_{\mathfrak{y}+4})d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\leq \prod_{i=1}^{\mathfrak{y}-k+2}\kappa_{\mathfrak{y}-i+2}+\prod_{i=1}^{\mathfrak{y}-k+3}\kappa_{\mathfrak{y}-i+3}+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k+2}\kappa_{\mathfrak{y}-i+2}\prod_{i=1}^{\mathfrak{y}-k+3}\kappa_{\mathfrak{y}-i+3}\Big)\\
&\quad d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1})
\end{aligned}$$

Inequality (3.14) subsequently follows

$$\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}+1}) &\leq \prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}+\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\\
&\quad\times\Big[\prod_{i=1}^{\mathfrak{y}-k+2}\kappa_{\mathfrak{y}-i+2}+\prod_{i=1}^{\mathfrak{y}-k+3}\kappa_{\mathfrak{y}-i+3}\Big]+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\\
&\quad\times\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k+2}\kappa_{\mathfrak{y}-i+2}\prod_{i=1}^{\mathfrak{y}-k+3}\kappa_{\mathfrak{y}-i+3}\Big)d_{\mathcal{B}}(\tau_{\mathfrak{y}+4},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\quad\vdots
\end{aligned}$$

$$
\begin{aligned}
&\leq \prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}+\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\\
&\times\Big[\prod_{i=1}^{\mathfrak{y}-k+2}\kappa_{\mathfrak{y}-i+2}+\prod_{i=1}^{\mathfrak{y}-k+3}\kappa_{\mathfrak{y}-i+3}\Big]\\
&\vdots\\
&+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\dots\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-4}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-4}\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-3}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-3}\Big)\\
&\times\Big[\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-2}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-2}+\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-1}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-1}\Big]\\
&+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\dots\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-2}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-2}\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-1}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-1}\Big)\\
&d_{\mathcal{B}}(\tau_{\mathfrak{y}+2\mathfrak{m}},\tau_{\mathfrak{y}+2\mathfrak{m}+1})\\
&\leq \prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}+\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\Big[\prod_{i=1}^{\mathfrak{y}-k+2}\kappa_{\mathfrak{y}-i+2}+\prod_{i=1}^{\mathfrak{y}-k+3}\kappa_{\mathfrak{y}-i+3}\Big]\\
&\vdots\\
&+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\dots\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-4}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-4}\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-3}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-3}\Big)\\
&\times\Big[\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-2}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-2}+\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-1}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-1}\Big]\\
&+\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}\Big)\dots\Big(1+\mu\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-2}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-2}\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-1}\kappa_{\mathfrak{y}-i+2\mathfrak{m}-1}\Big)\\
&\times\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}}\kappa_{\mathfrak{y}-i+2\mathfrak{m}}\,d_{\mathcal{B}}(\tau_k,\tau_{k+1})\\
&\leq \prod_{i=1}^{\mathfrak{y}-k}\kappa_{\mathfrak{y}-i}+\prod_{i=1}^{\mathfrak{y}-k+1}\kappa_{\mathfrak{y}-i+1}+\sum_{i=1}^{\mathfrak{m}-1}\Big[\prod_{l=1}^{\mathfrak{y}-k+2i}\kappa_{\mathfrak{y}-l+2i}+\prod_{l=1}^{\mathfrak{y}-k+2i+1}\kappa_{\mathfrak{y}-l+2i+1}\Big]\\
&\times\prod_{j=1}^{i}\Big(1+\mu\prod_{l=1}^{\mathfrak{y}-k+2i-2}\kappa_{\mathfrak{y}-l+2j-2}\prod_{l=1}^{\mathfrak{y}-k+2i-1}\kappa_{\mathfrak{y}-l+2j-1}\Big)\\
&+\prod_{i=1}^{\mathfrak{m}}\Big(1+\mu\prod_{l=1}^{\mathfrak{y}-k+2i-2}\kappa_{\mathfrak{y}-l+2i-2}\prod_{l=1}^{\mathfrak{y}-k+2i-1}\kappa_{\mathfrak{y}-l+2i-1}\Big)\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}}\kappa_{\mathfrak{y}-i+2\mathfrak{m}}
\end{aligned}
\tag{3.15}
$$

Let $\mathcal{W}_i=\prod_{j=1}^{i}\Big(1+\mu\prod_{l=1}^{\mathfrak{y}-k+2i-2}\kappa_{\mathfrak{y}-l+2j-2}\prod_{l=1}^{\mathfrak{y}-k+2i-1}\kappa_{\mathfrak{y}-l+2j-1}\Big)\Big[\prod_{l=1}^{\mathfrak{y}-k+2i}\kappa_{\mathfrak{y}-l+2i}+\prod_{l=1}^{\mathfrak{y}-k+2i+1}\kappa_{\mathfrak{y}-l+2i+1}\Big]$. Then, we get $\lim_{i\to\infty}\Big|\frac{\mathcal{W}_{i+1}}{\mathcal{W}_i}\Big|=\lim_{i\to\infty}\Big(1+\prod_{l=1}^{\mathfrak{y}-k+2i+2}\kappa_{\mathfrak{y}-l+2i+2}\prod_{l=1}^{\mathfrak{y}-k+2i+1}\kappa_{\mathfrak{y}-l+2i+1}\Big)\Big[\kappa_{\mathfrak{y}+2i+2}\kappa_{\mathfrak{y}+2i+1}+\kappa_{\mathfrak{y}+2i+1}\Big]<1$. Consequently, we determine by ratio test that $\sum_{i=1}^{\infty}\mathcal{W}_i$ converges and equation (3.15) thus leads to the conclusion that $d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}+1})$ tends to zero as $\mathfrak{y}$, $\mathfrak{m}$ tend to infinity.

Case 2: Now, suppose by considering an even number represented by $\mathfrak{z}=2\mathfrak{m}$, where $\mathfrak{m}\geq 1$, then the following is derived from inqualities (3.12) and (3.13):

$$
\begin{aligned}
d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+2\mathfrak{m}})&\leq d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1})+d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})+d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\quad+\mu d_{\mathcal{B}}(\tau_{\mathfrak{y}},\tau_{\mathfrak{y}+1})d_{\mathcal{B}}(\tau_{\mathfrak{y}+1},\tau_{\mathfrak{y}+2})d_{\mathcal{B}}(\tau_{\mathfrak{y}+2},\tau_{\mathfrak{y}+2\mathfrak{m}})
\end{aligned}
$$

$$
\begin{aligned}
&\leq \prod_{i=1}^{\mathfrak{y}-k} \kappa_{\mathfrak{y}-i} + \prod_{i=1}^{\mathfrak{y}-k+1} \kappa_{\mathfrak{y}-i+1} + \Big(1+\mu \prod_{i=1}^{\mathfrak{y}-k} \kappa_{\mathfrak{y}-i} \prod_{i=1}^{\mathfrak{y}-k+1} \kappa_{\mathfrak{y}-i+1}\Big) d_{\mathcal{B}}(\tau_{\mathfrak{y}+2}, \tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\vdots\\
&\leq \prod_{i=1}^{\mathfrak{y}-k} \kappa_{\mathfrak{y}-i} + \prod_{i=1}^{\mathfrak{y}-k+1} \kappa_{\mathfrak{y}-i+1} + \Big(1+\mu \prod_{i=1}^{\mathfrak{y}-k} \kappa_{\mathfrak{y}-i} \prod_{i=1}^{\mathfrak{y}-k+1} \kappa_{\mathfrak{y}-i+1}\Big)\\
&\times \Big[\prod_{i=1}^{\mathfrak{y}-k+2} \kappa_{\mathfrak{y}-i+2} + \prod_{i=1}^{\mathfrak{y}-k+3} \kappa_{\mathfrak{y}-i+3}\Big]\\
&\vdots\\
&+ \Big(1+\mu \prod_{i=1}^{\mathfrak{y}-k} \kappa_{\mathfrak{y}-i} \prod_{i=1}^{\mathfrak{y}-k+1} \kappa_{\mathfrak{y}-i+1}\Big) \ldots \Big(1+\mu \prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-6} \kappa_{\mathfrak{y}-i+2\mathfrak{m}-6} \prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-5} \kappa_{\mathfrak{y}-i+2\mathfrak{m}-5}\Big)\\
&\times \Big[\prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-4} \kappa_{\mathfrak{y}-i+2\mathfrak{m}-4} + \prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-3} \kappa_{\mathfrak{y}-i+2\mathfrak{m}-3}\Big]\\
&+ \Big(1+\mu \prod_{i=1}^{\mathfrak{y}-k} \kappa_{\mathfrak{y}-i} \prod_{i=1}^{\mathfrak{y}-k+1} \kappa_{\mathfrak{y}-i+1}\Big) \ldots \Big(1+\mu \prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-4} \kappa_{\mathfrak{y}-i+2\mathfrak{m}-4} \prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-3} \kappa_{\mathfrak{y}-i+2\mathfrak{m}-3}\Big)\\
&d_{\mathcal{B}}(\tau_{\mathfrak{y}+2\mathfrak{m}-2}, \tau_{\mathfrak{y}+2\mathfrak{m}})\\
&\leq \prod_{i=1}^{\mathfrak{y}-k} \kappa_{\mathfrak{y}-i} + \prod_{i=1}^{\mathfrak{y}-k+1} \kappa_{\mathfrak{y}-i+1} + \sum_{i=1}^{\mathfrak{m}-2} \Big[\prod_{l=1}^{\mathfrak{y}-k+2i} \kappa_{\mathfrak{y}-l+2i} + \prod_{l=1}^{\mathfrak{y}-k+2i+1} \kappa_{\mathfrak{y}-l+2i+1}\Big]\\
&\times \prod_{j=1}^{i} \Big(1+\mu \prod_{l=1}^{\mathfrak{y}-k+2i-2} \kappa_{\mathfrak{y}-l+2j-2} \prod_{l=1}^{\mathfrak{y}-k+2i-1} \kappa_{\mathfrak{y}-l+2j-1}\Big)\\
&+ \prod_{i=1}^{\mathfrak{m}-1} \Big(1+\mu \prod_{l=1}^{\mathfrak{y}-k+2i-2} \kappa_{\mathfrak{y}-l+2i-2} \prod_{l=1}^{\mathfrak{y}-k+2i-1} \kappa_{\mathfrak{y}-l+2i-1}\Big)\\
&\times \xi_{\mathfrak{y}+2\mathfrak{m}-2} \prod_{i=1}^{\mathfrak{y}-k+2\mathfrak{m}-3} \kappa_{\mathfrak{y}-i+2\mathfrak{m}-3}\, d_{\mathcal{B}}(\tau_k, \tau_{k+1})
\end{aligned} \tag{3.16}
$$

Let $\mathcal{W}_i = \prod_{j=1}^{i} \Big(1+\mu \prod_{l=1}^{\mathfrak{y}-k+2i-2} \kappa_{\mathfrak{y}-l+2j-2} \prod_{l=1}^{\mathfrak{y}-k+2i-1} \kappa_{\mathfrak{y}-l+2j-1}\Big) \Big[\prod_{l=1}^{\mathfrak{y}-k+2i} \kappa_{\mathfrak{y}-l+2i} + \prod_{l=1}^{\mathfrak{y}-k+2i+1} \kappa_{\mathfrak{y}-l+2i+1}\Big]$

Then, we see that

$$\lim_{i\to\infty} \Big|\frac{\mathcal{W}_{i+1}}{\mathcal{W}_i}\Big| = \lim_{i\to\infty} \Big(1+\prod_{l=1}^{\mathfrak{y}-k+2i} \kappa_{\mathfrak{y}-l+2i} \prod_{l=1}^{\mathfrak{y}-k+2i+1} \kappa_{\mathfrak{y}-l+2i+1}\Big) \Big[\kappa_{\mathfrak{y}+2i+2}\kappa_{\mathfrak{y}+2i+1} + \kappa_{\mathfrak{y}+2i+1}\Big] < 1.$$

Therefore, we establish by ratio test that $\sum_{i=1}^{\infty} \mathcal{W}_i$ converges and equation (3.16) thus leads to the conclusion that $d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+2\mathfrak{m}})$ tends to zero as $\mathfrak{y}$, $\mathfrak{m}$ tend to infinity. Accordingly, from both the cases, we obtain

$$d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}+\mathfrak{z}}) = 0, \forall \mathfrak{y}, \mathfrak{z} \in \mathbb{N}.$$

Thereby the sequence $\{\tau_{\mathfrak{y}}\}$ is Cauchy. The existence of some $\tau \in Y$ corresponding to $\tau_{\mathfrak{y}} \to \tau$ is ensured by the completeness of Y. Furthermore, we establish that τ is a fixed point of G.

Consider

$$
\begin{aligned}
d_{\mathcal{B}}(G\tau, \tau) &\leq d_{\mathcal{B}}(G\tau, G\tau_{\mathfrak{y}}) + d_{\mathcal{B}}(G\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau) + \mu d_{\mathcal{B}}(G\tau, G\tau_{\mathfrak{y}}) d_{\mathcal{B}}(G\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}}) d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau)\\
&\leq g_1(\tau, G\tau) d_{\mathcal{B}}(\tau, G\tau) + g_2(\tau_{\mathfrak{y}}, G\tau_{\mathfrak{y}}) d_{\mathcal{B}}(\tau_{\mathfrak{y}}, G\tau_{\mathfrak{y}}) + d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau) + \mu[g_1(\tau, G\tau)\\
&\quad d_{\mathcal{B}}(\tau, G\tau) + g_2(\tau_{\mathfrak{y}}, G\tau_{\mathfrak{y}}) d_{\mathcal{B}}(\tau_{\mathfrak{y}}, G\tau_{\mathfrak{y}})] d_{\mathcal{B}}(G\tau_{\mathfrak{y}}, \tau_{\mathfrak{y}}) d_{\mathcal{B}}(\tau_{\mathfrak{y}}, \tau)
\end{aligned}
$$

Letting $\eta \to \infty$, we get $d_{\mathcal{B}}(G\tau,\tau)[1-g_1(\tau,G\tau)] \leq 0$. Hence $d_{\mathcal{B}}(G\tau,\tau)=0$ i.e., $G\tau=\tau$. Thereby τ is a fixed point of G. Assuming that τ and ι are two fixed points of G, it can be seen that $G\tau=\tau$ and $G\iota=\iota$. Consider

$$\begin{aligned} &d_{\mathcal{B}}(\tau,\iota)=d_{\mathcal{B}}(G\tau,G\iota) \\ &\leq g_1(\tau,G\tau)d_{\mathcal{B}}(\tau,G\tau)+g_2(\iota,G\iota)d_{\mathcal{B}}(\iota,G\iota)=0, \end{aligned} \tag{3.17}$$

that conclude $\tau=\iota$. The fixed point of G is therefore unique. □

Considering $g_1(\tau,\iota)=g_2(\tau,\iota)=g<\frac{1}{2}$, for every $\tau,\iota\in Y$ in Theorem 3.2, we conclude with the Kannan fixed point theorem in the setting of Branciari suprametric space as stated below:

Corollary 3.3. *Let $(Y,d_{\mathcal{B}})$ be a complete Branciari suprametric space and $G:Y\to Y$ is a mapping. Assume that*

$$d_{\mathcal{B}}(G\tau,G\iota)\leq g[d_{\mathcal{B}}(\tau,G\tau)+d_{\mathcal{B}}(\iota,G\iota)], \tag{3.18}$$

for all $\tau,\iota\in Y$ and $g<\frac{1}{2}$. Then the fixed point of G is unique.

4. Existence-uniqueness of solution of the boundary value problem of nonlinear fractional differential equation of Riemann-Liouville

The application of Theorem 3.1 to investigate the existence and distinctiveness of solutions to a nonlinear fractional differential equation boundary value problem is the principal objective of this section:

$$\begin{aligned} &D^{\mathfrak{b}}_{0+}\mathfrak{z}(\mathfrak{u})+\rho(\mathfrak{u},\mathfrak{z}(\mathfrak{u}))=0, \quad 0<\mathfrak{u}<1, \\ &\mathfrak{z}(0)=\mathfrak{z}(1)=0, \end{aligned} \tag{4.1}$$

where $1<\mathfrak{b}\leq 2$ is a real number, $D^{\mathfrak{b}}_{0+}$ is the standard Riemann-Liouville differentiation and $\mathfrak{z}:[0,1]\times[0,\infty)\to[0,\infty)$ is a continuous function. Given that all continuous functions specified on [0, 1] are expressed by $Y=C([0,1],\mathbb{R})$, the complete Branciari suprametric on Y is defined according to the following with the constant $\mu = 6$,

$$d_{\mathcal{B}}(\mathfrak{z},\varkappa)=\kappa(\mathfrak{z},\varkappa)[1+\kappa(\mathfrak{z},\varkappa)], \tag{4.2}$$

where κ is a rectangular metric defined by

$$\kappa(\mathfrak{z},\varkappa)= \begin{cases} 2\,|\,\mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\,|, & \text{if } \mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\geq 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $\mathfrak{u}>0$ and $\mathfrak{z},\varkappa\in Y$. Note that $\mathfrak{z}\in Y$ solves equation (4.1) provided the subsequent integral equation will also be solved

$$\mathfrak{z}(\mathfrak{u})=\frac{1}{\Gamma(\mathfrak{b})}\int_0^1 \mathfrak{u}(1-s)^{\mathfrak{b}-1}\rho(s,\mathfrak{z}(s))ds-\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\rho(s,\mathfrak{z}(s))ds. \tag{4.3}$$

The readers can pertain to the research article [35] for a more thorough overview of the setting of the problem. The preceding theorem proves that there exists a solution to the nonlinear fractional differential equation (4.1)

Theorem 4.1. *Let the integral operator $G:Y\to Y$ be determined by*

$$G\mathfrak{z}(\mathfrak{u})=\frac{1}{\Gamma(\mathfrak{b})}\int_0^1 \mathfrak{u}(1-s)^{\mathfrak{b}-1}\rho(s,\mathfrak{z}(s))ds-\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\rho(s,\mathfrak{z}(s))ds, \tag{4.4}$$

where $g:[0,1]\times[0,\infty)\to[0,\infty)$ *corresponds to the following specific criteria:*

1. $|\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))|\leq|\mathfrak{z}(s)-\varkappa(s)|,\forall\mathfrak{z},\varkappa\in Y$
2. $\sup_{\mathfrak{u}\in(0,1)}\Big[\frac{\mathfrak{u}^{\mathfrak{b}-1}+\mathfrak{u}^{\mathfrak{b}}}{\Gamma(\mathfrak{b}+1)}\Big]^2<1.$

The nonlinear fractional differential equation (4.1) therefore has an unique solution in Y.

Proof. Consider

$$
\begin{aligned}
2\Big|G\mathfrak{z}(\mathfrak{u})-G\varkappa(\mathfrak{u})\Big|\Big(1+2\Big|G\mathfrak{z}(\mathfrak{u})-G\varkappa(\mathfrak{u})\Big|\Big) &= 2\Big|G\mathfrak{z}(\mathfrak{u})-G\varkappa(\mathfrak{u})\Big|+4\Big|G\mathfrak{z}(\mathfrak{u})-G\varkappa(\mathfrak{u})\Big|^2\\
&= 2\Big|\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}\Big[\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big]ds\\
&\quad-\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\Big[\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big]ds\Big|\\
&\quad+4\Big|\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}\Big[\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big]ds\\
&\quad-\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\Big[\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big]ds\Big|^2\\
&\leq 2\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}\Big|\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big|ds\\
&\quad+\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\Big|\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big|ds\Big)\\
&\quad+4\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}\Big|\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big|ds\\
&\quad+\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\Big|\rho(s,\mathfrak{z}(s))-\rho(s,\varkappa(s))\Big|ds\Big)^2\\
&\leq 2\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}\Big|\mathfrak{z}(s)-\varkappa(s)\Big|ds+\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\Big|\mathfrak{z}(s)-\varkappa(s)\Big|ds\Big)\\
&\quad+4\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}\Big|\mathfrak{z}(s)-\varkappa(s)\Big|ds+\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}\Big|\mathfrak{z}(s)-\varkappa(s)\Big|ds\Big)^2\\
&\leq 2\sup_{\mathfrak{u}\in(0,1)}\Big|\mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\Big|\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}ds+\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}ds\Big)\\
&\quad+4\sup_{\mathfrak{u}\in(0,1)}\Big|\mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\Big|^2\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}}{\Gamma(\mathfrak{b})}\int_0^1(1-s)^{\mathfrak{b}-1}ds+\frac{1}{\Gamma(\mathfrak{b})}\int_0^{\mathfrak{u}}(\mathfrak{u}-s)^{\mathfrak{b}-1}ds\Big)^2\\
&\leq\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}+\mathfrak{u}^{\mathfrak{b}}}{\Gamma(\mathfrak{b}+1)}\Big)^2\Big[2\sup_{\mathfrak{u}\in(0,1)}\Big|\mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\Big|+4\sup_{\mathfrak{u}\in(0,1)}\Big|\mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\Big|^2\Big]\\
&\leq\sup_{\mathfrak{u}\in(0,1)}\Big(\frac{\mathfrak{u}^{\mathfrak{b}-1}+\mathfrak{u}^{\mathfrak{b}}}{\Gamma(\mathfrak{b}+1)}\Big)^2 2\sup_{\mathfrak{u}\in(0,1)}\Big|\mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\Big|\Big[1+2\sup_{\mathfrak{u}\in(0,1)}\Big|\mathfrak{z}(\mathfrak{u})-\varkappa(\mathfrak{u})\Big|
\end{aligned}
$$

Consequently, the above-mentioned inequality yields

$$d_{\mathcal{B}}(G\mathfrak{z},G\varkappa)\leq g d_{\mathcal{B}}(\mathfrak{z},\varkappa),$$

where $g=\sup_{\mathfrak{u}\in(0,1)}\Big[\frac{\mathfrak{u}^{\mathfrak{b}-1}+\mathfrak{u}^{\mathfrak{b}}}{\Gamma(\mathfrak{b}+1)}\Big]^2$. In this regard, it is apparent that the presumptions of the Theorem 3.1 are verified. In accordance with the fact that G has an unique fixed point, the specified nonlinear fractional differential equation has a unique solution. □

Conclusion

The conceptual framework of Branciari suprametric spaces, which appears to be more effective than the ideas of rectangular metric spaces and suprametric spaces and constitutes an alternate perspective on the existence and uniqueness of the solutions to nonlinear fractional differential equations of the Riemann-Liouville type, was employed in this work. In the context of Branciari suprametric spaces, we presented an illustration and defined the terms convergence of sequences, Cauchy sequences, and completeness. A number of fixed point theorems, including the Banach fixed point theorem, were also proved in this space.

Declarations

Availability of data and material

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Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

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