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On uniform ideals and finite Goldie dimension in R-groups

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We consider an R-group G, where R is a (right) nearring. We introduce the notions relative uniform and strictly relative uniform ideals (or R-subgroup) which are not uniform, in general. We prove important properties and obtain a characterization for an R-subgroup to have finite Goldie dimension, in terms of strictly relative uniform R-subgroups. We provide the necessary examples.

Key words and phrases: Nearring, essential ideal, uniform ideal, finite dimension

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1. Introduction

The concept of uniform dimension in modules over rings is a generalization of the dimension of a vector space over a field. A module in which every non-zero submodule is essential is called uniform. Uniform submodules play a significant role in establishing various finite dimension conditions in modules over associative rings. Goldie [15] provided a characterization of equivalent conditions for a module to possess finite uniform dimension. In Bhavanari [4], the notion of uniform dimension was extended to modules over nearrings, (also known as, R-groups), with a characterization established for an R-group to have finite Goldie dimension (in short, $f.G.d$). Subsequently, aspects of Goldie dimension in modules over nearrings have been extensively explored by the authors [23, 7, 9]. However, the study of finite Goldie dimension in modules over rings, specifically in terms of pseudo uniform submodules, which do not necessarily adhere to the uniformity condition, was undertaken in [14]. In case of a module over a matrix nearring, the authors [10] introduced the concepts of essential ideals

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and uniform ideals, providing a characterization for a module over a matrix nearring to have *f.G.d.* Further exploration of prime and semiprime aspects in connection with *f.G.d.* in R-groups and matrix nearrings was conducted by the authors in [13, 16].

A (right) nearring (R, +, .) is an algebraic system (Pilz [22]), where R is an additive group (not necessarily Abelian), and a multiplicative semigroup, satisfying only one distributive axiom (say, right): $(p_1 + p_2)p_3 = p_1p_3 + p_2p_3$ for all $p_1, p_2, p_3 \in R$. In a right nearring, the properties such as $0p = 0$ and $(-p)q = -pq$ holds for all $p, q \in R$, though, in general, $p0 \neq 0$ for some $p \in R$. If $p0 = 0$ for all $p \in R$, then we say R is zero-symmetric (denoted as, $R = R_0$). An additive group (G, +) is called an R-group (or module over a nearring R), denoted by $_RG$ (or simply by G) if there exists a mapping $R \times G \to G$ (image $(k,g \to kg)$), satisfying: $(k + l)g = kg + lg$; $(kl)g = k(lg)$ for all $g \in G$ and $k, l \in R$. It is evident that every nearring is an R-group (over itself). Additionally, if R is a ring, then each (left) module over R is an R-group. Throughout, G represents an R-group where R is a right nearring.

A subgroup (H, +) of G with $RH \subseteq H$ is called an R-subgroup of G. A normal subgroup H of G is called an ideal if $n(g + h) - ng \in H$ for all $n \in R$, $h \in H$, $g \in G$. For any two R-groups G_1 and G_2, a map $f : G_1 \to G_2$ is called an R-homomorphism, $f(x + y) = f(x) + f(y)$ and $f(nx) = nf(x)$ hold for all $x, y \in G_1$ and $n \in R$. If f is one-one and onto, then f is an R-isomorphism.

In case of a zero symmetric nearring, for any ideals A and B of G, $A + B$ is an ideal of G ([22], Corollary 2.3).

For each $g \in G$, Rg is an R-subgroup of G. The ideal (or R-subgroup) generated by an element $g \in G$ is denoted by $\langle g \rangle$. For any subsets S, T of G, the noetherian quotient is defined as $(S : T) = \{n \in R : nT \subseteq S\}$, and if $S = \{0\}$, then $(0 : T)$ is called the annihilator of T. A proper ideal P of R is called semiprime, if an ideal I of R with $I^2 \subseteq P$, then $I \subseteq P$, R itself is semiprime, if (0) is a semiprime ideal.

An ideal T of an R-group G is essential (see, [23]), if for any ideal H of G, $T \cap H = (0)$ implies $H = (0)$. If every non trivial ideal $(0) \neq H$ of G is essential, then we say G is uniform. Further, an ideal (R-subgroup) T of G is said to be strictly uniform (see, [21]), if for any two R-subgroups P, Q of G, $P \subseteq T$, $Q \subseteq T$, $P \cap Q = (0)$ implies $P = (0)$ or $Q = (0)$.

In this paper, we consider the notions of uniform and strictly uniform ideal with respect to an arbitrary ideal (or R-subgroup) Ω of an R-group defined in [25]. We establish an equivalent condition for an R-subgroup to have a Ω-finite Goldie dimension (denoted by, Ω-*f.G.d.*) in terms of its strictly Ω-uniform R-subgroups.

For standard definitions and notations in nearrings, we direct the reader to [11, 22].

2. Uniform and strictly uniform ideals

We start this section with the definitions of Ω-uniform ideal (or R-subgroup) and strictly Ω-uniform ideal with suitable examples.

Definition 2.1. ([26], Definition 2.1) *An ideal H of G is said to be relative essential (resp. strictly relative essential), if there exists a proper ideal (resp. R-subgroup) Ω of G such that*

(a) $H \not\subseteq \Omega$,
(b) for any ideal (resp. R-subgroup) K of G, $H \cap K \subseteq \Omega$ implies $K \subseteq \Omega$.

We denote it by $H \leq_\Omega^e G$ (resp. $H \leq_\Omega^{se} G$), and read as H is Ω-essential in G (resp. H is strictly Ω-essential in G).

We denote $H_1 \leq_\Omega^e H_2$ when H_2 considered as an R-group. In case, $\Omega = (0)$, this is referred as G-essential, by the authors [4].

Definition 2.2. ([25], Definition 3.1)
An ideal I of G is called relative uniform (resp. strictly relative uniform) if J is an ideal (resp. R-subgroup) of G, $J \subseteq I$, then $J \leq_\Omega^e I$ (resp. $J \leq_\Omega^{se} I$).

Proposition 2.3. *Let I be an ideal of G and $\Omega \neq G$ be an ideal of G. I is Ω-uniform if and only if for any ideals H_1, H_2 contained in I such that $H_1 \cap H_2 \subseteq \Omega$ implies $H_1 \subseteq \Omega$* or *$H_2 \subseteq \Omega$.*

Proof. Suppose that I is Ω-uniform. Let H_1, H_2 be ideals of G such that $H_1 \subseteq I$, $H_2 \subseteq I$, and $H_1 \cap H_2 \subseteq \Omega$. Assume that $H_i \nsubseteq \Omega$. Since I is Ω-uniform, and $H_1 \subseteq I$, we have $H_1 \leq^e_\Omega I$, and since $H_1 \cap H_2 \subseteq \Omega$, we get $H_2 \subseteq \Omega$.

On the other hand, let J be any ideal of G such that $J \subseteq I$ and $J \nsubseteq \Omega$. To prove $J \leq^e_\Omega I$, let K be an ideal of G contained in I such that $J \cap K \subseteq \Omega$. Since $J \nsubseteq \Omega$, by converse hypothesis, we have $K \subseteq \Omega$. Therefore, $J \leq^e_\Omega I$.

We give an example that Ω-uniform need not be uniform, in general.

Example 2.4. *Let $R = (\mathbb{Z}_{12}, +_{12}, \cdot_{12})$ and $G = R$. Then R is considered as R-group (over itself). The ideals of $_RR$ are $H_1 = \{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8},\overline{10}\}$, $H_2 = \{\overline{0},\overline{6}\}$, $H_3 = \{\overline{0},\overline{3},\overline{6},\overline{9}\}$, $H_4 = \{\overline{0},\overline{4},\overline{8}\}$. Take $\Omega = H_3$. Then H_1 is Ω-uniform, but not uniform in G, as $H_2, H_4 \subseteq H_1$, $H_2 \cap H_4 = (\overline{0})$, and $H_2 \neq (\overline{0}) \neq H_4$.*

Example 2.5. *Let $G = (\mathbb{Z}_8 \times \mathbb{Z}_3, +)$. Then G is an R-group, where $R = (\mathbb{Z}, +, .)$. Take $\Omega = \mathbb{Z}_8 \times (0)$, an ideal of $_RG$. Then the ideal $(2) \times \mathbb{Z}_3$ is Ω-uniform, but not uniform, as the ideals $(4) \times (0)$, $(4) \times \mathbb{Z}_3$ contained in $(2) \times \mathbb{Z}_3$ such that $(4)\times(0) \cap (4)\times\mathbb{Z}_3 = (0)\times(0)$, but $(4) \times (0) \neq (0) \times (0) \neq (4) \times \mathbb{Z}_3$.*

The following corollary is straightforward.

Corollary 2.6. *Let Ω be a proper ideal of G. Then G is Ω-uniform if and only if for any two ideals K and L of G, $K \cap L \subseteq \Omega \Rightarrow K \subseteq \Omega$ or $L \subseteq \Omega$.*

Proposition 2.7. *Let Ω be a proper ideal of G, and G is Ω-uniform. Then any finite intersection of Ω-essential ideals of G is Ω-essential in G, and converse also holds.*

Proof. Let $\{H_i\}_{i=1}^n$ be a family of Ω-essential ideals of G. Write $H = \bigcap_{i=1}^n H_i$. Clearly $H_i \nsubseteq \Omega$ for each i, and since G is Ω-uniform, $H \nsubseteq \Omega$. To prove H is Ω-essential, we use the induction on the number n of Ω-essential ideals. Suppose that $n = 2$. Let L be an ideal of G such that $H \nsubseteq \Omega$, $H \cap L \subseteq \Omega$. Then $(H_1 \cap H_2) \cap L \subseteq \Omega$, implies that $H_1 \cap (H_2 \cap L) \subseteq \Omega$. Since $H_1 \leq^e_\Omega G$ and $H_2 \cap L$ is an ideal of G with $H_2 \cap L \subseteq \Omega$. Again, since $H_2 \leq^e_\Omega G$ and $H_2 \nsubseteq \Omega$, we get $L \subseteq \Omega$. Therefore the statement is true for $n = 2$. We assume the induction hypothesis for $(n - 1)$ ideals $\{H_i\}_{i=1}^{n-1}$ of G. Let L be an ideal of G such that $\left(\bigcap_{i=1}^n H_i\right) \cap L \subseteq \Omega$ and $H \nsubseteq \Omega$. Then, $\left(\bigcap_{i=1}^{n-1} H_i \cap H_n\right) \cap L \subseteq \Omega$. That is, $\bigcap_{i=1}^{n-1} H_i \cap (H_n \cap L) \subseteq \Omega$. Since $H \subsetneqq \bigcap_{i=1}^{n-1} H_i$ and $H \nsubseteq \Omega$, we have $\bigcap_{i=1}^{n-1} H_i \nsubseteq \Omega$, hence, by induction hypothesis, it follows that $H_n \cap L \subseteq \Omega$. Now since $H_n \leq^e_\Omega G$, and L is an ideal of G, we have $L \subseteq \Omega$, which shows that $H \leq^e_\Omega G$.

Conversely, suppose that $H = \bigcap_{i=1}^n H_i \leq^e_\Omega G$. Since $\bigcap_{i=1}^n H_i \nsubseteq \Omega$, we get $H_i \nsubseteq \Omega$, for all i. Then to show that $H_i \leq^e_\Omega G$ for every i, $1 \leq i \leq n$, let L be any ideal of G such that $H_i \cap L \subseteq \Omega$. Now $H \cap L \subseteq H_i \cap L \subseteq \Omega$ and since $H \leq^e_\Omega G$, we have that $L \subseteq \Omega$. Since $H_i (1 \leq i \leq n)$, is arbitrary, we conclude that $H_i \leq^e_\Omega G$ for every i.

Remark 2.8. *The converse of the Proposition 2.7 do not hold, in general. Consider the following example:*

Let $R = (\mathbb{Z}_{24}, +_{24}, \cdot_{24})$ and $G = R$. Then G is an R-group and the ideals are $H_1 = \langle\overline{2}\rangle, H_2 = \langle\overline{3}\rangle, H_3 = \langle\overline{4}\rangle$, $\Omega = H_4 = \langle\overline{6}\rangle$, $H_5 = \langle\overline{8}\rangle$, $H_6 = \langle\overline{12}\rangle$. Since $H_5 \cap H_3 = H_5$, we have

(*i*) $H_5 \cap H_1 \subseteq H_3$

(*ii*) $H_5 \cap H_1 \nsubseteq \Omega$

Then, $H_6 = \langle \overline{12} \rangle$ *is the only ideal satisfying* $H_6 \subseteq H_3$, $(H_5 \cap H_1) \cap H_6 \subseteq \Omega$, *implies that* $H_6 \subseteq \Omega$. *Therefore,* $H_5 \cap H_1 \leq_{\Omega}^{e} H_3$. *Since* $H_1 \not\subseteq H_3$, *we conclude that* $H_1 \not\leq_{\Omega}^{e} H_3$.

Definition 2.9. ([26], Definition 2.3) *An R-subgroup H of G is said to be relative essential (resp. strictly relative essential), if there exists a proper ideal (resp. R-subgroup)* Ω *of G such that*

(*a*) $H \not\subseteq \Omega$,
(*b*) *for any ideal (resp. R-subgroup) K of G,* $H \cap K \subseteq \Omega$ *implies* $K \subseteq \Omega$.

We denote it by $H \leq_{\Omega}^{e} G$ (*resp.* $H \leq_{\Omega}^{se} G$), *and read as H is* Ω*-essential in G* (*resp. H is strictly* Ω*-essential in G*).

Definition 2.10. *An R-subgroup I of G is called relative uniform (resp. strictly relative uniform) if there exists an ideal* (*resp. R-subgroup*) Ω *of G and any ideal* (*resp. R-subgroup*) *J of G,* $J \not\subseteq \Omega$ *and* $J \subseteq I$ *implies* $J \leq_{\Omega}^{e} I$ (*resp.* $J \leq_{\Omega}^{se} I$) (*here we consider I as an R-group*).

Furthermore, G is called relative uniform (*resp. strictly relative uniform*) *if there exists an ideal* (*resp. R-subgroup*) Ω *of G such that for each ideal* (*resp. R-subgroup*) *K of G and* $K \not\subseteq \Omega$, *then* $K \leq_{\Omega}^{e} G$ (*resp.* $K \leq_{\Omega}^{se} G$).

Table 1

+	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	a	2a	3a	b	a+b	2a+b	3a+b
a	a	2a	3a	0	a+b	2a+b	3a+b	b
2a	2a	3a	0	a	2a+b	3a+b	b	a+b
3a	3a	0	a	2a	3a+b	b	a+b	2a+b
b	b	3a+b	2a+b	a+b	0	3a	2a	a
a+b	a+b	b	3a+b	2a+b	a	0	3a	2a
2a+b	2a+b	a+b	b	3a+b	2a	a	0	3a
3a+b	3a+b	2a+b	a+b	b	3a	2a	a	0

Table 2

*1	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	0	0	0	0	0	0	0
a	0	a	2a	3a	b	a+b	2a+b	3a+b
2a	0	2a	0	2a	0	0	0	0
3a	0	3a	2a	a	b	a+b	2a+b	3a+b
b	0	b	2a	2a+b	b	a+b	2a+b	3a+b
a+b	0	a+b	0	a+b	0	0	0	0
2a+b	0	2a+b	2a	b	b	0	2a+b	3a+b
3a+b	0	3a+b	0	3a+b	0	0	0	0

Table 3

*2	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	0	0	0	0	0	0	0
a	0	a	2a	3a	b	a+b	2a+b	3a+b
2a	0	2a	0	2a	0	0	0	0
3a	0	3a	2a	a	b	a+b	2a+b	3a+b
b	0	b	2a	2a+b	b	0	2a+b	2a
a+b	0	a+b	0	a+b	0	a+b	+	a+b
2a+b	0	2a+b	2a	b	b	0	2a+b	2a
3a+b	0	3a+b	0	3a+b	0	a+b	0	a+b

Example 2.11. *Consider the nearring with addition and multiplication tables listed in K(135) and K(139) of p.418 of Pilz* [22]. *Let* $G = D_8 = \langle\{a, b \mid 4a = 2b = 0,\ a + b = b - a\}\rangle = \{a, 2a, 3a, 4a = 0, b, a + b, 2a + b, 3a + b\}$, *where a is the rotation in an anti-clockwise direction about the origin through* $\frac{\pi}{2}$ *radians and b is the reflection about the line of symmetry, and G = R. Then G is an R-group.*

(1) *Consider the operations in Table* 1 *and Table* 2. *The proper ideals are* $I_1 = \{0, 2a\}$, $I_2 = \{0, a + b, 2a, 3a + b\}$, *and R-subgroups are* $J_1 = \{0, 2a\}$, $J_2 = \{0, b\}$, $J_3 = \{0, a+b\}$, $J_4 = \{0, 2a+b\}$, $J_5 = \{0, 3a+b\}$, $J_6 = \{0, b, 2a, 2a+b\}$, $J_7 = \{0, 2a, a+b, 3a+b\}$. *Consider* $H = I_2 = \{0, 2a, a+b, 3a+b\}$ *and* $\Omega = I_1 = \{0, 2a\}$. *Now H is not strictly Ω-uniform, since the R-subgroups* $J_3 = \{0, a+b\}$ *and* $J_7 = \{0, 2a, a+b, 3a+b\}$ *which are contained in H with* $J_3 \cap J_7 \subseteq \Omega$, *but* $J_3 \not\subseteq \Omega$, $J_7 \not\subseteq \Omega$. *However, H is Ω-uniform, since the only ideals* $I_1 = \{0, 2a\}$ *and* $I_2 = \{0, 2a, a+b, 3a+b\}$ *contained in H satisfying* $I_1 \cap I_2 \subseteq \Omega$ *implies* $I_1 \subseteq \Omega$.

(2) *Consider the operations in Table* 1 *and Table* 3. *Then the proper ideals are* $I_1 = \{0, 2a\}$, $I_2 = \{0, 2a, b, 2a+b\}$, $I_3 = \{0, 2a, a+b, 3a+b\}$ *and R-subgroups are* $J_1 = \{0, 2a\}$, $J_2 = \{0, b\}$, $J_3 = \{0, a+b\}$, $J_4 = \{0, 2a+b\}$, $J_5 = \{0, b, 2a, 2a+b\}$, $J_6 = \{0, 2a, a+b, 3a+b\}$. *Consider* $H = I_3 = \{0, 2a, a+b, 3a+b\}$ *and* $\Omega = J_5 = \{0, 2a, b, 2a+b\}$. *Here, H is strictly Ω-uniform, since the R-subgroups* $J_1 = \{0, 2a\}$ *and* $J_3 = \{0, a+b\}$ *which are contained in H with* $J_1 \cap J_3 \subseteq \Omega$, *we have* $J_1 \subseteq \Omega$. *Also, H is Ω-uniform, since the only ideals* $I_1 = \{0, 2a\}$ *and* $I_3 = \{0, 2a, a+b, 3a+b\}$ *contained in H satisfying* $I_1 \cap I_3 \subseteq \Omega$ *implies* $I_1 \subseteq \Omega$.

Remark 2.12. *Let Ω be a proper ideal of G, and K a Ω-uniform ideal of G. If L is an ideal of G such that* $L \not\subseteq \Omega$ *and* $L \subseteq K$, *then L is Ω-uniform.*

Proof. Let L be an ideal of G and L_1, L_2 be ideals of G contained in L such that $L_1 \cap L_2 \subseteq \Omega$. Since $L_1, L_2 \subseteq L \subseteq K$, and K is Ω-uniform, we have $L_1 \subseteq \Omega$ or $L_2 \subseteq \Omega$. Hence L is Ω-uniform.

Proposition 2.13. *Let Ω be a proper ideal of G. If G is Ω-uniform, then G/Ω is uniform, and Ω is semiprime.*

Proof. Suppose G is Ω-uniform. Let K/Ω and L/Ω be ideals of G/Ω such that $K/\Omega \cap L/\Omega = (0)$ in G/Ω, where K and L are ideals of G, properly containing Ω. Then $K \cap L \subseteq \Omega$. Since G is Ω-uniform of G, we have $K \subseteq \Omega$ or $L \subseteq \Omega$. Therefore, $K/\Omega \subseteq (0)$ or $L/\Omega \subseteq (0)$ in G/Ω, hence G/Ω is uniform. Now to show Ω is semiprime, let I be an ideal of G such that $I^2 \subseteq \Omega$. Then $I \cap I \subseteq I^2 \subseteq \Omega$. Since G is Ω-uniform, we get $I \subseteq \Omega$.

Remark 2.14. *In Proposition* 2.13, *G not necessarily* Ω*-uniform, even if G/*Ω *is uniform.*

Consider the following examples.

(i) *Take* $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ *and* $R = \mathbb{Z}$. *Then G is a module over a nearring. Consider the ideal* $\Omega = \langle(\bar{1},\bar{1})\rangle$ *of G. Since* Ω *is maximal, G/*Ω *is simple, by* ([22], *Prop* 1.40), *and hence uniform. However, G is not* Ω*-uniform, since* $\langle(\bar{1},\bar{2})\rangle \cap \langle(\bar{0},\bar{3})\rangle \subseteq \Omega$, *but* $\langle(\bar{0},\bar{3})\rangle \nsubseteq \Omega$ *and* $\langle(\bar{1},\bar{2})\rangle \nsubseteq \Omega$.

(ii) *Take* $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $R = \mathbb{Z}_2$. *Then G is a module over a nearring. Consider an ideal* $\Omega = \{(x,x) \mid x \in R\}$ *of G. Clearly* Ω *is maximal ideal of G. Therefore,* $G/\Omega = \{(\bar{0},\bar{0}),(\bar{1},\bar{1})\}$ *is uniform. But* $\mathbb{Z}_2 \times (\bar{0})$ *is an ideal of G such that* $\mathbb{Z}_2 \times (\bar{0}) \nsubseteq \Omega$, *and* $\mathbb{Z}_2 \times (\bar{0})$ *is not* Ω*-essential in G, since* $(0,1 \in G \setminus \Omega)$ *for each* $r \in R$, $r(\bar{0},\bar{1}) \notin (\mathbb{Z}_2 \times (\bar{0})) \setminus \Omega$. *Hence, G is not* Ω*-uniform.*

(iii) *For each positive integer n,* $\mathbb{Z}/n\mathbb{Z}$ *is uniform if and only if* $\mathbb{Z}$ *is an* $n\mathbb{Z}$*-uniform.*

The following theorem provides a characterization for essentiality in R-subgroups of G, where G is an unitary R-group G (that is, $1 \in R$).

Theorem 2.15. *Let* $\Omega \subseteq H_1 \subseteq H_2$ *be R-subgroups of G and* $1 \in R$. *Then the following are equivalent.*

(1) $H_1 \leq_{\Omega}^{se} H_2$;

(2) *For each* $g \in H_2 \setminus \Omega$, *there exists* $n \in R$ *such that* $ng \in H_1 \setminus \Omega$.

Proof. (1) ⇒ (2): Let $g \in H_2 \setminus \Omega$. Since $Rg \subseteq H_2$ and as $1 \in R$, we have $Rg \nsubseteq \Omega$. Since $H_1 \leq_{\Omega}^{se} H_2$, we get $H_1 \cap Rg \nsubseteq \Omega$.

Let $x \in H_1 \cap Rg$ such that $x \notin \Omega$. Then there exists $n \in R$ such that $x = ng \in H_1$, and $ng \notin \Omega$, implies that $ng \in H_1 \setminus \Omega$..

(2) ⇒ (1): Let L be an R-subgroup of G such that $L \subseteq H_2$ and $H_1 \cap L \subseteq \Omega$. If $L \nsubseteq \Omega$, then there exists $a \in L \setminus \Omega \subseteq H_2 \setminus \Omega$. Now by (2), there exists $n \in R$ such that $na \in H_1 \setminus \Omega$, though $na \in H_1 \cap L \subseteq \Omega$, a contradiction. Hence, $H_1 \leq_{\Omega}^{se} H_2$.

Theorem 2.16. *Let* $\Omega \subseteq H_1 \subseteq H_2$ *be R-subgroups of G and* $1 \in R$. *Then the following are equivalent.*

(1) $H_1 \leq_{\Omega}^{se} H_2$;

(2) $(H_1 : g) \leq_{(\Omega:g)}^{se} (H_2 : g)$, *for each* $g \in H_2 \setminus \Omega$.

Proof. (1) ⇒ (2): Let $g \in H_2 \setminus \Omega$. By (2), there exists $n \in R$ such that $ng \in H_1 \setminus \Omega$, shows that $n \in (H_1 : g) \setminus (\Omega : g)$. Hence $(H_1 : g) \nsubseteq (\Omega : g)$.

Let I be an R-subgroup of R such that $(H_1 : g) \cap I \subseteq (\Omega : g)$. Clearly Ig is an R-subgroup of G. First we show that $H_1 \cap Ig \subseteq \Omega$. If $H_1 \cap Ig \nsubseteq \Omega$, then there exist $x \in H_1 \cap Ig$, but $x \notin \Omega$. So $x \in H_1$ and $x = ig$, for some $i \in I$. This means $x = ig \in H_1$, but $x = ig \notin \Omega$, implies $i \in (H_1 : g) \cap I$, but $i \notin (\Omega : g)$, a contradiction. Therefore, $H_1 \cap Ig \subseteq \Omega$. Since $H_1 \leq_{\Omega}^{se} H_2$, Ig is a R-subgroup of G contained in H_2, it follows that $Ig \subseteq \Omega$. So, $I \subseteq (\Omega : g) \subseteq (H_2 : g)$, as $\Omega \subseteq H_2$. Therefore, I is an R-subgroup of R contained in $(H_2 : g)$, proves $(H_1 : g) \leq_{(\Omega:g)}^{se} (H_2 : g)$.

(2) ⇒ (1): Suppose that K is a (proper) R-subgroup of G such that $K \subseteq H_2$ and $H_1 \cap K \subseteq \Omega$. If $K \nsubseteq \Omega$, then there exists $x \in K \setminus \Omega \subseteq H_2 \setminus \Omega$. Now by (2), we get $(H_1 : x) \leq_{(\Omega:x)}^{se} (H_2 : x)$. Since $(H_1 : x) \nsubseteq (\Omega : x)$, there exists $a \in (H_1 : x)$, but $a \notin (\Omega : x)$. That is, $ax \in H_1$, but $ax \notin \Omega$. Now since K be an R-subgroup of G, and $a \in R$, $x \in K$, we get $ax \in K$. Thus, $ax \in H_1 \cap K$, but $ax \notin \Omega$, a contradiction. Therefore, $H_1 \leq_{\Omega}^{se} H_2$.

Definition 2.17. ([18]) *Let G be an R-group. We say that R distributes over G if* $d(g_1+g_2) = dg_1 + dg_2$ *for all* $d \in R$, $g_1, g_2 \in G$.

In [6], the authors, used the condition that R distributes over G for obtaining aG an ideal of G, for any $a \in R$. In ([11], Remark 5.3.39), the authors provided the classes of nearrings where in every R-subgroup is an ideal. Among such classes, Boolean nearrings, and the classes of all strongly regular nearrings are the familiar ones. In [20], the authors extensively studied the class of all central Boolean rings. Further, if the R-group is tame ([22], Definition 9.165), then every R_0-subgroup is an ideal.

In the rest of the paper, we consider classes of nearrings wherein every R-subgroup is an ideal. Therefore, we assume that the sum of two R-subgroups is again an R-subgroup.

Theorem 2.18. *Let* Ω *be a proper R-subgroup of G and let* G_i, H_i *be R-subgroups of G,* $H_i \subseteq G_i$, *for* $i = 1, 2$ *such that* $H_1 \cap H_2 = \Omega = G_1 \cap G_2$. *If* $H_1 + H_2 \leq_{\Omega}^{se} G_1 + G_2$ *then* $H_i \leq_{\Omega}^{se} G_i$, *for* $i = 1, 2$, *and the converse holds if R distributes over G.*

Proof. Suppose $H_1 + H_2 \leq_{\Omega}^{se} G_1 + G_2$ and $H_1 \not\leq_{\Omega}^{se} G_1$. Then for some R-subgroup A of G such that $A \subseteq G_1, H_1 \cap A \subseteq \Omega$ and $A \not\subseteq \Omega$. We show that $(H_1 + H_2) \cap A \subseteq \Omega$. Let $p \in (H_1 + H_2) \cap A$. Then $p = h_1 + h_2$, where $h_1 \in H_1$, $h_2 \in H_2$ and $p \in A$. Now $h_2 = -h_1 + p \in (H_1 + A) \cap H_2 \subseteq G_1 \cap G_2 = \Omega \subseteq H_1$. That is, $h_2 \in H_1$. Hence $p = h_1 + h_2 \in H_1 \cap A \subseteq \Omega$, implies $(H_1 + H_2) \cap A \subseteq \Omega$, a contradiction. Hence, $H_1 \leq_{\Omega}^{se} G_1$. In a similar way, we will get $H_2 \leq_{\Omega}^{se} G_2$.

Conversely, suppose $H_1 \leq_{\Omega}^{se} G_1$, $H_2 \leq_{\Omega}^{se} G_2$. To show $H_1 + H_2 \leq_{\Omega}^{se} G_1 + G_2$. Let $l \in (G_1 + G_2) \setminus \Omega$. Then $l = x_1 + x_2$, for some $x_1 \in G_1$, $x_2 \in G_2$ and $x_1 + x_2 \notin \Omega$. This implies that $x_1 \notin \Omega$ or $x_2 \notin \Omega$. Suppose $x_1 \in G_1 \setminus \Omega$. Then by Theorem 2.15, there exists $n_1 \in R$ such that $n_1 x_1 \in H_1 \setminus \Omega$.

Case 1: If $n_1 x_2 \in H_2 \setminus \Omega$, then since R distributes over G, we have $n_1 l = n_1(x_1 + x_2) = n_1 x_1 + n_1 x_2 \in H_1 + H_2$. Now we show that $n_1 l \notin \Omega$. If $n_1 l \in \Omega \subseteq H_1$, then $n_1 x_2 = n_1 l - n_1 x_1 \in H_1$. Since $n_1 x_2 \in H_2$, we get $n_1 x_2 \in H_1 \cap H_2 \subseteq G_1 \cap G_2 \subseteq \Omega$, a contradiction. Therefore, $n_1 l \in (H_1 + H_2) \setminus \Omega$. If $n_1 x_2 \in \Omega \subseteq H_1$, then by the same argument as above we will get $n_1 l = n_1(x_1 + x_2) \in H_1 \subseteq H_1 + H_2$. Now if $n_1 l \in \Omega$, then $n_1 x_1 = n_1 l - n_1 x_2 \in \Omega$, a contradiction. Therefore, $n_1 l \in (H_1 + H_2) \setminus \Omega$..

Case 2: Now let $n_1 x_2 \notin H_2 \setminus \Omega$. Subcase (i): If $n_1 x_2 \in \Omega \subseteq H_1$, then $n_1 l = n_1(x_1 + x_2) = n_1 x_1 + n_1 x_2 \in H_1 \subseteq H_1 + H_2$. In this case if $n_1 l \in \Omega$, then $n_1 x_1 = n_1 l - n_1 x_2 \in \Omega$, a contradiction. Therefore, $n_1 l \in (H_1 + H_2) \setminus \Omega$. Subcase (ii): If $n_1 x_2 \in G_2 \setminus \Omega$, then by Theorem 2.15, there exists $n_2 \in R$ such that $n_2 \cdot (n_1 x_2) \in H_2 \setminus \Omega$. Therefore, by a similar argument, $(n_2 \cdot n_1) l \in (H_1 + H_2) \setminus \Omega$, hence $(H_1 + H_2) \leq_{\Omega}^{se} G_1 + G_2$.

Corollary 2.19. *Let* $\{G_i\}_{i=1}^n, \{H_i\}_{i=1}^n$ *be R-subgroups of G and* $H_i \subseteq G_i$ *for* $i = 1$ *to* n *such that* $\bigcap_{i=1}^n G_i = \Omega = \bigcap_{i=1}^n H_i$. *If* $\sum_{i=1}^n H_i \leq_{\Omega}^{se} \sum_{i=1}^n G_i$ *then* $H_i \leq_{\Omega}^{se} G_i$, $1 \leq i \leq n$ *and the converse holds if R distributes over G.*

Proof. By using Theorem 2.18 and induction on n.

Theorem 2.20. *Let* f: $G_1 \to G_2$ *be an R-isomorphism. Let K and* Ω *be proper ideals of* G_1. *Then K is* Ω-*uniform in* G_1 *if and only if* $f(K)$ *is* $f(\Omega)$-*uniform in* G_2.

Proof. Suppose K is Ω-uniform in G_1. To prove $f(K)$ is $f(\Omega)$-uniform in G_2, let L_2 and L_2 be ideals of G_2 contained in $f(K)$ such that $L_1 \cap L_2 \subseteq f(\Omega)$. This implies that $f^{-1}(L_1 \cap L_2) \subseteq \Omega$. Since f is an R-isomorphism, we have $f^{-1}(L_1) \cap f^{-1}(L_2) \subseteq \Omega$. Now $f^{-1}(L_1)$ and $f^{-1}(L_2)$ are ideals of G_1 contained in K, and K is Ω-uniform, we get $f^{-1}(L_1) \subseteq \Omega$ or $f^{-1}(L_2) \subseteq \Omega$. Hence, $L_1 \subseteq f(\Omega)$ or $L_2 \subseteq f(\Omega)$.

Conversely, suppose $f(K)$ that is $f(\Omega)$-uniform in G_2. To show K is Ω-uniform in G_1 let H_1, H_2 be ideals of G_1 contained in K such that $H_1 \cap H_2 \subseteq \Omega$. Since f is an R-isomorphism, $f(H_1 \cap H_2) \subseteq f(\Omega)$, implies, $f(H_1) \cap f(H_2) \subseteq f(\Omega)$. Since $f(K)$ is $f(\Omega)$-uniform, we have $f(H_1) \subseteq f(\Omega)$ *or* $f(H_2) \subseteq f(\Omega)$. As f is one-one, $H_1 \subseteq \Omega$ or $H_2 \subseteq \Omega$, desired.

The proof of the Corollary 2.21 follows from the fundamental homomorphism theorem for R-groups and Theorem 2.20.

Corollary 2.21. *If H and K are ideals of G and Ω a proper ideal of G such that $H \cap K = \Omega$. Then H is Ω-uniform if and only if $(H + K) / K$ is $f(\Omega)$-uniform in G/K, where $f : H \to (H + K) / K$ is a canonical epimorphism.*

Theorem 2.22. *Let Ω be a proper ideal of G. If every Ω-essential R-subgroup is strictly Ω-essential, then every Ω-uniform ideal of G is strictly Ω-uniform.*

Proof. Suppose H is Ω-uniform. In a contrary, suppose that H is not strictly Ω-uniform. Then, for some R-subgroups I, J, we have $I, J \subseteq H$, $I \cap J \subseteq \Omega$ but $I \nsubseteq \Omega$ and $J \nsubseteq \Omega$. Consider $\mathcal{S} = \{X : X \cap I \subseteq \Omega,\ X \text{ is an ideal of } G\}$. Clearly $(0), \Omega \in \mathcal{S}$ Now $\mathcal{S}$ is a non-empty partially ordered subset of ideals of G, in which every chain has an upper bound. Hence, by Zorn's lemma, let K be an ideal of G maximal with respect to $K \cap I \subseteq \Omega$. To show $I + K \leq^e_\Omega G$. Clearly, $I + K$ is an R-subgroup of G, by ([22], Proposition 2.15). Let L be any ideal of G such that $(I + K) \cap L \subseteq \Omega$. To show that $L \subseteq \Omega$, first we show that $I \cap (K + L) \subseteq \Omega$. Let $x \in I \cap (K + L)$. Then $x = i$ and $x = k + l$, for some $i \in I$, $k \in K$, $l \in L$, implies $l = i - k \in L \cap (I + K) \subseteq \Omega$. Hence, $l \in \Omega$, and so $i = k + l \in K + \Omega = K$, as K is maximal in $\mathcal{S}$. This implies $x = i \in I \cap K \subseteq \Omega$. Again by the maximality of K, it follows that $K + L \subseteq K$. Therefore, $L \subseteq K \subseteq \Omega$, which proves that $I + K \leq^e_\Omega G$. Now by hypothesis, $I + K \leq^{se}_\Omega G$. As $J \nsubseteq \Omega$ and $I + K \leq^{se}_\Omega G$, we have $(I + K) \cap J \nsubseteq \Omega$. Let $x \in (I + K) \cap J$ and $x \notin \Omega$. This implies that $x = i + k$ for some $i \in I$, $k \in K$, and $x = J$. Then $-i + x = k \in K \cap (I + J)$. Since $I \cap J \subseteq \Omega$ and $J \subseteq K$, $J \nsubseteq \Omega$, we get $k = -i + x \notin \Omega$. Write $K_1 = H \cap K$. Then $K_1 \nsubseteq \Omega$. Let T be an ideal of G such that $T \cap K_1 \subseteq \Omega$ and $T + K_1$ is Ω-essential. Since $T + K_1 \leq^e_\Omega G$ and $I \nsubseteq \Omega$, we have $I \cap (T + K_1) \nsubseteq \Omega$. As in the above similar argument, we get $T \cap H \nsubseteq \Omega$. Let $M = T \cap H \nsubseteq \Omega$. Then, $K_1, M \nsubseteq \Omega$ are ideals of G, and $K_1, M \subseteq H$ such that $K_1 \cap M = K_1 \cap (T \cap H) \subseteq K_1 \cap T \subseteq \Omega$, a contradiction to H is Ω-uniform.

3. Relative finite Goldie dimension

We define a finite Goldie dimension of an R-subgroup with respect to an arbitrary R-subgroup Ω. We provide examples and obtain a characterization for an R-subgroup to have Ω-*f.G.d*.

Definition 3.1. *Let Ω be a proper R-subgroup of G and let $\{I_i\}_{i \in I}$ be a family of R-subgroups of G. We say that $\{I_i\}_{i \in I}$ is Ω-direct if* $I_i \cap \left(\sum_{j \neq i} I_i \right) \subseteq \Omega$.

Definition 3.2. *Let Ω be a proper R-subgroup of G. An R-subgroup H of G is said to have Ω-finite Goldie dimension (denoted as, Ω-f.G.d) if H does not contain R-subgroups H_i's of infinite number with $H_i \nsubseteq \Omega$ and its sum is Ω-direct.*

An R-group G has Ω-f.G.d if G does not contain an infinite number of R-subgroups $H_i \nsubseteq \Omega$ whose sum is Ω-direct.

Example 3.3.

(1) Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$, $R = \mathbb{Z}$. Then G is an R-group. Let $\Omega = \langle (\bar{0}, \bar{2}) \rangle$. Consider $H_1 = \langle (\bar{1}, \bar{2}) \rangle$, $H_2 = \langle (\bar{1}, \bar{1}) \rangle$, $H_3 = \langle (\bar{0}, \bar{1}) \rangle$. Then $H_i \nsubseteq \Omega$ and $H_i \cap \left(\sum_{j \neq i} H_j \right) \subseteq \Omega$. Therefore, G has Ω-*f.G.d*.

(2) Let $R = (\mathbb{Z}_{24}, +_{24}, \cdot_{24})$ and $G = R$. Consider the R-subgroups $H = \langle \bar{2} \rangle$, $H_1 = \langle \bar{4} \rangle$, $H_2 = \langle \bar{6} \rangle$, $\Omega = \langle \overline{12} \rangle$. Clearly, $H_1 \cap H_2 \subseteq \Omega$, $H_1 \nsubseteq \Omega$, $H_2 \nsubseteq \Omega$. Hence, there exist no infinite R-subgroups whose sum is Ω-direct. Therefore, H has Ω-*f.G.d*.

Theorem 3.4. *Let H, Ω (proper) be R-subgroups of G where every R-subgroup of G contained in H, contains Ω. Then H has Ω-f.G.d if and only if for every strictly increasing sequence of R-subgroups $T_1, T_2, \ldots$ of G contained in H, there exists an integer 'i' such that $T_k \leq^{se}_\Omega T_{k+1}$ for all $k \geq i$.*

Proof. Suppose H has Ω-*f.G.d*. Let $T_1 \subsetneq T_2 \subsetneq \ldots$ be R-subgroups of G such that $T_i \nsubseteq \Omega$ for all i. In a contrary way, suppose that for every integer i, there exists $k \geq i$ such that $T_k \nleq_\Omega^{se} T_{k+1}$. For $i_1 = 1$, there exists $k_1 \geq 1$ such that $T_{k_1} \nleq_\Omega^{se} T_{k_1+1}$. For $i_2 = k_1 + 1$, there exists $k_2 \geq i_2$ such that $T_{k_2} \nleq_\Omega^{se} T_{k_2+1}$ and $k_2 \geq k_1 + 1$. Continuing the process, we get a subsequence $\{T_{k_i}\}_{i=1}^{\infty}$ *of* $\{T_i\}_{i=1}^{\infty}$ such that $T_{k_i} \nleq_\Omega^{se} T_{k_i+1}$ and $k_{i+1} \geq k_i + 1$. Since $T_1 \subsetneq T_2 \subsetneq \ldots$ is increasing we have that $Tk_{i+1} \supseteq Tk_{i+1}$. Since $T_{k_i} \nleq_\Omega^{se} T_{k_i+1}$ and $Tk_{i+1} \supseteq Tk_{i+1}$ we have Tk_i is not strictly Ω-essential in Tk_{i+1}. Thus we get a subsequence $\{T_{k_i}\}_{i=1}^{\infty}$ *of* $\{T_i\}_{i=1}^{\infty}$ such that $T_{k_i} \nleq_\Omega^{se} T_{k_i+1}$ for all i. Write $B_i = Tk_i$ for $i \geq 1$. Now $\{B_i\}_{i=1}^{\infty}$ is an increasing sequence of R-subgroups of G contained in H such that $\Omega \subset B_i$ and $B_i \nleq_\Omega^{se} B_{i+1}$, for all i. Therefore, for each i, there exists a non-zero R-subgroup $A_i \nsubseteq \Omega$ of G contained in H such that $\Omega \subset A_i \subseteq B_{i+1}$ and $B_i \cap A_i \subseteq \Omega$. Now we show that $A_i \cap (\sum_{i \neq j} A_j) \subseteq \Omega$. Let n be the number of such R-subgroups A_i's. Suppose $n = 2$, and let $x \in A_1 \cap A_2$. Since $A_1 \subseteq B_2$, we have $x \in A_1 \cap A_2 \subseteq B_2 \cap A_2 \subseteq \Omega$. For $n = 3$, let $x \in A_1 \cap (A_2 + A_3)$. Since $A_1 \subseteq B_3$ and $A_2 \subseteq B_3$, by modular law, we have $x \in A_1 \cap (A_2 + A_3) \subseteq B_3 \cap (A_2 + A_3) = A_2 + (B_3 \cap A_3)$. Also, since $(B_3 \cap A_3) \subseteq \Omega$ and $\Omega \subset A_2$, we have $x \in A_2 + \Omega \subseteq A_2$. Now, $x \in A_1 \cap A_2 \subseteq B_2 \cap A_2 \subseteq \Omega$, shows that $A_1 \cap (A_2 + A_3) \subseteq \Omega$. Continuing the process, we get $A_i \cap (\sum_{i \neq j} A_j) \subseteq \Omega$. Therefore, $\sum_{i=1}^{\infty} A_i$ is Ω-direct, a contradiction to H has Ω-*f.G.d*. Converse follows from the definition of Ω-*f.G.d*.

Example 3.5. *Let* $G = (\mathbb{Z}_{48}, +_{48})$ *and* $R = (\mathbb{Z}, +, \cdot)$. *Then* $\cdot_{48}$: $R \times G \to G$ *is a R-group with respect to the external operation* $\cdot_{48}$. *Consider the R-subgroups* $H_1 = \langle \overline{2} \rangle$, $H_2 = \langle \overline{3} \rangle$, $H_3 = \langle \overline{4} \rangle$, $H_4 = \langle \overline{6} \rangle$, $H_5 = \langle \overline{8} \rangle$, $H_6 = \langle \overline{12} \rangle$, $H_7 = \langle \overline{16} \rangle$, $H_8 = \langle \overline{24} \rangle$. *Take* $H = H_2$ *and* $\Omega = H_8$. *Now we have*

Chain 1: $H_8 \subsetneq H_6 \subsetneq H_4 \subsetneq H_2$. *Then by the notation in the proof of Theorem* 3.4 *for i* = 2, *k* = 2, 3, *we have* $H_6 \leq_\Omega^{se} H_4$ *and* $H_3 \leq_\Omega^{se} H_2$. *Similarly,*

Chain 2: $H_8 \subsetneq H_4 \subsetneq H_2$. *For i* = 2, *k* = 2, *we have* $H_4 \leq_\Omega^{se} H_2$. *Therefore, H has* Ω-*f.G.d.*

Example 3.6. *Let* $G = (\mathbb{Z}_{p^n}, +_p)$, *where p is prime and* $R = (\mathbb{Z}, +, \cdot)$. *Then* $\cdot p$: $R \times G \to G$ *is a R-group with respect to the external operation* $\cdot p$. *Consider the R-subgroups* $H_i = p^{n-i}\mathbb{Z}_{p^n}$, $i \geq 0$. Take $\Omega = p^{n-1}\mathbb{Z}_{p^n}$ *and* $H = p\mathbb{Z}_{p^n}$. *Then* $H_k \leq_\Omega^{se} H_{k+1}$, *for all* $k \geq 2$. *Therefore, H* is Ω-*f.G.d.*

Example 3.7. *Let* $R = \left(\begin{pmatrix} 0 & \mathbb{Z}_{q^n} \\ 0 & 0 \end{pmatrix}, +_{q^n}, \cdot_{q^n} \right)$, *where q is prime and* $n \in \mathbb{Z}^+$. *Here R non-commutative ring and let G = R. Now G is considered as an R-group. The ideals as well as R-subgroups of G are* $K_i = \{ \begin{pmatrix} 0 & q^i\mathbb{Z}_{q^n} \\ 0 & 0 \end{pmatrix} :\ n \in \mathbb{Z}^+\}$. *Consider* $K = K_{n-1}$ *and* $\Omega = K_1$. *Then,* $H_k \leq_\Omega^{se} H_{k+1}$, *for all i* ≥ 2. *Therefore, K has* Ω-*f.G.d.*

Example 3.8. *Consider the nearring with addition and multiplication tables listed below* ([1], *Table no* 6/2(18)). *Let* $G = S_3$, *the symmetric group, and G = R. Then G is an R-group.*

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	1	4	0	2
4	4	2	5	0	3	1
5	5	3	4	1	2	0

+	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	0	0	1
2	0	2	2	0	0	2
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	5	5	0	0	5

The R-subgroups of G are $H_1 = \{0\}$, $H_2 = \{0,1\}$, $H_3 = \{0,2\}$, $H_4 = \{0, 3, 4\}$, $H_5 = \{0, 5\}$, $H_6 = G$. *Consider* $\Omega = H_4$. *Then* H_2 *is not* Ω*-f.G.d, as* $H_3,\ H_5 \not\subseteq H_2$, *but the sum is not* Ω*-direct.*

Example 3.9. *Let R is a nearring and* $R_i = R$, *for all* $i \in \mathbb{N}$. *Then* $\overset{\infty}{\underset{i=1}{\oplus}} R_i$ *is an R-group which has neither finite dimension not* Ω*-f.G.d.*

For instance, let $R = \mathbb{Z}$ *and* $G = \overset{\infty}{\underset{i=1}{\oplus}} \mathbb{Z}_i,\ \mathbb{Z}_i = \mathbb{Z}(i \geq 1)$. *Then G is an R-group. Take* $\Omega = 2\mathbb{Z} \times 2\mathbb{Z} \times \ldots 2\mathbb{Z} \times \ldots$ *Then* $I_i = (0, \ldots, \mathbb{Z}, \ldots, 0, \ldots)$. *It is clear that* $I_i \not\subseteq \Omega$ *for each i. Also* $I_i \cap (\sum_{i\neq j}^{\infty} I_j) = (0)\times(0)\times\cdots \subseteq \Omega$.

Therefore, $\{I_i\}_{i=1}^{\infty}$ *forms an infinite* Ω*-direct sum, hence G does not have* Ω*-f.G.d.*

Lemma 3.10. *If G has* Ω*-f.G.d, then every R-subgroup H of G,* $H \not\subseteq \Omega$, *contains a strictly* Ω*-uniform R-subgroup.*

Proof. Suppose that G has Ω-*f.G.d.* On a contrary, suppose H contains no strictly Ω-uniform R-subgroup. Then H is not strictly Ω-uniform. So there exist R-subgroups H_1 and H_1' of G contained in H, and $H_1, H_1' \not\subseteq \Omega$ such that $H_1 \cap H_1' \subseteq \Omega,\ H_1 + H_1' \subseteq H$. Then by supposition H_1' is not strictly Ω-uniform, which implies that there exist R-subgroups H_2, H_2' contained in H_1' and $H_2,\ H_2' \not\subseteq \Omega$ such that $H_2 \cap H' \subseteq \Omega, H_2 + H_2' \subseteq H_1'$. If we continue, then we get $\{H_i\}_1^{\infty},\ \{H_i'\}_1^{\infty}$ of two infinite sequences of R-subgroups of G, not contained in Ω such that $H_i \cap H_i' \subseteq \Omega$ and $H_i + H_i' \subseteq H_{i-1}'$, for $i \geq 2$. Thus, the sum $\sum_{i=1}^{\infty} H_i$ is infinite Ω-direct, which contradict the fact that G has Ω-*f.G.d.*

Theorem 3.11. *If G has* Ω*-f.G.d, then there exist finite number of strictly* Ω*-uniform R-subgroups of G, such that the sum is* Ω*-direct and strictly* Ω*-essential in G.*

Proof. Since G has Ω-*f.G.d*, by Lemma 3.10, G contains a strictly Ω-uniform R-subgroup, say H_1. If H_1 is strictly Ω-essential in G, then the conclusion is obvious. Suppose that H_1 is not strictly Ω-essential in G. Then there exists an R-subgroup K_1 of G with $H_1 \cap K_1 \subseteq \Omega$ but $K_1 \not\subseteq \Omega$. By Lemma 3.10, K_1 contains a strictly Ω-uniform R-subgroup, say H_2. If $H_1 + H_2 \not\leq_{\Omega}^{se} G$, then there exists an R-subgroup K_2 of G such that $(H_1 + H_2) \cap K_2 \subseteq \Omega$, but $K_2 \not\subseteq \Omega$. Again by Lemma 1, there exists a strictly Ω-uniform R-subgroup, $H_3 \subseteq K_2$. Continuing this process, we get a strictly increasing chain $H_1 \subsetneq H_1 + H_2 \subsetneq H_1 + H_2 + H_3 \subsetneq \cdots$, which must terminate as G has Ω-*f.G.d.* Hence $\sum_{i=1}^{n} H_i \leq_{\Omega}^{se} G$, for some n.

Lemma 3.12. *Let* $H_1 \subseteq H_2 \subseteq H_3$, *and* Ω (*proper*) *be R-subgroups of G. Then* $H_1 \leq_{\Omega}^{se} H_3$ *if and only if* $H_1 \leq_{\Omega}^{se} H_2$ *and* $H_2 \leq_{\Omega}^{se} H_3$.

Proof. Let $H_1 \leq_{\Omega}^{se} H_3$ and $H_1 \cap L \subseteq \Omega$, where L is R-subgroup of G and $L \subseteq H_2$. Since $L \subseteq H_2 \subseteq H_3$ and $H_1 \leq_{\Omega}^{e} H_3$, we have that $L \subseteq \Omega$. Therefore, $H_1 \leq_{\Omega}^{se} H_2$. Next, let L be an R-subgroup of G such that $H_2 \cap L \subseteq \Omega$ and $L \subseteq H_3$. Now $H_1 \cap L \subseteq H_2 \cap L \subseteq \Omega$ and since $H_1 \leq_{\Omega}^{se} H_3$, we have $L \subseteq \Omega$. Therefore, $H_2 \leq_{\Omega}^{se} H_3$. Conversely, assume $H_1 \leq_{\Omega}^{se} H_2$ and $H_2 \leq_{\Omega}^{se} H_3$. Let L be an R-subgroup of G such that $H_1 \cap L \subseteq \Omega$ and $L \subseteq H_3$. We have $H_1 \cap (H_2 \cap L) \subseteq H_1 \cap L \subseteq \Omega$. Since $H_2 \cap L$ is R-subgroup of G, $H_2 \cap L \subseteq H_2$,, and $H_1 \leq_{\Omega}^{e} H_2$, it follows that $H_2 \cap L \subseteq \Omega$. Also, as $H_2 \leq_{\Omega}^{se} H_3$, it follows $L \subseteq \Omega$.

Theorem 3.13. *Let R be distributes over G and* Ω *a proper R-subgroup of G. If G has strictly* Ω*-uniform R-subgroups* $H_1, H_2, \ldots, H_n$ *containing* Ω *such that* $\sum_{i=1}^{n} H_i$ *is* Ω*-direct and* $\sum_{i=1}^{n} H_i \leq_{\Omega}^{se} G$, *then G has* Ω*-f.G.d* (*here,* $n \in \mathbb{Z}^+$ *is independent of the choice of* H_i's).

Proof. Suppose G has strictly Ω-uniform R-subgroups $H_1, H_2, \ldots, H_n$ such that its sum is Ω-direct and $\sum_{i=1}^{n} H_i \leq_{\Omega}^{se} G$. Let $L_1, L_2, \ldots, L_m$ be R-subgroups of G such that $L_i \not\subseteq \Omega$, and $\sum_{i=1}^{m} L_i$ is Ω-direct.

Now to show $m \leq n$, first we show that if T is an R-subgroup of G such that $T \cap H_i \not\subseteq \Omega$ for all i, then $T \leq_{\Omega}^{se} G$. Suppose $T \cap H_i \not\subseteq \Omega$. Since H_i is strictly Ω-uniform, by definition, every R-subgroup contained in H_i is strictly Ω-essential. In particular, $T \cap H_i$ is an R-subgroup contained in H_i and so $T \cap H_i \leq_{\Omega}^{se} H_i$. Now by Theorem 2.18, $\sum_{i=1}^{n}(T \cap H_i) \leq_{\Omega}^{se} \sum_{i=1}^{n} H_i$ and $\sum_{i=1}^{n} H_i \leq_{\Omega}^{se} G$. Hence, by Lemma 3.12, we have $\sum_{i=1}^{n}(T \cap H_i) \leq_{\Omega}^{se} G$. Again by Lemma 2.18, since $\sum_{i=1}^{n}(T \cap H_i) \subseteq T \subseteq G$ and $\sum_{i=1}^{n}(T \cap H_i) \leq_{\Omega}^{se} G$, we get $T \leq_{\Omega}^{se} G$. Now if $\sum_{i=2}^{m} L_i \leq_{\Omega}^{se} G$, then since $\sum_{i=2}^{m} L_i$ is Ω-direct, we have $\sum_{i=2}^{m} L_i \cap L_1 \subseteq \Omega$, but $L_i \not\subseteq \Omega$, a contradiction. Hence, $\sum_{i=2}^{m} L_i \not\leq_{\Omega}^{se} G$. So there exists an $j \in \{1, 2, ..., n\}$ such that $\sum_{i=2}^{m} L_i \cap H_j \subseteq \Omega$, and $H_j \not\subseteq \Omega$. Suppose $j = 1$, then $\sum_{i=2}^{m} L_i \cap H_1 \subseteq \Omega$, which shows that $\sum_{i=2}^{m} L_i + H_1$ is Ω-direct. Again, since $H_1 + \sum_{i=3}^{m} L_i \not\leq_{\Omega}^{se} G$, there exists $j \in \{2, ..., n\}$ such that $\sum_{i=3}^{m} L_i + H_j \subseteq \Omega$, and $H_j \not\subseteq \Omega$, say $j = 2$, which implies that $\sum_{i=3}^{m} L_i + H_1 + H_2$ is Ω-direct. Continuing this process, we get $m \leq n$. Hence, G has Ω-*f.G.d.*

4. Conclusion

We have introduced the concept of uniform ideal (strictly uniform ideal) with respect to an arbitrary ideal Ω (or R-subgroup) in R-groups. Several properties of Ω-uniform ideals were proved and exhibited suitable examples or counterexamples. Finally, we have obtained Goldie theorems analog in terms of Ω-uniform R-subgroups. One can extend to study various dimensions properties involving quotient R-groups.

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