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Mathematical analysis of a fractional order two strain SEIR epidemic model

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Abstract

In this paper, a fractional order two-strain SEIR epidemic model is studied and analyzed. This model will be presented in the form of a system containing six fractional order equations, that illustrate the interactions between susceptible, strain-1 exposed, strain-2 exposed, strain-1 infected, strain-2 infected and removed individuals. The proposed model has four equilibrium points: the disease-free equilibrium point, the strain-1 equilibrium point, the strain-2 equilibrium point and the total equilibrium point. By determining the new generation matrix, we have shown that our model has two basic reproduction numbers R_0^1 and R_0^2; the first one is associated with the strain-1 and the second one is related to the strain-2. Using the Lyapunov method and La-Salle's invariance principle, we have proved the global stability of the different equilibrium points, this stability depends on the strain-1 reproduction number R_0^1 and on the strain-2 reproduction number R_0^2. Finally, numerical simulations are presented to value our theoretical results. More precisely, if the two basic reproduction numbers are less than or equal 1, then the disease-free equilibrium point is globally asymptotically stable, if one of the basic reproduction numbers is less than or equal 1 and the other is greater than 1, then the equilibrium point associated with the greatest basic reproduction number is globally asymptotically stable, and if the two basic reproduction numbers are greater than 1, then the last equilibrium point is globally asymptotically stable. Moreover, we have shown that the change in the fractional order value has no effect on the stability of the

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steady states. However, the time of convergence toward these states depends on the value of the fractional order derivative.

1. Introduction

The transmission and spread of most infectious diseases can be expressed by the mean of mathematics language in the form of a system of differential equations [1]. In the middle of the 18th century, Daniel Bernoulli established a mathematical model concerning smallpox infection; that is considered the first mathematical contribution in the field of epidemiology [2].

Currently, mathematical modeling in epidemiology serves to study the dynamics of human infectious diseases such as Human immunodeficiency virus (VIH), coronavirus (COVID-19), hepatitis B virus (HBV) and many others [3–5]. These infectious diseases can be described by the simple SIR compartmental models with S as the compartment of susceptible individuals, I for infected individuals and R for removed individuals (see [6–9]). These models are concerned with diseases when there is no incubation period for the infection; that is to say that the recently contaminated individuals becomes directly infectious. However, in most general cases, the infectious diseases may require some time for the contaminated individual to become infectious; in this situation, the model can be described by SEIR with the new compartment E representing the compartment of exposed individuals, those who are contaminated but not yet infectious. They will become infectious after a certain incubation period (see[10–13]). The SEIR models are usually represented by systems of ordinary differential equations (ODE) [14–18].

Nowadays, fractional differential equations present an interesting tool to model the dynamics of infectious diseases. Indeed, the used integer derivative in ODE can be generalized by a fractional derivative.Many works have used systems of FDE to better describe the biological systems [19–30]. For instance, in a recent work Danane et al. [31] have proposed a model which describes the dynamics of an HBV viral infection by taking into account the memory effect, the model contains five fractional order differential equations which describe the interaction between the healthy phagocytes, the infected ones the capsids, the viruses and the antibodies. The authors have demonstrated the positivity, the boundedness of the solutions and the global stability of the different equilibrium points. The transmission of most viruses such as COVID-19, HIV and other, can be described using multi-strain SIR or SEIR compartmental models because these viruses are characterized by their multiple strains, and the choice of incidence functions for these models plays a very important role in better describing how diseases are transmitted. An incidence function can be defined as the number of newly infected people in a specific time period. Most models use bilinear incidence functions in the form $\beta_1 SI$ with β_1 as the infection coefficient. Most mathematical models in the literature deal with the case of two strains as [32–43], with the infected compartment as divided into two sub-compartments generally denoted by I_0 and I_2 with the first representing the infected individuals for the first strain and the second one stands for infected individuals for the second strain 2. In the same manner, the exposed compartment is divided into two sub-compartments by $E1$ and $E2$ representing the exposed individuals for the first and the second strain, respectively. These models can have two bilinear incidence rates in the form of $\beta_1 S_1 I_1$ and $\beta_2 S_2 I_2$. Recently, Baba et al. in [44] suggested a two-strain epidemic model. The authors begin the analysis of the model by giving the different theorems of existence, positivity and boundedness of the model solutions, they gave the different theorems of the global stability of equilibria in order to give some numerical simulation to value their theoretical results. More, recently Yaagoub et al. in [45] suggested a two-strain epidemic model with a quarantine strategy. The authors gave the theorems of existence, positivity and boundedness of solutions. The authors gave also discussed the global stability of equilibria. Some numerical simulations are given in their work to value the theoretical results and to show the effect of quarantine. But, the previously suggest two-strain epidemic models use only the classical ordinary differential

equations (ODE). In the present work, we will continue two-strain epidemic model investigations by suggesting a new two-strain epidemic model. In our model, a fractional derivative order will describe the memory effect in the different acting components which is represented by the following nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}S(t) = \Lambda - \beta_1 SI_1 - \beta_2 SI_2 - \mu S, \\ D^{\alpha}E_1(t) = \beta_1 SI_1 - (\delta_1 + \mu)E_1, \\ D^{\alpha}E_2(t) = \beta_2 SI_2 - (\delta_2 + \mu)E_2, \\ D^{\alpha}I_1(t) = \delta_1 E_1 - (\gamma_1 + \mu)I_1, \\ D^{\alpha}I_2(t) = \delta_2 E_2 - (\gamma_2 + \mu)I_2, \\ D^{\alpha}R(t) = \gamma_1 I_1 + \gamma_2 I_2 - \mu R. \end{cases} \tag{1.1}$$

Where D^{α} denotes the Caputo fractional derivative of order α with $0 < \alpha \leq 1$. This derivative is defined for a given function as $\Psi(t)$ by [43]

$$D^{\alpha}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi'(s)}{(t-s)^{\alpha}}ds, \tag{1.2}$$

where $\Gamma(.)$ means Gamma function. In the system (1.1), $S(t)$, $E_1(t)$, $E_2(t)$, $I_1(t)$, $I_2(t)$ and $R(t)$ represented the numbers of susceptible, strain-1 exposed individuals, strain-2 exposed individuals, strain-1 infected individuals, strain-2 infected individuals and removed individuals at time t, respectively. Λ is the recruitment rate, β_1 is the infection coefficient of the strain-1, β_2 is the infection coefficient of the strain-2, μ natural mortality rate of the population and it is fixed for all compartments, $\frac{1}{\delta_1}$ is the incubation period of the strain-1, $\frac{1}{\delta_2}$, is the incubation period of the strain-2, $\frac{1}{\gamma_1}$, is infection period of strain-1 and $\frac{1}{\gamma_2}$ is infection period of strain-2 with incidence functions are bilinear under the form $\beta 1$ SI_1 and β_2 SI_2 . We can summarize all this in the following diagram:

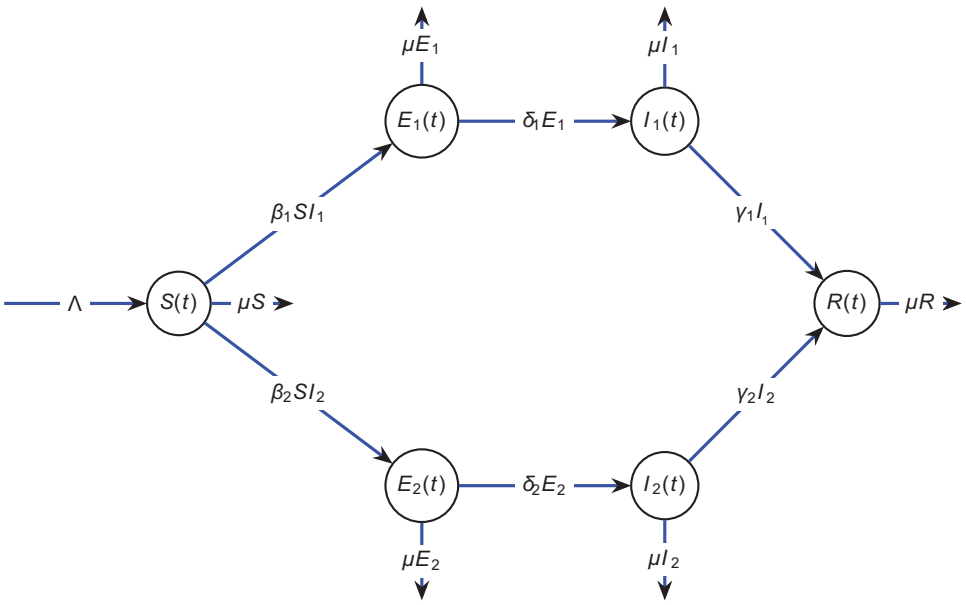

Figure 1: The diagram of a fractional two strain SEIR epidemic model.

This present work is divided into five sections. In Section 2 we give mathematical preliminary on the fractional derivative. In Section 3 we prove the positivity and boundedness of the model (1.1) variables. The existence of steady states and the global stability of these states is studied in Section 4. In Section 5 we give some numerical simulations with some discussions to value our theoretical results. The last section concludes this work.

2. Preliminary results

In this section, we will give some preliminaries and basic notions of fractional derivatives such as the definition of fractional order integral, Caputo fractional derivative and Mittage-Leffler function (see[43]).

Definition 1. *The fractional integral of order* $a > 0$ *of function* $\Psi : \mathbb{R}_+ \to \mathbb{R}$ *is defined by*

$$I^{\alpha}\Psi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Psi(s)ds, \tag{2.1}$$

where $\Gamma(.)$ *stands for Gamma function.*

Definition 2. *The Mittag-Leffler function of parameter a, with* $\alpha > 0$ *noted by* E_α *and defined by*

$$E_{\alpha}(t) = \sum_{j=0}^{+\infty} \frac{t^j}{\Gamma(\alpha j + 1)}, \tag{2.2}$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ where $n \geq 1$. We consider the following fractional order system given by:

$$D^{\alpha}X(t) = f(X), \tag{2.3}$$

with $X(0) = X_0$, $X_0 \in \mathbb{R}^n$ and $0 < \alpha \leq 1$. To demonstrate the global stability of solutions of system (2.3), we will need the following lemma:

Lemma 3. *If the function f satisfies the two following conditions:*

1. $f(X)$ *and* $\dfrac{\partial f}{\partial X}(X)$ *are continuous on* $\mathbb{R}^n$.
2. $\| f(X) \| \leq K_1 + K_2 \| X \|$ *for all* $X \in \mathbb{R}^n$, *with* K_1 *and* K_2 *are two positive constants.*

Then, the system (2.3) has a only solution defined on $\mathbb{R}^n_+$

3. Positivity and Boundedness

In this section, we will show that the solutions of model (1.1) remain positive and bounded. For a biological reasons we will assume that $S(0)$, $E_1(0)$, $E_2(0)$, $I_1(0)$, $I_2(0)$ and $R(0)$ are greater than or equal to zero. This result is presented in the form of the following proposition:

Proposition 4. *For any positive initial conditions* $S(0)$, $E_1(0)$, $E_2(0)$, $I_1(0)$, $I_2(0)$ *and* $R(0)$, *the system* (1.1) *has an unique bounded and positive solution defined on* $[0, +\infty[$ *for all* $t \geq 0$.

Proof.
We can reduce our model (1.1) as follow:

$$D^{\alpha}U(t) = \mathcal{F}(U), \tag{3.1}$$

with,

$$U=\begin{pmatrix} S \\ E_1 \\ E_2 \\ I_1 \\ I_2 \\ R \end{pmatrix}, \text{and } \mathcal{F}(U)=\begin{pmatrix} \Lambda-\beta_1 SI_1-\beta_2 SI_2-\mu S \\ \beta_1 SI_1-(\delta_1+\mu)E_1 \\ \beta_2 SI_2-(\delta_2+\mu)E_2 \\ \delta_1 E_1-(\gamma_1+\mu)I_1 \\ \delta_2 E_2-(\gamma_2+\mu)I_2 \\ \gamma_1 I_1+\gamma_2 I_2-\mu R \end{pmatrix}.$$

If $a = 1$, the system(3.1) will be a system with ordinary differential equations. In this case of a system with fractional differential equations, let

$$\mathcal{A}_1=\begin{pmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \mathcal{A}_2=\begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\delta_1+\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\delta_2+\mu) & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & -(\gamma_1+\mu) & 0 & 0 \\ 0 & 0 & \delta_2 & 0 & -(\gamma_2+\mu) & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & -\mu \end{pmatrix}$$

and

$$\mathcal{A}_3=\begin{pmatrix} -\beta_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_4=\begin{pmatrix} -\beta_2 & 0 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

therefore

$$f(X)=\mathcal{A}_1+\mathcal{A}_2X+I_1\mathcal{A}_3X+I_2\mathcal{A}_4X;$$

this implied that

$$\| f(X)\| \leq \| A_1\| +(\| A_2\| +\| V\| \| A_3\|)\| X\| .$$

Hence, the proprieties of Lemma 3 are satisfied. Then the system (1.1) has a unique solution on $[0, +\infty[$.

Now we will prove the positivity of model solutions. First, we will assume that all model parameters are positive, we have

$$D^\alpha S|_{S=0}=\Lambda\geq 0,$$
$$D^\alpha E_1|_{E_1=0}=\beta_1 SI_1\geq 0,$$
$$D^\alpha E_2|_{E_2=0}=\beta_2 SI_2\geq 0,$$
$$D^\alpha I_1|_{I_1=0}=\delta_1 E_1\geq 0,$$
$$D^\alpha I_2|_{I_2=0}=\delta_2 E_2\geq 0,$$
$$D^\alpha R|_{R=0}=\gamma_1 I_1+\gamma_2 I_2\geq 0.$$

So all the solutions of the model remain positive.

Finally, we will show the boundedness of the solutions. Let the total population

$$N(t) = S(t) + E_1(t) + E_2(t) + I_1(t) + I_2(t) + R(t).$$

By adding all equations of the system (1.1), we will have

$$\begin{aligned} D^\alpha N(t) &= D^\alpha S(t) + D^\alpha E_1(t) + D^\alpha E_2(t) + D^\alpha I_1(t) + D^\alpha I_2(t) + D^\alpha R(t) \\ &= \Lambda - \mu N(t), \end{aligned}$$

then

$$N(t) \le \frac{\lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu} \right) E_\alpha(-\mu t^\alpha)$$

where $E_\alpha(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j + 1)}$ is the Mittaj-Leffler function of parameter α. Since $0 \le E_\alpha(-\mu t^\alpha) \le 1$, we have

$$N(t) \le N(0) + \frac{\Lambda}{\mu}.$$

This completes the proof,

4. Global stability of the steady states

In this section, we show that there exists a disease-free equilibrium point and three endemic equilibrium points, and we will study the global stability of these equilibrium points using some suitable Lyapunov functions method's and LaSalle's invariant principle. Since the first five equations of the system (1.1) are independent of R and knowing that the number of the total population verifies the equation (4.2), we can omit the sixth equation and the system can be reduced to:

$$\begin{cases} D^\alpha S = \Lambda - \beta_1 S I_1 - \beta_2 S I_2 - \mu S, \\ D^\alpha E_1 = \beta_1 S I_1 - (\delta_1 + \mu) E_1, \\ D^\alpha E_2 = \beta_2 S I_2 - (\delta_2 + \mu) E_2, \\ D^\alpha I_1 = \delta_1 E_1 - (\gamma_1 + \mu) I_1, \\ D^\alpha I_2 = \delta_2 E_2 - (\gamma_2 + \mu) I_2. \end{cases} \tag{4.1}$$

With

$$R = N - V - E_1 - E_2 - I_1 - I_3. \tag{4.2}$$

4.1. The basic reproduction number calculation

The basic reproduction number can be defined as the average number of new cases of an infection caused by one typical infected individual in a population consisting of susceptible individuals only. We use the next generation FV^{-1} to calculate the basic reproduction R_0. The formula that gives us the basic reproduction number is: $R_0 = \rho(FV^{-1})$, where ρ stands for the spectral radius, F is the non-negative matrix of new infection cases, and V is the matrix of the transition infections associated to the model (4.1).

Let the disease-free equilibrium

$$\mathcal{E}_f = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right),$$

and let

$$F = \begin{pmatrix} 0 & 0 & \dfrac{\beta_1 \Lambda}{\mu} & 0 \\ 0 & 0 & 0 & \dfrac{\beta_2 \Lambda}{\mu} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \delta_1 + \mu & 0 & 0 & 0 \\ 0 & \delta_2 + \mu & 0 & 0 \\ -\delta_1 & 0 & \gamma_1 + \mu & 0 \\ 0 & -\delta_2 & 0 & \gamma_2 + \mu \end{pmatrix}.$$

Let calculate FV^{-1} we have

$$\det(V) = (\delta_1 + \mu)(\delta_2 + \mu)(\gamma_1 + \mu)(\gamma_2 + \mu) \neq 0,$$

so V is invertible, furthermore

$$FV^{-1} = \begin{pmatrix} \dfrac{\beta_1 \Lambda \delta_1}{\mu(\gamma_1 + \mu)(\delta_1 + \mu)} & 0 & \dfrac{\beta_1 \Lambda}{\mu(\gamma_1 + \mu)} & 0 \\ 0 & \dfrac{\beta_2 \Lambda \delta_2}{\mu(\gamma_2 + \mu)(\delta_2 + \mu)} & 0 & \dfrac{\beta_2 \Lambda}{\mu(\gamma_2 + \mu)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then

$$R_0 = \max\{R_0^1; R_0^2\},$$

with

$$\begin{cases} R_0^1 = \dfrac{\beta_1 \Lambda \delta_1}{\mu(\gamma_1 + \mu)(\delta_1 + \mu)}, \\ R_0^2 = \dfrac{\beta_2 \Lambda \delta_2}{\mu(\gamma_2 + \mu)(\delta_2 + \mu)}, \end{cases} \tag{4.3}$$

we note

$$a = \delta_1 + \mu; b = \delta_2 + \mu; c = \gamma_1 + \mu; e = \gamma_2 + \mu,$$

then

$$\begin{cases} R_0^1 = \dfrac{\beta_1 \Lambda \delta_1}{\mu a c}, \\ R_0^2 = \dfrac{\beta_2 \Lambda \delta_2}{\mu b e}. \end{cases} \tag{4.4}$$

4.2. Steady states

The model (4.1) has for steady states, one disease-free equilibrium and three endemic equilibrium points as follows:

1. The disease-free equilibrium

$$\mathcal{E}_f = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0 \right),$$

2. The strain 1 endemic equilibrium $\mathcal{E}_{s_1} = (S_{s_1}^*, E_{1,s_1}^*, E_{2,s_1}^*, I_{1,s_1}^*, I_{2,s_1}^*)$, where:

$$S_{s_1}^* = \frac{ac}{\beta_1 \delta_1},$$

$$E_{1,s_1}^* = \frac{c}{\delta_1} I_{1,s_1}^*, E_{2,s_1}^* = 0,$$

$$I_{1,s_1}^* = \frac{\mu}{\beta_1}(R_1^0 - 1), I_{2,s_1}^* = 0.$$

3. The strain 2 endemic equilibrium $\mathcal{E}_{s_2} = (S_{s_2}^*, E_{1,s_2}^*, E_{2,s_2}^*, I_{1,s_2}^*, I_{2,s_2}^*)$, where:

$$S_{s_2}^* = \frac{be}{\beta_2 \delta_2},$$

$$E_{1,s_2}^* = 0, E_{2,s_2}^* = \frac{e}{\delta_2} I_{2,s_2}^*,$$

$$I_{1,s_2}^* = 0, I_{2,s_2}^* = \frac{\mu}{\beta_2}\left(R_0^2 - 1 \right).$$

4. The total endemic equilibrium $\mathcal{E}_{s_t} = (S_t^*, E_{1,t}^*, E_{2,t}^*, I_{1,t}^*, I_{2,t}^*)$, where:

$$S_t^* = \frac{be}{\beta_2 \delta_2} = \frac{ac}{\beta_1 \delta_1},$$

$$E_{1,t}^* = \frac{c}{\delta_1} I_{1,t}^*, E_{2,t}^* = \frac{e}{\delta_2} I_{2,t}^*,$$

$$I_{1,t}^* = \frac{\Lambda}{\beta_1 S_t^*} - \frac{\mu}{\beta_1} - \frac{\beta_2 I_{2,t}^*}{\beta_1}, I_{2,t}^* = \frac{\delta_2 E_{2,t}^*}{e}.$$

Remark 5. *From the components of the equilibrium points, we conclude that points exist when* $R_0^1 > 1$ *and* $R_0^2 > 1$.

4.3. Global stability

In this section, we will give some theorems of the global stability of the different diseases equilibrium found.

Firstly, for the global stability of the disease-free equilibrium, we have the following result:

Theorem 6. *If* $R_0^1 \le 1$ *and* $R_0^2 \le 1$*, then the disease-free equilibrium* ε_f *is globally asymptotically stable.*

Proof. We consider the following Lyapunov function L_f in $\mathbb{R}_+^5$:

$$L_f(S,E_1,E_2,I_1,I_2) = S_0\left(\frac{S}{S_0} - \ln\left(\frac{S}{S_0}\right) - 1\right) + E_1 + E_2 + \frac{a}{\delta_1} I_1 + \frac{b}{\delta_2} I_2.$$

The a derivative of L_f is given by:

$$\begin{aligned}
D^\alpha L_f &= \frac{d^\alpha L_f}{d^\alpha S} D^\alpha S + \frac{d^\alpha L_f}{d^\alpha E_1} D^\alpha E_1 + \frac{d^\alpha L_f}{d^\alpha E_2} D^\alpha E_2 + \frac{d^\alpha L_f}{d^\alpha I_1} D^\alpha I_1 + \frac{d^\alpha L_f}{d^\alpha I_2} D^\alpha I_2 \\
&\le \left(1 - \frac{S_0}{S}\right) D^\alpha S + D^\alpha E_1 + D^\alpha E_2 + \frac{\alpha}{\delta_1} D^\alpha I_1 + \frac{b}{\delta_2} D^\alpha I_1 \\
&\le \left(1 - \frac{S_0}{S}\right)\left(\Lambda - \beta_1 S I_1 - \beta_2 S I_2 - \mu S\right) + \left(\beta_1 S I_1 - a E_1\right) + \left(\beta_2 S I_2 - b E_2\right) + \frac{a}{\delta_1}\left(\delta_1 E_1 - c I_1\right) \\
&\quad + \frac{b}{\delta_2}\left(\delta_2 E_2 - e I_2\right) \\
&\le \mu S_0\left(2 - \frac{S}{S_0} - \frac{S_0}{S}\right) + \frac{ac}{\delta_1}\left(R_0^1 - 1\right) + \frac{be}{\delta_2}\left(R_0^2 - 1\right).
\end{aligned}$$

Since the arithmetic mean is greater than or equal to the geometric mean, we have

$$2 - \frac{S}{S_0} - \frac{S_0}{S} \le 0.$$

So when $R_0^1 \le 0$ and $R_0^2 \le 0$ then, $D^\alpha L_f \le 0$.

We denote by V_0 the largest invariant set in $\{(S, E_1, E_2, I_1, I_2)\ D^\alpha L_f = 0\}$. We notice that $D^\alpha L_f(t) = 0$ if and only if $S = S_0 = \frac{\Lambda}{\mu}, E_1 = 0, E_2 = 0 I_1 = 0$ and $I_2 = 0$. Then, $V_0 = \{\varepsilon_f\} = \left\{\left(\frac{\Lambda}{\mu}, 0,0,0,0\right)\right\}$. So since La-Salle's invariant principal that the disease-free equilibrium ε_f is globally asymptotically stable if $R_0^1 \le 0$ and $R_0^2 \le 0$.

Secondly, for the global stability of the strain-1 endemic equilibrium, we have the following result:

Theorem 7. *If* $R_0^2 \le 1 < R_0^1$*. Then the steady state* ε_s *is globally asymptotically stable.*

We consider the following Lyapunov function L_1 in $\mathbb{R}_+^5$:

$$\begin{aligned}
L_1(S,E_1,E_2,I_1,I_2) &= S_{s_1}^*\left(\frac{S}{S_{s_1}^*} - \ln\left(\frac{S}{S_{s_1}^*}\right) - 1\right) + E_{1,s_1}^*\left(\frac{E_1}{E_{1,s_1}^*} - \ln\left(\frac{E_1}{E_{1,s_1}^*}\right) - 1\right) \\
&\quad + E_2 + \frac{a}{\delta_1} I_{1,s_1}^*\left(\frac{I_1}{I_{1,s_1}^*} - \ln\left(\frac{I_1}{I_{1,s_1}^*}\right) - 1\right) + \frac{b}{\delta_2} I_2.
\end{aligned}$$

The α derivative of L_1 is given by:

$$\begin{aligned}
D^{\alpha}L_1 &= \frac{d^{\alpha}L_1}{d^{\alpha}S}D^{\alpha}S + \frac{d^{\alpha}L_1}{d^{\alpha}E_1}D^{\alpha}E_1 + \frac{d^{\alpha}L_1}{d^{\alpha}E_2}D^{\alpha}E_2 + \frac{d^{\alpha}L_1}{d^{\alpha}I_1}D^{\alpha}I_1 + \frac{d^{\alpha}L_1}{d^{\alpha}I_2}D^{\alpha}I_2 \\
&\leq \left(1-\frac{S^*_{s_1}}{S}\right)D^{\alpha}S + \left(1-\frac{E^*_{1,s_1}}{E_1}\right)D^{\alpha}E_1 + D^{\alpha}E_2 + \frac{\alpha}{\delta_1}\left(1-\frac{I^*_{1,s_1}}{I_1}\right)D^{\alpha}I_1 + \frac{b}{\delta_2}D^{\alpha}I_2 \\
&\leq \left(1-\frac{S^*_{s_1}}{S}\right)\left(\Lambda-\beta_1 SI_1-\beta_2 SI_2-\mu S\right) + \left(1-\frac{E^*_{1,s_1}}{E_1}\right)\left(\beta_1 SI_1-aE_1\right) + \left(\beta_2 SI_2-bE_2\right) \\
&\quad + \frac{a}{\delta_1}\left(1-\frac{I^*_{1,s_1}}{I_1}\right)\left(\delta_1 E_1-cI_1\right) + \frac{b}{\delta_2}\left(\delta_2 E_2-eI_2\right),
\end{aligned}$$

we know that the point $(S^*_{s_1}, E^*_{1,s_1}, E^*_{2,s_1}, I^*_{1,s_1}, I^*_{2,s_1})$ is an equilibrium point of the system (4.1), so we will have: $E^*_{2,s_1}=0, I^*_{2,s_1}=0$ and

$$\begin{cases}
\Lambda = \beta_1 S^*_{s_1} I^*_{1,s_1} + \mu S^*_{s_1}, \\
\beta_1 S^*_{s_1} I^*_{1,s_1} = aE^*_{1,s_1} = \dfrac{ac}{\delta_1} I^*_{1,s_1}.
\end{cases}$$

Therefore

$$\begin{aligned}
D^{\alpha}L_1 \leq \mu S^*_{s_1}\left(2-\frac{S^*_{s_1}}{S}-\frac{S}{S^*_{s_1}}\right) + aE^*_{1,s_1}\left(3-\frac{E_1}{E^*_{1,s_1}}\frac{I^*_{1,s_1}}{I_1}-\frac{S^*_{s_1}}{S}-\frac{I_1}{I^*_{1,s_1}}\frac{S}{S^*_{s_1}}\frac{E^*_{1,s_1}}{E_1}\right) \\
+\beta_1 I_1\left(S^*_{s_1}-\frac{ac}{\beta_1\delta_1}\right) + \beta_2 I_2\left(S^*_{s_1}-\frac{be}{\beta_2\delta_1}\right).
\end{aligned}$$

If $R_0^2 \leq 1$, then we will have

$$\frac{\Lambda}{\mu} \leq \frac{be}{\beta_2\delta_2},$$

and since $R_0^1 > 1$,

$$\frac{\Lambda}{\mu} > \frac{ac}{\beta_1\delta_1} = S^*_{s_1},$$

so we will have

$$S^*_{s_1} - \frac{be}{\beta_2\delta_2} \leq 0,$$

we have also

$$S^*_{s_1} - \frac{ac}{\beta_1\delta_1} = 0.$$

We know that the arithmetic mean is greater than or equal to the geometric mean, we will have

$$2-\frac{S}{S^*_{s_1}}-\frac{S^*_{s_1}}{S}\le 0,$$

and

$$3-\frac{E_1}{E^*_{1,s_1}}\frac{I^*_{1,s_1}}{I_1}-\frac{S^*_{s_1}}{S}-\frac{I_1}{I^*_{1,s_1}}\frac{S}{S^*_{s_1}}\frac{E^*_{1,s_1}}{E_1}\le 0.$$

We denote by V_1 the largest invariant set in $\{(S, E_1, E_2, I_1, I_2)\, D^\alpha L_1 = 0\}$. We notice that $D^\alpha L_1(t) = 0$ if and only if $S = S^*_{s_1}, E_1 = E^*_{1,s_1}, E_2 = E^*_{2,s_1} = 0, I_1 = I^*_{1,s_1}$ and $I_2 = I^*_{2,s_1} = 0$ Then, $V_1 = \{\mathcal{E}_{s_1}\} = \{(S^*_{s_1}, E^*_{1,s_1}, E^*_{2,s_1}, I^*_{1,s_1}, I^*_{2,s_1})\}$. So since La-Salle's invariant principal that the strain-1 endemic equilibrium $\mathcal{E}_{s_1}$ is globally asymptotically stable if $R^2_0 \le 1$ and $R^1_0 > 1$.

Thirdly, for the global stability of the strain-2 endemic equilibrium, we have the following result:

Theorem 8. *If $R^1_0 \le 1 < R^2_0$. Then the steady state $\mathcal{E}_{s_2}$ is globally asymptotically stable.*

Proof. We consider the following Lyapunov function L_2 in $\mathbb{R}^5_+$:

$$L_2(S,E_1,E_2,I_1,I_2) = S^*_{s_2}\left(\frac{S}{S^*_{s_2}}-\ln\left(\frac{S}{S^*_{s_2}}\right)-1\right)+E_1+E^*_{2,s_2}\left(\frac{E_2}{E^*_{2,s_2}}-\ln\left(\frac{E_2}{E^*_{2,s_2}}\right)-1\right)+\frac{a}{\delta_1}I_1$$
$$+\frac{b}{\delta_2}\left(\frac{I_2}{I^*_{2,s_2}}-\ln\left(\frac{I_2}{I^*_{2,s_2}}\right)-1\right).$$

The α derivative of L_2 is given by:

$$\begin{aligned}
D^\alpha L_2 &= \frac{d^\alpha L_2}{d^\alpha S}D^\alpha S+\frac{d^\alpha L_2}{d^\alpha E_1}D^\alpha E_1+\frac{d^\alpha L_2}{d^\alpha E_2}D^\alpha E_2+\frac{d^\alpha L_2}{d^\alpha I_1}D^\alpha I_1+\frac{d^\alpha L_2}{d^\alpha I_2}D^\alpha I_2\\
&\le \left(1-\frac{S^*_{s_2}}{S}\right)D^\alpha S+D^\alpha E_1+\left(1-\frac{E^*_{2,s_2}}{E_2}\right)D^\alpha E_2+\frac{a}{\delta_1}D^\alpha I_1+\frac{b}{\delta_2}\left(1-\frac{I^*_{2,s_2}}{I_2}\right)D^\alpha I_2\\
&\le \left(1-\frac{S^*_{s_2}}{S}\right)\left(\Lambda-\beta_1 SI_1-\beta_2 SI_2-\mu S\right)+\left(1-\frac{E^*_{2,s_2}}{E_2}\right)\left(\beta_2 SI_2-bE_2\right)+\left(\beta_1 SI_1-aE_1\right)\\
&\quad+\frac{a}{\delta_1}\left(\delta_1 E_1-cI_1\right)+\left(1-\frac{I^*_{2,s_2}}{I_2}\right)\frac{b}{\delta_2}\left(\delta_2 E_2-eI_2\right),
\end{aligned}$$

we know that the point $(S^*_{s_2}, E^*_{1,s_2}, E^*_{2,s_2}, I^*_{1,s_2}, I^*_{2,s_2})$ is an equilibrium point of the system (4.1), so we will have: $E^*_{1,s_2} = 0, I^*_{1,s_2} = 0$ and

$$\begin{cases}
\Lambda = \beta_2 S^*_{s_2} I^*_{2,s_2} + \mu S^*_{s_2},\\
\beta_2 S^*_{s_2} I^*_{2,s_2} = bE^*_{1,s_2} = \dfrac{be}{\delta_2} I^*_{2,s_2}.
\end{cases}$$

Therefore

$$D^{\alpha}L_2 \le \mu S^*_{s_2}\left(2-\frac{S^*_{s_2}}{S}-\frac{S}{S^*_{s_2}}\right)+bE^*_{2,s_2}\left(3-\frac{E_2}{E^*_{2,s_2}}\frac{I^*_{2,s_2}}{I_2}-\frac{S^*_{s_2}}{S}-\frac{I_2}{I^*_{2,s_2}}\frac{S}{S^*_{s_2}}\frac{E^*_{2,s_2}}{E_2}\right)$$
$$+\beta_1 I_1\left(S^*_{s_2}-\frac{ac}{\beta_1\delta_1}\right)+\beta_2 I_2\left(S^*_{s_2}-\frac{be}{\beta_2\delta_2}\right),$$

if $R^1_0 \le 1$, then we will have

$$\frac{\Lambda}{\mu}\le\frac{ac}{\beta_1\delta_1},$$

and since $R^2_0 > 1$,

$$\frac{\Lambda}{\mu}>\frac{be}{\beta_2\delta_2}=S^*_{s_2},$$

so we will have

$$S^*_{s_2}-\frac{ac}{\beta_1\delta_1}\le 0,$$

we have also

$$S^*_{s_2}-\frac{be}{\beta_2\delta_2}=0.$$

Since the arithmetic mean is greater than or equal to the geometric mean, we will have

$$2-\frac{S}{S^*_{s_2}}-\frac{S^*_{s_2}}{S}\le 0$$

and

$$3-\frac{E_2}{E^*_{2,s_2}}\frac{I^*_{2,s_2}}{I_2}-\frac{S^*_{s_2}}{S}-\frac{I_2}{I^*_{2,s_2}}\frac{S}{S^*_{s_2}}\frac{E^*_{2,s_2}}{E_2}\le 0.$$

We denote by V_2 the largest invariant set in $\{(S,\ E_1\ ,\ E_2\ ,\ I_1\ ,\ I_2\ )\ D^{\alpha}L_2 = 0\}$. We notice that $D^{\alpha}L2\ (t) = 0$ if and only if $S = S^*_{s_2}, E_1 = E^*_{1,s_2} = 0, E_2 = E^*_{2,s_2}, I_1 = I^*_{1,s_1} = 0$ and $I_2 = I^*_{2,s_1}$. Then, $V_2 = \{\mathcal{E}_{s_2}\} = \{(S^*_{s_2}, E^*_{1,s_2}, E^*_{2,s_2}, I^*_{1,s_2}, I^*_{2,s_2})\}$. So since La-Salle's invariant principal that the strain-2 endemic equilibrium $\mathcal{E}_{s_2}$ is globally asymptotically stable if $R^1_0 > 1$ and $R^2_0 > 1$.

Theorem 9. *If $R^1_0 = R^2_0 > 1$. Then the steady state of the model $\mathcal{E}_t$ is globally asymptotically stable.*

We consider the following Lyapunov function L_3 in $\mathbb{R}^5_+$:

$$L_3(S,E_1,E_2,I_1,I_2)=S^*_{s_t}\left(\frac{S}{S^*_t}-\ln\left(\frac{S}{S^*_t}\right)-1\right)+E^*_{1,t}\left(\frac{E_1}{E^*_{1,t}}-\ln\left(\frac{E_1}{E^*_{1,t}}\right)-1\right)+E^*_{2,t}\left(\frac{E_2}{E^*_{2,t}}-\ln\left(\frac{E_2}{E^*_{2,t}}\right)-1\right)$$
$$+\frac{a}{\delta_1}I^*_{1,t}\left(\frac{I_1}{I^*_{1,t}}-\ln\left(\frac{I_1}{I^*_{1,t}}\right)-1\right)+\frac{b}{\delta_2}I^*_{2,t}\left(\frac{I_2}{I^*_{2,s_2}}-\ln\left(\frac{I_2}{I^*_{2,s_2}}\right)-1\right).$$

The α derivative of L_3 is given by:

$$\begin{aligned}
D^{\alpha}L_3 &= \frac{d^{\alpha}L_3}{d^{\alpha}S}D^{\alpha}S + \frac{d^{\alpha}L_3}{d^{\alpha}E_1}D^{\alpha}E_1 + \frac{d^{\alpha}L_3}{d^{\alpha}E_2}D^{\alpha}E_2 + \frac{d^{\alpha}L_3}{d^{\alpha}I_1}D^{\alpha}I_1 + \frac{d^{\alpha}L_3}{d^{\alpha}I_2}D^{\alpha}I_2 \\
&\leq \left(1-\frac{S_t^*}{S}\right)D^{\alpha}S + \left(1-\frac{E_{1,t}^*}{E_1}\right)D^{\alpha}E_1 + \left(1-\frac{E_{2,t}^*}{E_2}\right)D^{\alpha}E_2 + \frac{\alpha}{\delta_1}\left(1-\frac{I_{1,t}^*}{I_1}\right)D^{\alpha}I_1 + \frac{b}{\delta_2}\left(1-\frac{I_{2,t}^*}{I_2}\right)D^{\alpha}I_2 \\
&\leq \left(1-\frac{S_t^*}{S}\right)\left(\Lambda-\beta_1 SI_1-\beta_2 SI_2-\mu S\right) + \left(1-\frac{E_{1,t}^*}{E_1}\right)\left(\beta_1 SI_1-aE_1\right) + \left(1-\frac{E_{2,t}^*}{E_2}\right)\left(\beta_2 SI_2-bE_2\right) \\
&\quad + \frac{a}{\delta_1}\left(1-\frac{I_{1,t}^*}{I_1}\right)\left(\delta_1 E_1-cI_1\right) + \frac{b}{\delta_2}\left(1-\frac{I_{2,t}^*}{I_2}\right)\left(\delta_2 E_2-eI_2\right),
\end{aligned}$$

we know that the point $(S_t^*, E_t^*, E_{2,t}^*, I_{1,t}^*, I_{2,t}^*)$ is an equilibrium point of the system (4.1), so we will have:

$$\begin{cases}
\Lambda = \beta_1 S_t^* I_{1,t}^* + \beta_2 S_t^* I_{2,t}^* + \mu S_{s_1}^*, \\
aE_{1,t}^* = \beta_1 S_t^* I_{1,t}^* = \dfrac{ac}{\delta_1} I_{1,t}^*, \\
aE_{2,t}^* = \beta_2 S_t^* I_{2,t}^* = \dfrac{be}{\delta_2} I_{2,t}^*.
\end{cases}$$

Therefore

$$\begin{aligned}
D^{\alpha}L_3 \leq \mu S_t^*\left(2-\frac{S_t^*}{S}-\frac{S}{S_t^*}\right) &+ aE_{1,t}^*\left(3-\frac{S_t^*}{S}-\frac{E_1}{E_{1,t}^*}\frac{I_{2,t}^*}{I_2}-\frac{I_2}{I_{2,t}^*}\frac{S}{S_t^*}\frac{E_{2,t}^*}{E_2}\right) \\
&+ bE_{2,t}^*\left(3-\frac{S_t^*}{S}-\frac{E_2}{E_{2,t}^*}\frac{I_{2,t}^*}{I_2}-\frac{I_2}{I_{2,t}^*}\frac{S}{S_{s_1}^*}\frac{E_{2,t}^*}{E_1}\right) \\
&+ \beta_1 I_1\left(S_t^*-\frac{ac}{\beta_1\delta_1}\right) + \beta_2 I_2\left(S_t^*-\frac{be}{\beta_2\delta_2}\right).
\end{aligned}$$

According to the formula of S_t^*, we have

$$S_t^* = \frac{be}{\beta_2\delta_2} = \frac{ac}{\beta_1\delta_1} \Leftrightarrow R_0^1 = R_0^2,$$

$$S_t^* - \frac{ac}{\beta_1\delta_1} = 0$$

and

$$S_t^* - \frac{be}{\beta_2\delta_2} = 0.$$

Since the arithmetic mean is greater than or equal to the geometric mean, we will have

$$2-\frac{S_t^*}{S}-\frac{S}{S_t^*}\leq 0,$$

$$3-\frac{S_t^*}{S}-\frac{E_1}{E_{1,t}^*}\frac{I_{2,t}^*}{I_2}-\frac{I_2}{I_{2,t}^*}\frac{S}{S_t^*}\frac{E_{2,t}^*}{E_2}\leq 0.$$

We denote by V_3 the largest invariant set in $\{(S, E_1, E_2, I_1, I_2)\ D^\alpha L_3 = 0\}$. We notice that $D^\alpha L_3(t) = 0$ if and only if $S = S_t^*, E_1 = E_{1,t}^*, E_2 = E_{2,t}^*, I_1 = I_{1,t}^*$ and $I_2 = I_{2,t}^*$. Then, $V_3 = \{\mathcal{E}_t\} = \{(S_t^*, E_{1,t}^*, E_{2,t}^*, I_{1,t}^*, I_{2,t}^*)\}$.So since La-Salle's invariant principle that the total endemic equilibrium $\mathcal{E}_t$ is globally asymptotically stable if $R_0^1 = R_0^2 > 1$.

5. Numerical simulations and discussions

In this section, we will give some numerical simulations to validate the theoretical results found in the previous section. For this purpose, we will use an algorithm used in [46] by Odibat et al. To solve our model (1.1) numerically, we know that according to the equation (3.1), we can transform it into:

$$\begin{cases} D^\alpha U(t) = \mathcal{F}(U(t)), \\ U(0) = U_0, U_0 \in \mathbb{R}_+^6. \end{cases} \tag{5.1}$$

Where $U_0 = \begin{pmatrix} S(0) \\ E_1(0) \\ E_2(0) \\ I_1(0) \\ I_2(0) \\ R(0) \end{pmatrix}$ is the initial condition. So the problem (5.1) can be solved by using the following numerical scheme:

$$\begin{aligned} U(t_j) = {} & \frac{h^\alpha}{\Gamma(\alpha+2)}\left((j-1)^{\alpha+1} - (j-\alpha-1)j^\alpha\right)\mathcal{F}(U(t_0)) + U(t_0) \\ & + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{i=1}^{j-1}\left((j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1}\right)\mathcal{F}(U(t_i)) \\ & + \frac{h^\alpha}{\Gamma(\alpha+2)}\mathcal{F}(U(t_{j-1}) + \frac{h^\alpha}{\Gamma(\alpha+1)}\mathcal{F}(U(t_{j-1}))), \end{aligned}$$

where $t_j = t(j-1) + h$, for $j = 0, 1, ..., N-1$. and $U_0 = \begin{pmatrix} 10 \\ 5 \\ 4.5 \\ 3 \\ 2.8 \\ 2 \end{pmatrix}$

Figure 2 describes the time evolution of the infection for the different compartments of the model. We observe that all the curves converge toward the equilibrium point $\mathcal{E}_f = (5, 0, 0, 0, 0, 0)$. Here, the

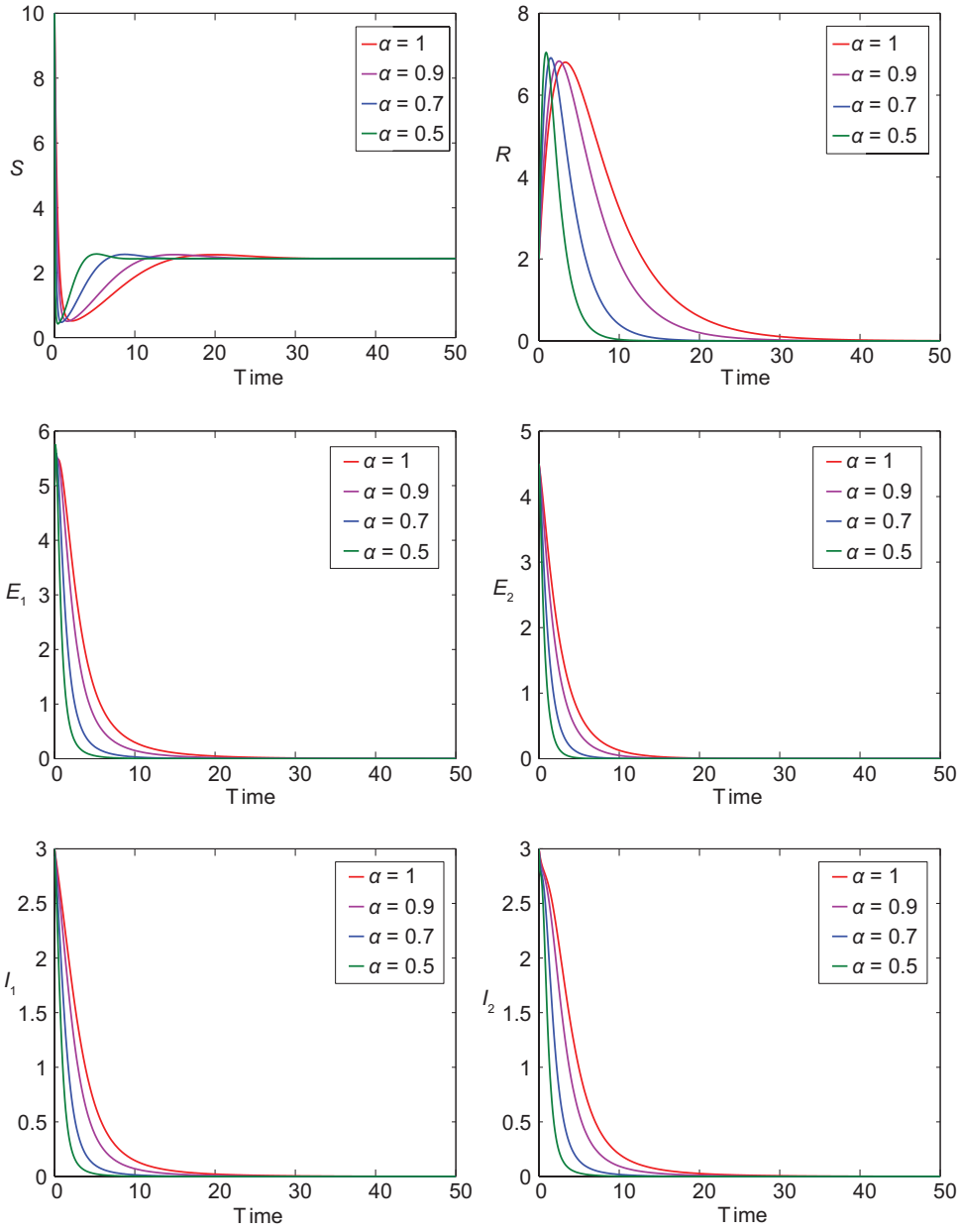

Figure 2: Behavior of the infection as function of time for $\Lambda = 1$, $\beta_1 = 0.17$, $\beta_2 = 0.12$, $\delta_1 = 0.4$, $\delta_2 = 0.3$, $\gamma_1 = 0.65$, $\gamma_2 = 0.75$, $\mu = 0.2$ and $h = 0.1$, which corresponds to the free-equilibrium point $\mathcal{E}_f$.

two basic reproductions are less or equal to 1 ($R_0^1 = 0.66 \leq 1$ and $R_0^2 = 0.37 \leq 1$), which confirms the results found in the Theorem 6 concerning the stability of the equilibrium point $\mathcal{E}_f$.

Figure 3 also shows the dynamics of the infection, we can see that all the curves converge toward the strain-1 endemic equilibrium $\mathcal{E}_{S_1}$ = (1.8001, 1.0666, 0, 0.7111, 0, 1.4231), with the basic reproduction number that corresponds to the first strain is greater than 1 ($R_0^1 = 2.77 > 1$), the basic reproduction number corresponding to the second strain is less than or equal to 1 ($R_0^2 = 0.60 \leq 1$). We notice that the components representing the first strain stay at a strictly positive level and the others components representing the second strain vanish. This numerical result coincides with our results mentioned in Theorem 7 concerning the stability of the strain 1 endemic equilibrium $\mathcal{E}_{S_1}$.

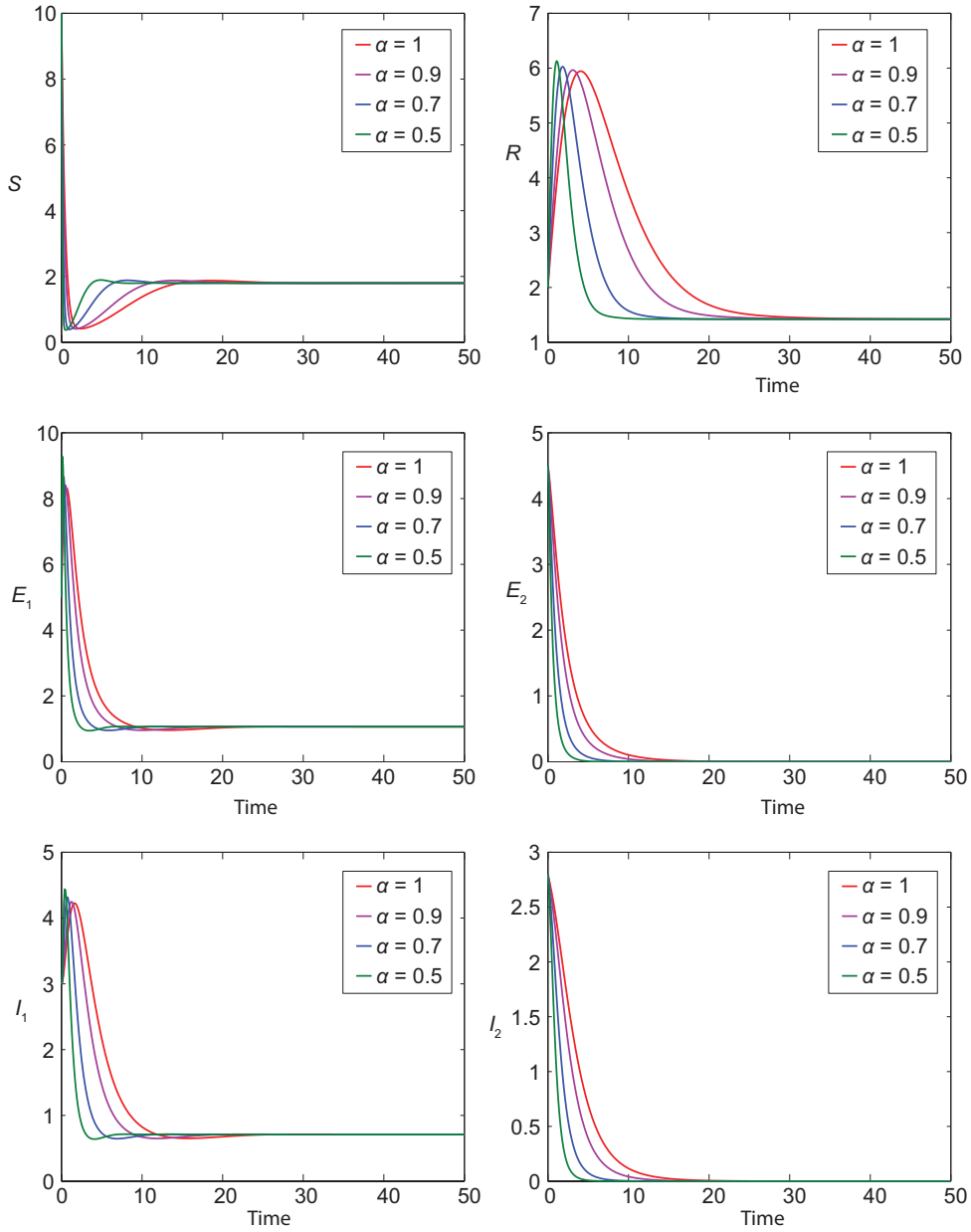

Figure 3: Behavior of the infection as function of time for $\Lambda = 1$, $\beta_1 = 0.15$, $\beta_2 = 0.12$, $\delta_1 = 0.4$, $\delta_2 = 0.3$, $\gamma_1 = 0.4$, $\gamma_2 = 0.4$, $\mu = 0.2$ and $h = 0.1$, which corresponds to the strain-1 endemic equilibrium point $\mathcal{E}_{S_1}$.

Figure 4 depicts the dynamics of the infection for the strain-2 endemic equilibrium $\mathcal{E}_{S_2}$, this figure shows that all the curves converge toward $\mathcal{E}_{S_2}$ = (1.5416, 0, 1.3833, 0, 1.1217, 0.9543), with the basic reproduction number that corresponds to the first strain is less or equal to 1 ($R_0^1 = 0.85 \leq 1$) and the basic reproduction number corresponding to the second strain is greater than 1 ($R_0^2 = 3.24 > 1$). The same observation as the previous figure, we remark that the elements corresponding to the second strain stay at a strictly positive level and the others representing the first strain vanish. This coincides with our results mentioned in Theorem 8 concerning the global stability of the strain 2 endemic equilibrium $\mathcal{E}_{S_2}$.

Finally, the last Figure 5 shows that all the curves converge toward the total endemic equilibrium $\mathcal{E}_{S_t}$ = (1.1100, 0.6757, 0.6210, 0.7305, 0.6713, 1.1924), with the basic reproduction numbers are

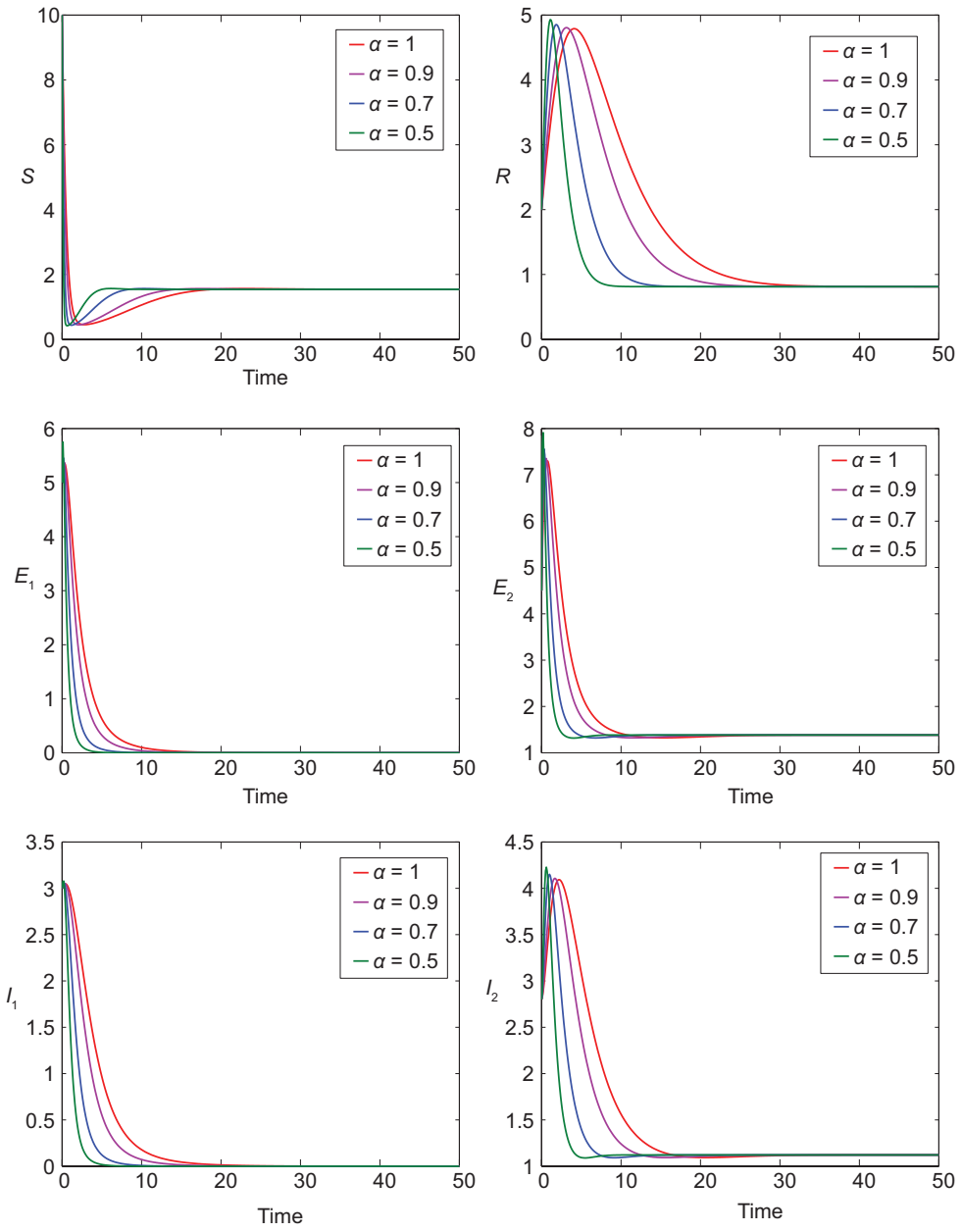

Figure 4: Behavior of the infection as function of time for $\Lambda = 1$, $\beta_1 = 0.17$, $\beta_2 = 0.4$, $\delta_1 = 0.4$, $\delta_2 = 0.3$, $\gamma_1 = 0.4$, $\gamma_2 = 0.17$, $\mu = 0.2$ and $h = 0.1$, which corresponds to the strain-2 endemic equilibrium point $\mathcal{E}_{S_2}$.

equal ($R_0^1 = R_0^2 = 4.5$). We notice in this case that the elements of the two strains stay at a strictly positive level, which coincides with the theoretical results obtained in Theorem 9 concerning the global stability of the total endemic equilibrium $\mathcal{E}_{S_t}$.

According to the previous figures, we can notice that for the small values of the derivative order α the model converges very quickly to the steady states. However, for high values of the fractional derivative parameter, we notice that the model converges very slowly toward the steady states, with corresponds to a long memory effect. In addition, the different values of α have no effect on the stability of the steady states but especially on the duration of convergence of the model to its steady states.

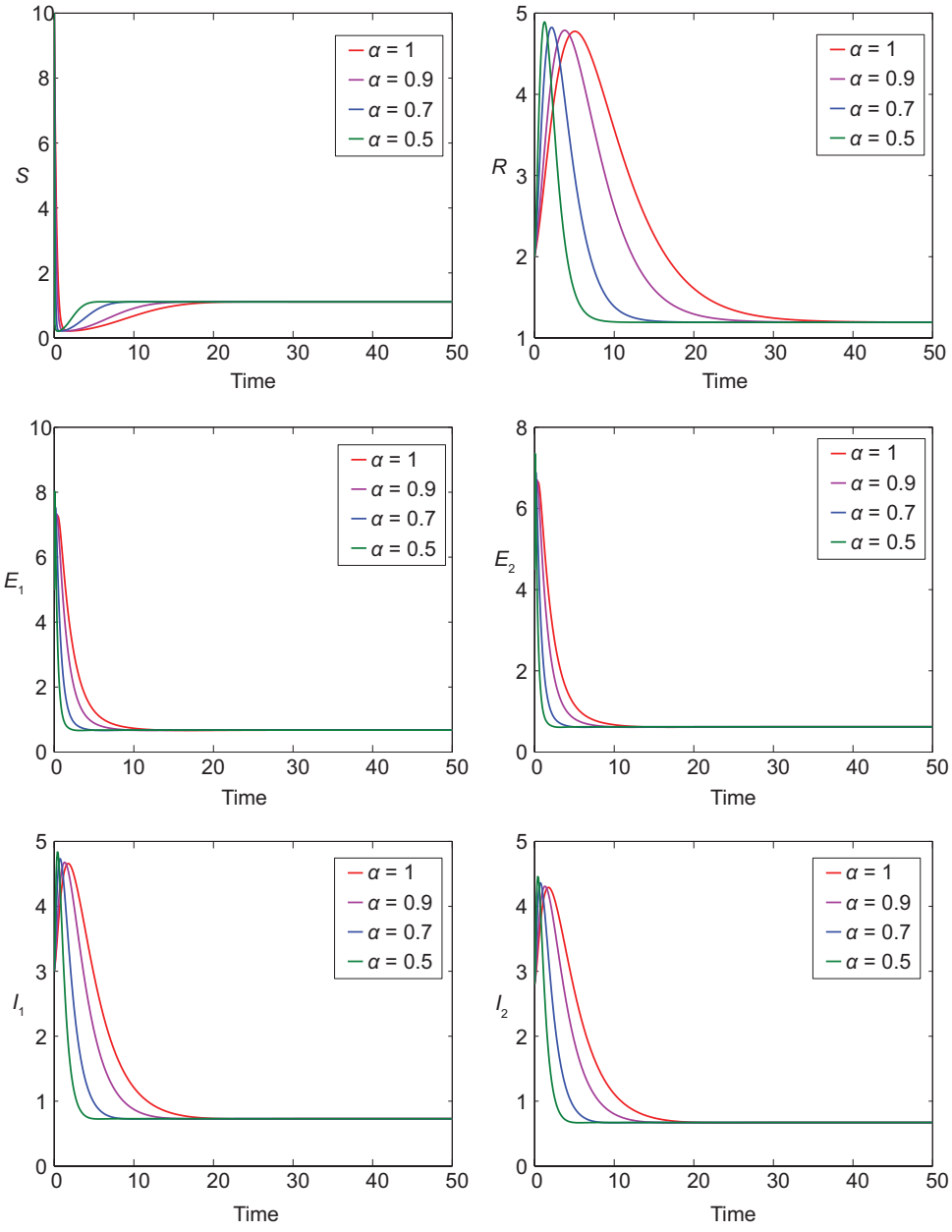

Figure 5: Behavior of the infection as function of time for $\Lambda = 1$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $\delta_1 = 0.4$, $\delta_2 = 0.4$, $\gamma_1 = 0.17$, $\gamma_2 = 0.17$, $\mu = 0.2$ and $h = 0.1$, which corresponds to the total endemic equilibrium point $\mathcal{E}_t$.

6. Conclusion

In this work, we have proposed a fractional SEIR epidemic model with two bilinear incidence rates, we started the analysis of this model by proving the positivity and the boundedness of the solutions. Via the new generation matrix method, we have shown that the model has two basic reproductions numbers R_0^1 and R_0^2, with the first one associated with the strain-1 and the second related to the strain-2. Using the Lyapunov functions method's and La-Salle invariance principle, we have shown the global stability of the steady states, this stability depends only on the two basic reproduction numbers. More precisely, if R_0^1 and R_0^2 are less than 1, the disease-free equilibrium $\mathcal{E}_f$ is globally asymptotically stable; if $R_0^1 \leq 1$ and $R_0^2 > 1$ (respectively $R_0^2 \leq 1$ and $R_0^1 > 1$) then the strain 1 endemic equilibrium $\mathcal{E}_{S_2}$ is globally asymptotically stable (respectively the strain-2 endemic equilibrium $\mathcal{E}_{S_2}$ is globally asymptotically stable) which means that the elements of the strain that has the large basic

reproduction number stay at a strictly positive level and the other vanish. Finally, we have demonstrated that if $R_0^1 = R_0^2$, then the total endemic equilibrium $\mathcal{E}_{S_t}$ is globally asymptotically stable. A numerical simulation was given to validate our theoretical results and to demonstrate the effect of fractional derivative α on the stability. We conclude from all the figures that the different values of the fractional derivative order have no effect on the stability of the equilibria, but only on the convergence time of the solutions toward the steady states. For the high value of this parameter, which means the long memory, the solutions converge very slowly to the equilibrium points and for the small value of this parameter, which means the short memory, the solutions converge very quickly to the steady states.

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