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Local well-posedness and blow-up solutions of a fourth-order pseudo-parabolic equation

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In this study, the initial and periodic boundary value problem were solved for the following fourth-order pseudo-parabolic equation with gradient non-linearity and pseudo-term

$$u_t - a\Delta u_t - \Delta u + \Delta^2 u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u)$$

where $a \geq 0$. A local existence-uniqueness result for mild solutions was found for any initial data in $L^2(\Omega)$. In addition, the existence of blow-up solutions was proved and a lower bound for the blow-up time was obtained.

Key words and phrases: Fourth-order pseudo-parabolic equation, Gradient non-linearity, Existence-uniqueness, Blow-up and Lower blow-up time.

Mathematics Subject Classification (2020): 35A01, 35A02, 35B44

1. Introduction

In this paper, the following fourth-order pseudo-parabolic equation was solved:

$$u_t - a\Delta u_t - \Delta u + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad (x,t) \in \Omega \times (0,T) \tag{1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad u_0 \in L^2(\Omega), \quad x \in \Omega, \tag{2}$$

and periodic boundary conditions

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$$\forall x \in \Gamma_i,\ 0 < t < T,\quad i = 1,2,\quad u(x,t) = u(x + L_i e_i, t), \tag{3}$$

$$u_{x_i}(x,t) = u_{x_i}(x + L_i e_i, t), \tag{4}$$

where

$$\Omega = (0, L_1) \times (0, L_2),\quad \Gamma_i = \partial\Omega \cap \{x_i = 0\}.$$

Here, $\partial\Omega$ is the boundary of $\Omega, a \geq 0$, and $p > 2$ cases were considered. For $u_0 \not\equiv 0$, it is assumed that

$$\int_\Omega u_0(x) = 0. \tag{5}$$

It is clear from equation (1) and condition (5) that $\frac{d}{dt}\int_\Omega u dx = 0$, which means that the average zero periodic initial value functions produce average zero periodic solutions. In this study, the aim is to analyze the effect of the pseudo-parabolic term $-a\Delta u_t$ and the diffusion term $-\Delta u$ in equation (1). When $a = 0$, equation (1) turns into a form of thin-film equations:

$$u_t + A_1\Delta u + A_2\Delta^2 u + A_3\nabla\cdot(|\nabla u|^2\,\nabla u) + A_4\Delta|\nabla u|^2 = 0, \tag{6}$$

where $u(x,t)$ and $A_1\Delta u$ denote the height of a film in epitaxial growth and the diffusion due to evaporation condensation, respectively, the terms $A_2\Delta^2 u$ and $A_3\nabla\cdot(|\nabla u|^2\,\nabla u)$ are the capillarity-driven surface diffusion and atomic displacements, respectively, and the term $A_4\Delta|\nabla u|^2$ describes the motion of an atom to a neighbor effects. For a detailed description of this model, please see [13]. Thin film equations are long-standing topics of research, as shown in [7, 13, 14, 16, 21, 23, 24].

Blow-up solutions for the nonlinear parabolic initial-boundary value problems have been studied by many researchers; please see [7, 9–12, 17–22]. For the fourth-order nonlinear parabolic equations, see the articles: [7, 8, 20, 29]. In [7], Feng and Xu studied the problem:

$$u_t + (-\Delta)^2 u = -\nabla\cdot(|\nabla u|^{p-2}\,\nabla u) \tag{7}$$

with the initial condition

$$u(x,0) = u_0(x), u_0 \in L^2(\Omega), x \in \Omega, u \equiv 0 \tag{8}$$

on a two-dimensional torus. They derived an existence-uniqueness result when $2 < p > 3$. Moreover, they obtained a result for the existence of finite-time blow-up solutions for (7). In [29], Zhou derived a blow-up result for the initial boundary value problem for fourth-order reaction-diffusion equation with a non-local source term under the assumption that the initial energy is positive. In [20], Philippin studied the existence-uniqueness of solutions of the following initial-boundary value problem:

$$\begin{aligned}
&u_t + \Delta^2 u = k(x)|u|^{p-1}u, x \in \Omega, 0 < t < T,\\
&u(x,t) = 0, \frac{\partial u}{\partial n} = 0, for\ x \in \partial\Omega, 0 < t < T,\\
&u(x,0) = u_0(x), x \in \Omega, \Omega \in \mathbb{R}^n, n \geq 2, and\ k(x) > 0.
\end{aligned}$$

In addition, this work obtained a blow-up result and a lower bound for the blow-up time.
The effect of the pseudo-parabolic term was studied by many researchers; please see [4, 15, 25–28, 30]. In [25], Showalter and Ting constructed connections between pseudo-parabolic and parabolic equations for a mixed boundary value problem for the partial differential equation.

$$u_t - \eta\Delta u_t = k\Delta u, \tag{9}$$

where the cases $\eta = 0$ and $\eta \neq 0$ were considered. For $\eta = 0$, this reduces to

$$u_t = k\Delta u. \tag{10}$$

They showed the existence-uniqueness and regularity of their solution. Moreover, they found that the solution continuously depended on η, and when η approached to zero, the solution of (9) converged to the solution of (10).

The rest of this paper is organized as follows: in the next section, the required phase spaces and some preliminaries are given. In Section 3, the local existence-uniqueness of a mild solution of (1)-(2) is obtained. In Section 4, the blow-up solution and a lower bound for the blow-up time are obtained.

2. Notations and Preliminaries

In this study, $L^2(\Omega)$ is specified in the regular and periodical Hilbert spaces as given by

$$H^2(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \; for \, |\, \alpha \,| \leq 2\},$$
$$\dot{H}^2_{per}(\Omega) := \{u \in H^2_{per}(\Omega) : \int_\Omega u \, dx = 0\},$$

respectively. The inner product and its norm are given as

$$(u,v) = \int_0^a u(x)v(x)dx, \quad \|u\|^2 = \int_\Omega u^2 dx.$$

The pair $(\dot{H}^2(\Omega), \|.\|_{\dot{H}^2(\Omega)})$ denotes Hilbert space with the inner product of gradients:

$$(u,v) := \int_\Omega \nabla u . \nabla u \, dx + \int_\Omega \Delta u \Delta v \, dx,$$

and the norm

$$\| u \|^2_{H^2(\Omega)} := \| \nabla u \|^2 + \| \Delta u \|^2 .$$

In addition, $L^q(\Omega)$ is defined as the Lebesque space with the norm $\| u \|_q = \left(\int_\Omega |\, u \,|^q \, dx \right)^{\frac{1}{q}}$.

The Sobolev Space $W^{s,q}(\Omega)$ for $1 \leq q < \infty$ is defined as the subset of functions *f* in $L^q(\Omega)$ such that *f* and its weak derivatives up to an integer order *s* are in L^q. That is,

$$W^{s,q}(\Omega) = \left\{ u : u \in L^q(\Omega), u_x \in L^q(\Omega), \ldots, \frac{\partial^s u}{\partial x^s} \in L^q(\Omega), \right.$$
$$\left. \left(\int_\Omega \left[|\, u \,|^q + |\, u_x \,|^q + \ldots + |\, \frac{\partial^s u^q}{\partial x^s} dx \,| \right] \right)^{\frac{1}{q}} < \infty \right\}.$$

For the case $q \to \infty$, the norm is defined by

$$\| f \|_{k,\infty} = \max_{i=0,\ldots,k} \| f^{(i)} \|_\infty = \max_{i=0,\ldots,k} \left(ess \sup_t |\, f^{(i)}(t) \,| \right).$$

In the inequality above, *ess*sup means essential supremum. This allows us to generalize the maximum of a function in a useful way. Let $f \in L^q(\Omega)$ with $q \geq 1$ and $\hat{f}(k)$ be The Fourier transform of *f*(*k*) at the frequencies $k \in \mathbb{Z}^2$ is given by

$$\widehat{F}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx. \tag{11}$$

Moreover, when $q = 2$ and $s \in \mathbb{R}$, the Sobolev space by $H^s(\Omega)$ and the homogeneous Sobolev space by $\dot{H}^s(\Omega)$ are denoted by the norms

$$\| f \|^2_{H^s} = \sum_{k \in \mathbb{Z}^2} (1 + |\, k \,|^2)^s \, |\, \hat{f}(k) \,|^2 = \| (I - \Delta)^{s/2} f \|^2_{L^2},$$
$$\| f \|^2_{\dot{H}^s} = \sum_{k \in \mathbb{Z}^2} (|\, k \,|^2)^s \, |\, \hat{f}(k) \,|^2 = \| (-\Delta)^{s/2} f \|^2_{L^2},$$

respectively. Here, *I* is the identity operator and $(-\Delta)^{s/2}$ shows Fourier multiplier with symbol $|\, k \,|^s$, $k \neq 0$.

The strongly continuous semi-group of operators, which are generated by $\mathcal{L} = (I - a\Delta)^{-1}(-\Delta + \Delta^2)$ on L^2, are defined by

$$e^{-t\mathcal{L}} f = F^{-1}\left(te^{-t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} f \right),$$

where F^{-1} denotes the inverse Fourier transform on Ω.

Definition 2.1 *A function* $u : C([0,T]; L^2(\Omega))$ *for* $0 \le T < \infty$ *is called a mild solution of (1)-(2) on* $[0,T]$ *with initial data* $u_0 \in L^2(\Omega)$, *if it satisfies*

$$\eta(u)(t) = u(t) = e^{-t\mathcal{L}} u_0 - \int_0^t e^{-(t-s)\mathcal{L}} (I - a\Delta)^{-1} (\nabla \cdot (|\nabla u|^{p-2} \nabla u)) ds \tag{12}$$

for $0 \le t < T$.

In order to be able to define the non-linearity that needs ∇u to be locally integrable, the following Banach spaces are defined for

$u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$,

$$E_s = \begin{cases} \sup\limits_{0 \le t \le T} t^{\frac{1}{2}} \| \nabla u \|_2 < \infty, & \text{if } a \ge 1, \\ \sup\limits_{0 \le t \le T} t^{\frac{1}{4}} \| \nabla u \|_2 < \infty, & \text{if } 0 \le a < 1, \end{cases}$$

and

$$\overline{E}_S = C([0,T]; L^2(\Omega) \cap E_S),$$

with the norm

$$\| u \|_{\overline{E}_S} = \begin{cases} \max\left(\sup\limits_{0 \le t \le T} \| u \|_2, \sup\limits_{0 \le t \le T} t^{\frac{1}{2}} \| \nabla u \|_2 \right), & \text{if } a \ge 1, \\ \max\left(\sup\limits_{0 \le t \le T} \| u \|_2, \sup\limits_{0 \le t \le T} t^{\frac{1}{4}} \| \nabla u \|_2 \right), & \text{if } 0 \le a < 1. \end{cases}$$

Lemma 2.2. *There exist positive constants* C^1 *and* C^2 *such that*

$$\| e^{-t\mathcal{L}} (I - a\Delta)^{-1} (\nabla \cdot f) \|_2 \le \begin{cases} C^1 t^{-\frac{1}{4}} \| f \|_1 & \text{if } a \ge 1, \\ \dfrac{C^2}{(1+a)} t^{-\frac{1}{2}} \| f \|_1 & \text{if } 0 \le a < 1. \end{cases}$$

Proof. By using Plancherel's identity and the definition of the operator $e^{-t\mathcal{L}}$, one finds

$$\begin{aligned} \| e^{-t\mathcal{L}} (I - a\Delta)^{-1} (\nabla \cdot f) \|_2^2 &= \sum_{k \in \mathbb{Z}^2} \frac{|k|^2}{(1 + a|k|^2)^2} e^{-2t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} |\hat{f}(k)|^2 \\ &\le \sup_{k \in \mathbb{Z}^2} |\hat{f}|(k)|^2 \sum_{k \in \mathbb{Z}^2} \frac{|k|^2}{(1 + a|k|^2)^2} e^{-2t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} \\ &\le \| f \|_1^2 \int_{\mathbb{R}^2} \left(\frac{|x|^2}{(1 + a|x|^2)^2} e^{-2t|x|^2 \frac{1+|x|^2}{1+a|x|^2}} \right) dx. \end{aligned} \tag{13}$$

The above integral in polar coordinates is split into two parts:

$$\begin{aligned} I: &= \int_{\mathbb{R}^2}\left(\frac{|x|^2}{(1+a|x|^2)^2} e^{-2t|x|^2\frac{1+|x|^2}{1+a|x|^2}}\right)dx \\ &= \int_0^{2\pi}\int_0^{\infty}\left(\frac{r^2}{(1+ar^2)^2} e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdrd\theta \\ &= 2\pi\int_0^{\infty}\left(\frac{r^2}{(1+ar^2)^2} e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdr \\ &= 2\pi\int_0^{1}\left(\frac{r^2}{(1+ar^2)^2} e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdr + 2\pi\int_1^{\infty}\left(\frac{r^2}{(1+ar^2)^2} e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdr. \end{aligned} \tag{14}$$

In (14), the first integral is proper and it converges to a positive number, C_1. The second integral can be written as

$$I_2 = 2\pi\int_1^{\infty}\left(\frac{r^2 r}{(1+ar^2)^2} e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)dr.$$

Since $\frac{r^2(1+r^2)}{1+ar^2} \geq \frac{r^2}{a}$ for $a \geq 1$, and for $0 < t \leq 1$,

$$I_2 \leq 2\pi\int_1^{\infty}\left(\frac{r}{a^2r^2} e^{-2t\frac{r^2}{a}}\right)dr = \frac{2\pi}{a^2}\int_1^{\infty}\left(\frac{\sqrt{r}\sqrt{r}}{r^2} e^{-2t\frac{r^2}{a}}\right)dr. \tag{15}$$

Applying Hölder inequality to the right side of (15), we obtain

$$\frac{2\pi}{a^2}\int_1^{\infty}\left(\frac{\sqrt{r}\sqrt{r}}{r^2} e^{-2t\frac{r^2}{a}}\right)dr \leq \frac{2\pi}{a^2}\left(\int_1^{\infty} re^{-4t\frac{r^2}{a}}dr\right)^{\frac{1}{2}}\left(\int_1^{\infty}\frac{r}{r^4}dr\right)^{\frac{1}{2}}. \tag{16}$$

By computing the above integrals, the following is obtained:

$$I_2 \leq C_2 t^{-\frac{1}{2}} e^{-\frac{2t}{a}} \leq C_2 t^{-\frac{1}{2}} \quad for \quad C_2 = \frac{\pi}{2a^{\frac{3}{2}}}, \tag{17}$$

and

$$I \leq C_1 + C_2 t^{-\frac{1}{2}}.$$

For $C^1 = max\{C_1, C_2\}$, one gets

$$I \leq C^1(1+t^{-\frac{1}{2}}). \tag{18}$$

If equation (18) is used in (13), then for $0 < t \leq 1$ one gets

$$\| e^{-t\mathcal{L}}(I-a\Delta)^{-1}(\nabla.f)\|_2^2 \leq C^1(1+t^{-\frac{1}{2}})\|f\|_1^2 \leq C^1 t^{-\frac{1}{2}}\|f\|_1^2.$$

In addition, $0 \leq a < 1$, $\frac{r^2+r^4}{1+ar^2} > r^2$ and $0 < t \leq 1$

$$I_2 \leq 2\pi\int_1^{\infty} e^{-2tr^2}\frac{r^2}{(1+ar^2)^2}rdr.$$

Taking $s = 1 + ar^2$, the right-hand side turns into

$$\frac{\pi}{a^2} e^{\frac{2t}{a}} \int_{1+a}^{\infty} e^{-\frac{2ts}{a}} \frac{s-1}{s^2} ds = \frac{\pi}{2t} e^{-2t} \frac{1}{(1+a)^2} \leq \frac{C^2}{(1+a)^2 t},$$

where $C^2 = \frac{\pi}{2}$. Hence,

$$\| e^{-t\mathcal{L}} (I - a\Delta)^{-1} (\nabla . f) \|_2^2 \leq \frac{C^2}{(1+a)^2} t^{-1} \| f \|_1^2.$$

Lemma 2.3. *For any $s \geq 0$, there exists constants C^3 and C^4 such that*

$$\| (-\Delta)^{-\frac{s}{2}} e^{-t\mathcal{L}} f \|_2 \leq \begin{cases} C^3 t^{-\frac{s}{2}} \| f \|_2 & \text{if } a \geq 1, \\ C^4 t^{-\frac{s}{4}} \| f \|_2 & \text{if } 0 \leq a < 1. \end{cases}$$

Proof. By using Plancherel's identity and the definition of the operator $e^{-t\mathcal{L}}$ for $a \geq 1$, one can write:

$$\| (-\Delta)^{\frac{s}{2}} e^{-t\mathcal{L}} f \|_2^2 = \sum_{k \in \mathbb{Z}^2} | k |^{2s} e^{-2t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} | \hat{f}(k) |^2 \leq \sum_{k \in \mathbb{Z}^2} | k |^{2s} e^{-2t\frac{|k|^2}{a}} | \hat{f}(k) |^2$$

$$\leq C^3 \left(\sup_{x \in \mathbb{R}^+} x^{2s} e^{-t\frac{x^2}{a}} \right) \sum_{k \in \mathbb{Z}^2} | \hat{f}(x) |^2 \leq C^3 t^{-s} \| f \|_2^2.$$

For $0 \leq a < 1$, one can write:

$$\| (-\Delta)^{\frac{s}{2}} e^{-t\mathcal{L}} f \|_2^2 = \sum_{k \in \mathbb{Z}^2} | k |^{2s} e^{-2t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} | \hat{f}(k) |^2 \leq \sum_{k \in \mathbb{Z}^2} | k |^{2s} e^{-2ta|k|^4} | \hat{f}(k) |^2$$

$$\leq C^4 \left(\sup_{x \in \mathbb{R}^+} x^{2s} e^{-tax^4} \right) \sum_{k \in \mathbb{Z}^2} | \hat{f}(x) |^2 \leq C^4 t^{-\frac{s}{2}} \| f \|_2^2.$$

3. Local Existence and Uniqueness

In this study, the following lemmas were used, which are crucial in proving the existence-uniqueness of the mild solution:

Lemma 3.1. ***(i)*** *For $2 < p < \frac{5}{2}, a \geq 1,$ and $0 < T \leq 1,$ there exists a positive constant C_1 such that the operator $\eta : \overline{E}_s \to \overline{E}_s$ satisfies*

$$\| \eta(u) \|_{\overline{E}_S} \leq C_1 \left(\| u_0 \|_2 + T^{\frac{5-2p}{4}} \| u \|_{\overline{E}_S}^{p-1} \right). \tag{19}$$

(ii) *For $2 < p < 3, 0 \leq a < 1,$ and $0 < T \leq 1,$ there exists a positive constant $\widehat{C}_1$ such that the operator $\eta : \overline{E}_s \to \overline{E}_s$ satisfies*

$$\| \eta(u) \|_{\overline{E}_S} \leq \widehat{C}_1 \left(\| u_0 \|_2 + (1+a)^{-1} T^{\frac{3-p}{4}} \| u \|_{\overline{E}_S}^{p-1} \right). \tag{20}$$

Proof. To prove **(i)** in this lemma, it is sufficient to show the following two assertions hold:

Assertion 1 If $u \in \overline{E}_S$, *then* $\eta(u) \in C([0,T]; L^2(\Omega))$;

Assertion 2 If $u \in \overline{E}_S$, then $\sup_{0\le t\le T} t^{\frac{1}{2}} \| \nabla(\eta(u)) \|_2 \le \infty$.

In the rest of the computations, we shall use T to denote the operator $(I - a\Delta)^{-1}$. In the proofs of these two inequalities, the equality $s = t\xi$ will be used when needed. To prove Assertion 1, Lemma 2.2 and Lemma 2.3 are used with $2 < p < \frac{5}{2}$ as follows:

$$
\begin{aligned}
\| \eta(u) \|_2 &\le C\left(\| u_0 \|_2 + \int_0^t \| e^{-(s-t)\mathcal{L}} T \nabla \cdot (|\nabla u|^{p-2} \nabla u) \|_2 ds \right) \\
&\le C\left(\| u_0 \|_2 + \int_0^t (t-s)^{-\frac{1}{4}} \| (|\nabla u|^{p-2} \nabla u) \|_1 ds \right) \\
&\le C\left(\| u_0 \|_2 + \int_0^t (t-s)^{-\frac{1}{4}} \| \nabla u \|_2^{p-1} ds \right) \\
&\le C\left(\| u_0 \|_2 + \int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{(p-1)}{2}} (s^{\frac{1}{2}} \| \nabla u \|_2)^{p-1} ds \right) \\
&\le C\left(\| u_0 \|_2 + t^{\frac{5-2p}{4}} \int_0^1 (1-\xi)^{-\frac{1}{4}} \xi^{-\frac{(p-1)}{2}} \| u \|_{E_S}^{p-1} d\xi \right) \\
&\le C\left(\| u_0 \|_2 + t^{\frac{5-2p}{4}} \| u \|_{E_S}^{p-1} \right).
\end{aligned}
$$

Similarly, for Assertion 2,

$$
\begin{aligned}
\| \nabla \eta(u)(t) \|_2 &\le \| \nabla e^{-t\mathcal{L}} u_0 \|_2 + \int_0^t \| \nabla e^{-\frac{(t-s)}{2}\mathcal{L}} \|_{2\to 2} \| e^{-\frac{(t-s)}{2}\mathcal{L}} T(\nabla \cdot (|\nabla u|^{p-2} \nabla u)) \|_2 ds \\
&\le C t^{\frac{-1}{2}} \| u_0 \|_2 + C\int_0^t (t-s)^{-\frac{1}{2}} (t-s)^{-\frac{1}{4}} \| |\nabla u|^{p-2} \nabla u) \|_1 ds \\
&\le C t^{\frac{-1}{2}} \| u_0 \|_2 + C\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{(p-1)}{2}} (s^{\frac{1}{2}} \| \nabla u \|_2)^{p-1} ds \\
&\le C t^{\frac{-1}{2}} \| u_0 \|_2 + C t^{\frac{3-2p}{4}} \int_0^1 (1-\xi)^{-\frac{3}{4}} \xi^{-\frac{(p-1)}{2}} \| u \|_{E_S}^{p-1} d\xi \\
&\le C\left(t^{\frac{-1}{2}} \| u_0 \|_2 + t^{\frac{3-2p}{4}} \| u \|_{E_S}^{p-1} \right).
\end{aligned} \tag{21}
$$

Multiplying the inequality (21) by $t^{\frac{1}{2}}$, one obtains

$$
t^{\frac{1}{2}} \| \nabla \eta(u)(t) \|_2 \le C\left(\| u_0 \|_2 + t^{\frac{5-2p}{4}} \| u \|_{E_S}^{p-1} \right),
$$

for any $t \in [0,T]$. Combining these two assertions, the proof of **(i)** is completed where $\eta : \overline{E}_s \to \overline{E}_s$ is a bounded operator.

Similarly, to prove the second part of this lemma, it is sufficient to show the following two assertions hold:

Assertion 1 If $u \in \overline{E}_S$, *then* $\eta(u) \in C([0,T]; L^2(\Omega))$;

Assertion 2 If $u \in \overline{E}_S$, then $\sup_{0\le t\le T} t^{\frac{1}{4}} \| \nabla(\eta(u)) \|_2 \le \infty$.

In the proofs of these two inequalities, the equality $s = t\xi$ will be used when needed. To prove Assertion 1, Lemma 2.2 and Lemma 2.3 are used with $2 < p < 3$,

$$\begin{aligned}\|\eta(u)\|_2 &\le C\left(\|u_0\|_2+\int_0^t \|e^{-(s-t)\mathcal{L}}T\nabla\cdot(|\nabla u|^{p-2}\nabla u)\|_2 ds\right)\\ &\le C\left(\|u_0\|_2+(1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{2}}\|(|\nabla u|^{p-2}\nabla u)\|_1 ds\right)\\ &\le C\left(\|u_0\|_2+(1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla u\|_2^{p-1} ds\right)\\ &\le C\left(\|u_0\|_2+(1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{(p-1)}{4}}(s^{\frac{1}{4}}\|\nabla u\|_2)^{p-1} ds\right)\\ &\le C\left(\|u_0\|_2+(1+a)^{-1}t^{\frac{3-p}{2}}\int_0^1 (1-\xi)^{-\frac{1}{2}}\xi^{-\frac{(p-1)}{4}}\|u\|_{\overline{E}_S}^{p-1} d\xi\right)\\ &\le C\left(\|u_0\|_2+(1+a)^{-1}t^{\frac{3-p}{4}}\|u\|_{\overline{E}_S}^{p-1}\right).\end{aligned}$$

Similarly, for Assertion 2,

$$\begin{aligned}\|\nabla\eta(u)(t)\|_2 &\le \|\nabla e^{-t\mathcal{L}}u_0\|_2+\int_0^t \|\nabla e^{-\frac{(t-s)}{2}\mathcal{L}}\|_2\|e^{-\frac{(t-s)}{2}\mathcal{L}}T(\nabla\cdot(|\nabla u|^{p-2}\nabla u))\|_2 ds\\ &\le Ct^{-\frac{1}{4}}\|u_0\|_2+C(1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{4}}(t-s)^{-\frac{1}{2}}\||\nabla u|^{p-2}\nabla u)\|_1 ds\\ &\le Ct^{-\frac{1}{4}}\|u_0\|_2+C(1+a)^{-1}\int_0^t (t-s)^{-\frac{3}{4}}s^{-\frac{(p-1)}{4}}(s^{\frac{1}{4}}\|\nabla u\|_2)^{p-1} ds\\ &\le Ct^{-\frac{1}{4}}\|u_0\|_2+C(1+a)^{-1}t^{\frac{2-p}{4}}\int_0^1 (1-\xi)^{-\frac{3}{4}}\xi^{-\frac{(p-1)}{4}}\|u\|_{\overline{E}_S}^{p-1} d\xi\\ &\le C\left(t^{-\frac{1}{4}}\|u_0\|_2+(1+a)^{-1}t^{\frac{2-p}{4}}\|u\|_{\overline{E}_S}^{p-1}\right).\end{aligned} \tag{22}$$

Multiplying the inequality (22) by $t^{\frac{1}{4}}$, one gets

$$t^{\frac{1}{4}}\|\nabla\eta(u)(t)\|_2 \le C\left(\|u_0\|_2+(1+a)^{-1}t^{\frac{3-p}{4}}\|u\|_{\overline{E}_S}^{p-1}\right),$$

for any $t\in[0,T]$. Combining these two assertions, the proof of **(ii)** is completed. This means $\eta:\overline{E}_s\to\overline{E}_s$ is a bounded operator.

Lemma 3.2. ***(i)*** *For* $2<p<\frac{5}{2}, a\ge 1$, *and* $0<T\le 1$, *there exists a constant* C_2 *such that the operator* $\eta:\overline{E}_s\to\overline{E}_s$ *is a Lipschitz continuous map. That is, it satisfies:*

$$\|\eta(u_1)-\eta(u_2)\|_{\overline{E}_S}\le C_2T^{\frac{5-2p}{4}}(\|u_1\|_{\overline{E}_S}^{p-2}+\|u_2\|_{\overline{E}_S}^{p-2})\|u_1-u_2\|_{\overline{E}_S}. \tag{23}$$

(ii) *For* $2<p<3, 0\le a<1$, *and* $0<T\le 1$, *there exists a constant* $\widehat{C}_2$ *such that the operator* $\eta:\overline{E}_s\to\overline{E}_s$ *is a Lipschitz continuous map. That is, it satisfies:*

$$\|\eta(u_1)-\eta(u_2)\|_{\overline{E}_S}\le \widehat{C}_2T^{\frac{3-p}{4}}(\|u_1\|_{\overline{E}_S}^{p-2}+\|u_2\|_{\overline{E}_S}^{p-2})\|u_1-u_2\|_{\overline{E}_S}. \tag{24}$$

Proof. To prove **(i)**, it is sufficient to show the following two inequalities:

$$\mathbf{I}_1 \quad \| \eta(u_1) - \eta(u_2) \|_2 \le C t^{\frac{5-2p}{4}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S};$$

$$\mathbf{I}_2 \quad \sup_{t\in[0,T]} t^{\frac{1}{2}} \| \nabla\eta(u_1) - \nabla\eta(u_2) \|_2 \le C t^{\frac{5-2p}{4}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S}.$$

In the proofs of these two inequalities, the equality $s = t\xi$ will be used when needed. To prove $\mathbf{I}_1$, Lemma 2.2 and Lemma 2.3 are used with $2 < p < \frac{5}{2}$,

$$\begin{aligned}
\| \eta(u_1) - \eta(u_2) \|_2 &\le C \int_0^t (t-s)^{-\frac{1}{4}} \| | \nabla u_1 |^{p-2} \nabla u_1 - | \nabla u_2 |^{p-2} \nabla u_2 \|_1 ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{4}} \| | \nabla u_1 - \nabla u_2 | (| \nabla u_1 |^{p-2} + | \nabla u_2 |^{p-2}) \|_1 ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{4}} \| \nabla u_1 - \nabla u_2 \|_2 (\| | \nabla u_1 |^{p-2} \|_2 + \| | \nabla u_2 |^{p-2} \|_2) ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{1}{2}} \| u_1 - u_2 \|_{\overline{E}_S} s^{-\frac{(p-2)}{2}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) ds \\
&\le C t^{\frac{5-2p}{4}} \int_0^1 (1-\xi)^{-\frac{1}{4}} \xi^{\frac{2-p}{2}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S} d\xi, \\
&\le C t^{\frac{5-2p}{4}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S}.
\end{aligned}$$

Similarly, for $\mathbf{I}_2$, one can write

$$\begin{aligned}
\| \nabla\eta(u_1) - \nabla\eta(u_2) \|_2 &\le \int_0^t \| \nabla e^{-\frac{(t-s)}{2}\mathcal{L}} \|_2 \| e^{-\frac{(t-s)}{2}\mathcal{L}} T(\nabla \cdot (| \nabla u_1 |^{p-2} \nabla u_1 - | \nabla u_2 |^{p-2}) \nabla u_2) \|_2 ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{2}} (t-s)^{-\frac{1}{4}} \| | \nabla u_1 |^{p-2} \nabla u_1 - | \nabla u_2 |^{p-2} \nabla u_2 \|_1 ds \\
&\le C \int_0^t (t-s)^{-\frac{3}{4}} \| \nabla u_1 - \nabla u_2 \|_2 (\| | \nabla u_1 |^{p-2} \|_2 + \| | \nabla u_2 |^{p-2} \|_2) ds \\
&\le C \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} \| u_1 - u_2 \|_{\overline{E}_S} s^{-\frac{(p-2)}{2}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) ds \\
&\le C t^{\frac{3-2p}{4}} \int_0^1 (1-\xi)^{-\frac{3}{4}} \xi^{\frac{1-p}{2}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S} d\xi, \\
&\le C t^{\frac{3-2p}{4}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S}.
\end{aligned} \tag{25}$$

Multiplying the inequality (25) by $t^{\frac{1}{2}}$, one gets

$$t^{\frac{1}{2}} \| \nabla\eta(u_1) - \nabla\eta(u_2) \|_2 \le C t^{\frac{5-2p}{4}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S},$$

for any $t \in [0,T]$. Combining the above two inequalities, the proof of **(i)** is completed. That is, $\eta : \overline{E}_s \to \overline{E}_s$ is a Lipschitz continuous map.

To prove **(ii)**, similar to the previous inequality, it is sufficient to show the following two inequalities hold:

$$\mathbf{I}_3 \quad \| \eta(u_1) - \eta(u_2) \|_2 \le C(1+a)^{-1} t^{\frac{3-p}{4}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S};$$

$$\mathbf{I}_4 \quad \sup_{t\in[0,T]} t^{\frac{1}{2}} \| \nabla\eta(u_1) - \nabla\eta(u_2) \|_2 \le C(1+a)^{-1} t^{\frac{3-p}{4}} (\| u_1 \|_{\overline{E}_S}^{p-2} + \| u_2 \|_{\overline{E}_S}^{p-2}) \| u_1 - u_2 \|_{\overline{E}_S}.$$

In the proofs of these two inequalities, the equality $s=t\xi$ will be used when needed. To prove $\mathbf{I}_3$, Lemma 2.2 and Lemma 2.3 are used and $2<p<3$ is assumed:

$$\begin{aligned}
\|\eta(u_1)-\eta(u_2)\|_2 &\le C^*\int_0^t (t-s)^{-\frac{1}{2}} \| |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\|_1\, ds \\
&\le C^*\int_0^t (t-s)^{-\frac{1}{2}} \| |\nabla u_1-\nabla u_2| \left(|\nabla u_1|^{p-2}+|\nabla u_2|^{p-2}\right)\|_1\, ds \\
&\le C^*\int_0^t (t-s)^{-\frac{1}{2}} \|\nabla u_1-\nabla u_2\|_2 \left(\| |\nabla u_1|^{p-2}\|_2+\| |\nabla u_2|^{p-2}\|_2\right) ds \\
&\le C^*\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{4}} \|u_1-u_2\|_{\bar{E}_S}\, s^{-\frac{(p-2)}{4}} \left(\|u_1\|_{\bar{E}_S}^{p-2}+\|u_2\|_{\bar{E}_S}^{p-2}\right) ds \\
&\le C^* t^{\frac{3-p}{4}} \int_0^1 (1-\xi)^{-\frac{1}{2}} \xi^{\frac{2-p}{4}} \left(\|u_1\|_{\bar{E}_S}^{p-2}+\|u_2\|_{\bar{E}_S}^{p-2}\right) \|u_1-u_2\|_{\bar{E}_S}\, d\xi \\
&\le C(1+a)^{-1} t^{\frac{3-p}{4}} \left(\|u_1\|_{\bar{E}_S}^{p-2}+\|u_2\|_{\bar{E}_S}^{p-2}\right) \|u_1-u_2\|_{\bar{E}_S},
\end{aligned}$$

where $C^*=C(1+a)^{-1}$. Similarly, for $\mathbf{I}_4$, one has

$$\begin{aligned}
\|\nabla\eta(u_1)-\nabla\eta(u_2)\|_2 &\le \int_0^t \|\nabla e^{-\frac{(t-s)}{2}\mathcal{L}}\|_2 \| e^{-\frac{(t-s)}{2}\mathcal{L}} T\left(\nabla\cdot\left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\right)\|_2\, ds \\
&\le C\int_0^t (t-s)^{-\frac{1}{2}} (t-s)^{-\frac{1}{4}} \| |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\|_1\, ds \\
&\le C^*\int_0^t (t-s)^{-\frac{3}{4}} \|\nabla u_1-\nabla u_2\|_2 \left(\| |\nabla u_1|^{p-2}\|_2+\| |\nabla u_2|^{p-2}\|_2\right) ds \\
&\le C^*\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{4}} \|u_1-u_2\|_{\bar{E}_S}\, s^{-\frac{(p-2)}{4}} \left(\|u_1\|_{\bar{E}_S}^{p-2}+\|u_2\|_{\bar{E}_S}^{p-2}\right) ds \\
&\le C^* t^{\frac{2-p}{4}} \int_0^1 (1-\xi)^{-\frac{3}{4}} \xi^{\frac{1-p}{4}} \left(\|u_1\|_{\bar{E}_S}^{p-2}+\|u_2\|_{\bar{E}_S}^{p-2}\right) \|u_1-u_2\|_{\bar{E}_S}\, d\xi \\
&\le C(1+a)^{-1} t^{\frac{2-p}{4}} \left(\|u_1\|_{\bar{E}_S}^{p-2}+\|u_2\|_{\bar{E}_S}^{p-2}\right) \|u_1-u_2\|_{\bar{E}_S}, \qquad (26)
\end{aligned}$$

where $C^*=C(1+a)^{-1}$. Multiplying the inequality (26) with $t^{\frac{1}{4}}$, one gets:

$$t^{\frac{1}{4}} \|\nabla\eta(u_1)-\nabla\eta(u_2)\|_2 \le C(1+a)^{-1} t^{\frac{3-p}{4}} \left(\|u_1\|_{\bar{E}_S}^{p-2}+\|u_2\|_{\bar{E}_S}^{p-2}\right) \|u_1-u_2\|_{\bar{E}_S}$$

for any $t\in[0,T]$. Combining these two inequalities, the proof of **(ii)** is completed. That is, $\eta:\bar{E}_s\to\bar{E}_s$ is a Lipschitz continuous map.

The main result in this study is given below.

Theorem 3.3. ***(i)*** *Let $u_0\in L^2(\Omega)$ such that $a\ge 1$, $\|u_0\|_2\le \dfrac{R}{C_0}$, where $C_0=\max\{C_1,C_2\}$ for the positive constants C_1 and C_2 provided by Lemma 3.1 and Lemma 3.2, and $2<p<\dfrac{5}{2}$. Then there exists $0\le T\le 1$, depending on $\|u_0\|_2$, such that (1)-(2) admits a unique mild solution u on $[0,T]\in\bar{E}_S$.*

(ii) *Let $u_0\in L^2(\Omega)$ such that $0\le a<1$, $\|u_0\|_2\le \dfrac{R}{C_0^*}$, where $C_0^*=\max\{\hat{C}_1,\hat{C}_2\}$ for the positive constants $\hat{C}_1$ and $\hat{C}_2$ provided by Lemma 3.1 and Lemma 3.2, and $2<p<3$. Then there exists $0\le T\le 1$, depending on $\|u_0\|_2$, such that* (1)-(2) *admits a unique mild solution u on $[0,T]\in\bar{E}_S$.*

Proof. **(i)** Let $B_R(0)$ be the closed ball with radius R, and

$$T \le \min\{1,(2C_0 R^{p-2})^{-\frac{4}{5-2p}}\}.$$

Then, Lemma 3.1 suggests that, for every $u \in B_R(0)$,

$$\| \eta(u) \|_{\bar{E}_S} \le R.$$

Moreover, Lemma 3.2 states that, for every $u_1, u_2 \in B_R(0)$ and $l_1 < 1$,

$$\| \eta(u_1) - (u_2) \|_{\bar{E}_S} \le l_1 \| u_1 - u_2 \|_{\bar{E}_S} .$$

By using the Banach Contraction Mapping Theorem, a unique fixed point of η in $B_R(0)$ is obtained. Thus, this fixed point u is the unique mild solution of (1)-(2) with the initial data u_0.

(ii) Let $B_R(0)$ be the closed ball with radius R, and

$$T \le min\{1,((1+a)^{-1} 2C_0^* R^{p-2})^{-\frac{4}{3-p}}\}.$$

Then, Lemma 3.1 implies that, for every $u \in B_R(0)$,

$$\| \eta(u) \|_{\bar{E}_S} \le R.$$

Moreover, Lemma 3.2 denotes that for every $u_1, u_2 \in B_R(0)$ and $l_2 < 1$,

$$\| \eta(u_1) - (u_2) \|_{\bar{E}_S} \le l_2 \| u_1 - u_2 \|_{\bar{E}_S} .$$

By using the Banach Contraction Mapping Theorem, a unique fixed point of η in $B_R(0)$ is obtained. Thus, this fixed point u is the unique mild solution to (1)-(2) with the initial data u_0.

4. The blow-up solutions and an estimate for the lower limit of blow-up time

In this section, the blow-up result for the solutions of (1)-(2) is presented. To achieve this goal, the following functionals are defined:

$$\Phi = \| u \|_2^2 + a \| \nabla u \|_2^2,$$

$$\Psi = \frac{1}{p} \| \nabla u \|_p^p - \frac{1}{2} \| \nabla u \|_2^2 - \frac{1}{2} \| \Delta u \|_2^2 .$$

The main theorem of this study is given as follows:

Theorem 4.1. *Assume that* $p > 2$, $u_0 \in H^2(\Omega)$, $u_0 \neq 0$, *and*

$$\Psi(0) = \frac{1}{p} \| \nabla u_0 \|_p^p - \frac{1}{2} \| \nabla u_0 \|_2^2 - \frac{1}{2} \| \Delta u_0 \|_2^2 .$$

Then for the solution $u(x,t)$ *of* (1)-(2), *there exists some* $T^* > 0$ *such that*

$$\lim_{t \to T^*} \Phi(t) = \infty$$

where

$$T^* = \frac{\Phi^{\frac{2-p}{2}}(0)}{p(p-2)}.$$

Theorem 4.1. was formulated using the ideas of [20].

Proof. Using (1)-(2), one can obtain

$$\Phi'(t) = 2\left[\| \nabla u \|_p^p - \| \nabla u \|_2^2 - \| \Delta u \|_2^2 \right] \tag{27}$$

$$\Psi'(t) = \| u_t \|_2^2 + a \| \nabla u_t \|_2^2 .$$

By Schwarz inequality, one has

$$\Phi(t)\Psi'(t) = \left(\| u \|_2^2 + a \| \nabla u \|_2^2 \right)\left(\| u_t \|_2^2 + a \| \nabla u_t \|_2^2 \right) \geq \frac{1}{4}(\Phi'(t))^2. \tag{28}$$

Combining (27) and (28), one gets

$$\Phi(t)\Psi'(t) \geq \frac{1}{4}\Phi'(t)\Phi'(t) \geq \frac{p}{2}\Phi'(t)\Psi(t), \tag{29}$$

which turns into

$$\Phi(t)\Psi'(t) - \frac{p}{2}\Phi'(t)\Psi(t) \geq 0.$$

Now consider

$$\left(\Psi(t)\Phi(t)^{-\frac{p}{2}} \right)' = \Phi(t)^{\frac{-p-2}{2}} \left(\Psi'(t)\Phi(t) - \frac{p}{2}\Phi'(t)\Psi(t) \right) \geq 0. \tag{30}$$

From this one gets

$$\Psi(t)\Phi(t)^{-\frac{p}{2}} \geq \Psi(0)\Phi(0)^{-\frac{p}{2}} := M, \tag{31}$$

and

$$\Psi(t) \geq M\Phi(t)^{\frac{p}{2}}.$$

Using (27), the following is obtained:

$$\frac{\Phi'(t)\Phi(t)^{-\frac{p}{2}}}{2p} \geq M. \tag{32}$$

This is equivalent to

$$\frac{\left(\Phi(t)^{\frac{2-p}{2}} \right)'}{(2-p)p} \geq M,$$

and

$$\frac{1}{(2-p)p}\left((\Phi(t))^{\frac{2-p}{2}} - (\Phi(0))^{\frac{2-p}{2}} \right) \geq Mt.$$

Thus, one gets

$$(\Phi(t))^{\frac{2-p}{2}} \geq \left(\Phi^{\frac{2-p}{2}}(0) - p(p-2)t \right). \tag{33}$$

Taking the roots of both sides of (33), one gets the following:

$$\Phi(t) \geq \left(\Phi^{\frac{2-p}{2}}(0) - p(p-2)t \right)^{\frac{2}{2-p}}.$$

This expression gives the upper bound of the time interval, i.e, $t \leq T^* = \dfrac{\Phi^{\frac{2-p}{2}}(0)}{p(p-2)}$ for the solution blow-up.

Now, the following theorem is given to specify a finite time interval $(0,T_*)$ on which the quantity $\| \nabla u \|_2^2 + a \| \Delta u \|_2^2$ remains bounded and this is inspired by [17, 20]. Indeed, T_* is a lower bound for t because, by the Poincaré inequality, one has

$$\| u \|_2^2 + a \| \nabla u \|_2^2 \leq \lambda_1 (a \| \Delta u \|_2^2 + \| \nabla u \|_2^2), t \in (0,T_*),$$

where λ_1 is the first eigenvalue of the $-\Delta u = \lambda u$ under the periodic of boundary conditions.

Theorem 4.2 *Let $u(x,t)$ be solution of the problem (1)-(2). Then there exists a positive number* $T_* = \dfrac{\beta^{2-p}(0)}{(p-2)C^{2p-2}(\Omega)}$ *such that*

$$\beta(t) = \int_\Omega (|\nabla u|^2 + a|\Delta u|^2) dx \tag{34}$$

remains bounded in $(0,T_*)$.

Proof. By using Green's identity and using (1)-(2)

$$\begin{aligned} \beta'(t) &= 2\int_\Omega \Delta u(-u_t + a\Delta u_t) dx \\ &= 2\int_\Omega \Delta u(-\Delta u + (-\Delta)^2 u + \nabla \cdot (|\nabla u|^{p-2} \nabla u)) dx, \\ &= -2 \| \Delta u \|^2 - 2 \| \nabla \Delta u \|^2 - 2\int_\Omega \nabla \Delta u \cdot (|\nabla u|^{p-2} \nabla u)) dx. \end{aligned} \tag{35}$$

By the inequality $|2|\delta u||u|^p| \leq |\delta u|^2 + |u|^{2p}$ the last term above gives

$$|-2\int_\Omega |\nabla \Delta u||\nabla u|^{p-1} dx| \leq \| \nabla \Delta u \|_2^2 + \| \nabla u \|_{2p-2}^{2p-2}.$$

Substituting this into (35) $\beta'(t) \leq \| \nabla u \|_{2p-2}^{2p-2}$. Now, by using the Sobolev inequality

$$\| \nabla u \|_{2p-2}^{2p-2} \leq C^{2p-2}(\Omega)(\| \nabla u \|_2^2 + a \| \Delta u \|_2^2)^{p-1}. \tag{36}$$

So, one gets

$$\beta'(t) \leq C^{2p-2}(\Omega)(\| \nabla u \|^2 + a \| \Delta u \|^2)^{p-1} = C^{2p-2}(\Omega)\beta^{p-1}(t).$$

If this inequality is solved, the following is obtained:

$$\beta^{2-p}(t) \geq \beta^{2-p}(0) - (p-2)C^{2p-2}(\Omega)t. \tag{37}$$

Taking the root of both sides of (37), one finds

$$\beta(t) \geq \left(\beta^{2-p}(0) - (p-2)C^{2p-2}(\Omega)t\right)^{\frac{1}{2-p}}.$$

Therefore, $T_* = \dfrac{\beta^{2-p}(0)}{(p-2)C^{2p-2}(\Omega)}$ as obtained.

5. Conclusion

Higher-order pseudo-parabolic equations can be obtained from parabolic equations by adding the term $-a\Delta u_t$. Pseudo-parabolic equations play a crucial role in modeling complex physical phenomena where traditional parabolic or hyperbolic equations are insufficient. Their ability to capture memory effects and non-local behavior and to ensure the smoothness of solutions makes them valuable in various scientific and engineering applications, see [1–3, 5, 6].

By setting $a = 0$ in the term $-a\Delta u_t$, these equations generalize the thin-film equation, which arises in various fields of science such as biology and physics. This includes the spreading of low-amplitude long waves and heat conduction, describing nonlinear phenomena in the seepage of homogeneous fluids through fissured rock, the unidirectional propagation of nonlinear dispersive long waves, and numerous other phenomena.

In this research, it was observed that these two different types of equations have different solution spaces and these spaces converge to each other as a approaches zero. It was also observed that the solutions of pseudo-parabolic equations are smoother than the solutions of parabolic equations.

For $a \geq 1$, the solutions are in

$$E_S = \{u : \mathbb{R}_+ \times \Omega \to \mathbb{R} \,|\sup_{0\leq t\leq T} t^{\frac{1}{2}} \| \nabla u \|_2 < \infty\}$$

and for $0 \leq a < 1$, the solution space is given by

$$E_S = \{u : \mathbb{R}_+ \times \Omega \to \mathbb{R} \,|\sup_{0\leq t\leq T} t^{\frac{1}{4}} \| \nabla u \|_2 < \infty\}.$$

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