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Fekete-Szego results for certain BI-univalent functions involving q-analogues of logarithmic functions

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Abstract

In this paper, we discuss a novel type of analytic bi-univalent functions by utilizing specialized q-Salagean differential operators. Then, we use the q-analogue of the logarithmic function to introduce definition and provide properties of a class of bi-univalent functions. Further, we use the subordination principle to estimate the initial Taylor and Maclaurin coefficients for these given univalent functions. Additionally, we introduce new operators to demonstrate practical applications of the existing theory and establish Fakte-Szego results for each function in the defined sets. Further, we discuss certain coefficient inequalities in detail.

Keywords: Salagean differential operator; geometric function theory; inequalities; bi-univalent functions.

MSC 2010: Primary 14E20; Secondary 46E25.

1. Introduction

Recently, the fractional q-calculus has emerged as an extension of the conventional fractional calculus. It has found applications in several scientific fields, such as hypergeometric functions, optimal control, q-difference operators, q-integral equations, ordinary fractional calculus, and univalent functions of complex analysis (see, e.g., [1–5]). For a complex valued function θ and a real number $0 < q < 1$, Jackson invented the so-called q-difference operator as [6]

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$$D_q\theta(\xi)=\frac{\theta(\xi)-\theta(q\xi)}{\xi-q\xi},\ \text{ where } \xi\in D(\ D=:\{\xi,|\xi|<1\}).$$

Srivastava [7] examined the q-difference operator within the framework of geometric function theory, whereas Kanas and Raducanu [8] talked about a q-Ruscheweyh differential operator. Srivastava invented the q-Noor integral operator in [9] by utilizing the Hadamard product concept. Conversely, Kanas and colleagues [10] explored a new class of univalent functions and suggested a symmetric operator for the q-derivative. As an alternative, Srivastava et al. [11] used q-calculus operators to construct many new families of k-symmetric harmonic functions. Nonetheless, readers are referred to [12–17] and citations therein for up-to-date information on the fractional theory. The definition of the q-logarithm function is given as [18]

$$ln_q(1-\xi)=\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\xi^k=\frac{1}{1-q}\sum_{k=1}^{\infty}\frac{q^k}{[k]_q}\xi^k,[k]_q=\frac{1-q^k}{1-q}.$$

Next, Yamano [19] examined the q-exponential and q-logarithmic functions' characteristics and their use in Tsallis statistics. The function $\theta\in\mathcal{A}$ may be represented in the expansion form if $\mathcal{A}$ represents the set of all functions θ on the unite disc $D=\{\xi,|\xi|<1\}$, normalized by $\theta(0)=0$ and $\theta'(0)=1$, then the function $\theta\in\mathcal{A}$ can be expressed in the expansion form

$$\theta(\xi)=\xi+\sum_{n=2}^{\infty}a_n\xi^n. \tag{1}$$

The classes of univalent functions in $\mathcal{A}$ are represented by S, starlike functions by S^*, and convex functions by K in this instance (See [20] and [21–24]). Generally speaking, if there exists an analytic Schwartz function ω on D such that $\omega(0)=0$ and $|w(\xi)|<1(\xi\in D), f(\xi)=g(w(\xi))$ [20, 23], then the function f is subordinate to a function g on D, $f(\xi)\prec g(\xi)$ where $\xi\in D$. An equation for subordination is provided by Miller and Mocanu in [23], with significant implications for complex analysis and univalent functions. All elements $\theta\in S$ enfolding a disc of radius 1/4 are known to have an inverse θ^{-1} such that $\theta^{-1}(\theta(\xi))=\xi,(\xi\in D)$, thanks to the Koebe one-quarter theorem [24]

$$\theta(\theta^{-1}(w))=w,\ \left(w\in\Delta=\left\{w\in\mathbb{C}:|w|<\frac{1}{4}\right\}\right).$$

If θ and θ^{-1} are both univalent on D, then the function θ is regarded as bi-univalent on D. If the class Σ of bi-univalent functions on D is represented by the Taylor-Maclaurin series expansion (1), then the illustrative examples of such functions in the class Σ may be

$$\frac{\xi}{1-\xi},\qquad -\log(1-\xi),\qquad \frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right).$$

However, the Koebe function is not in Σ and examples of functions in S can be

$$\xi-\frac{\xi^2}{2},\qquad \frac{\xi}{1-\xi^2},$$

which are also not members of Σ. Various bi-univalent sets of functions are explored in [25–27]. Further, authors in the aforementioned papers obtain coefficient estimates and study certain Fekete-Szego results.

For a function $\theta\in\mathcal{A}$ mentioned by (1), the Salagean differential operator $D^n\theta$ has the form [28]

$$D^n\theta(\xi)=\xi+\sum_{n=2}^{\infty}k^n a_k\xi^k. \tag{2}$$

In the literature, there are several differential operators introduced to generalize (2), whilst a variety of classes of univalent functions involved with the generalized Salagean differential operators are discussed by [29–32]. In what follows, we define the differential operator

$$D_q^0 f(z) = f(z),$$
$$D_{q,\alpha,\lambda} f(z) = (\alpha - \lambda) f(z) + (\lambda - \alpha + 1) z D_q f(z),$$
$$\vdots$$
$$D_{q,\alpha,\lambda}^n f(z) = D_{q,\alpha,\lambda}(D_{q,\alpha,\lambda}^{n-1} f(z)),$$

where $\alpha \geq 0, \lambda \geq 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in D$. It is clear that $D_{q,\alpha,\lambda}^n f(0) = 0$. If *f* is given by (1), then

$$D_{q,\alpha,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} \Lambda_k^n a_k z^k, \tag{3}$$

where

$$\Lambda_k^n = \left[(\alpha - \lambda) + (\lambda - \alpha + 1)[k]_q\right]^n, \quad k = 2,3,4,\ldots, n \in \mathbb{N}. \tag{4}$$

Based on the definition, we can infer that if q approaches 1 from the negative side, then we have

$$\lim_{q \to 1^-} D_{q,\alpha,\lambda}^n f(z) = \lim_{q \to 1^-} \left[z + \sum_{k=2}^{\infty} \left[(\alpha - \lambda) + (\lambda - \alpha + 1)[k]_q\right]^n a_k z^k \right]$$
$$= z + \sum_{k=2}^{\infty} \left[(n-1)(\lambda - \alpha) + k\right]^n a_k z^k = D_{\lambda,\alpha}^n f(z),$$

where $D_\lambda f$ is the Salagean differential operator presented by [33]. Note that for $\alpha = \lambda$ as $q \to 1^-$, we attain the Salagean differential operator (see [28]). Now, we define

$$\varphi_L(z) = \frac{1 + ln_q(1-z)}{1 - ln_q(1-z)}, \quad (z \in D). \tag{5}$$

Therefore, we have

$$\varphi_L(z) = 1 + \frac{2q}{1-q} z + \left[\frac{2q^2}{1-q^2} + 2\left(\frac{q}{1-q}\right)^2 \right] z^2 + \left[\frac{2q^3}{1-q^3} + 4 \frac{q}{1-q} \frac{q^2}{1-q^2} + 2\left(\frac{q}{1-q}\right)^3 \right] z^3 + \ldots. \tag{6}$$

Lemma 1.1. *[20,23] Assume the function $r \in \mathcal{P}$ is expressed by*

$$r(z) = 1 + r_1 z + r_2 z^2 + r_3 z^3 + \ldots \quad (z \in D),$$

then, the coefficient estimate is given by

$$|r_n| \leq 2 \qquad (n \in \mathbb{N}).$$

Our aim in this paper is to give a broad introduction to the q-analogue of the logarithmic operator and its application to the definition of bi-univalent function subclasses via the generalized Salagean differential operator. For the coefficients $|a_2|$ and $|a_3|$, we also obtain limits and prove the Fekete-Szego inequality for these functions. We reviewed a few preliminary findings and supplementary findings from the geometric field theory in Section 1. We estimate coefficient bounds and go into great depth on several Fekete-Szego issues in Sections 2 and 3.

2. Certain class of coefficient inequalities

In this section, we examine coefficient inequalities for various categories of bi-univalent functions. We divide this section into two subsections to discuss coefficient estimates for the function class $\sum_q(\lambda, \alpha, \varphi_L)$ and the function class $\sum_q^\mu(\lambda, \alpha, \varphi_L)$ as well.

2.1. Coefficient estimates for the function class $\sum_q(\lambda,\alpha,\varphi_L)$. *Following is a definition discussing the class* $\sum_q(\lambda,\alpha,\varphi_L)$ *of functions.*

Definition 2.1. Let $\alpha \geq 0, 0 \leq q \leq 1$ and $\lambda \geq 0$. An element $f \in \Sigma$ is planned to be in the class $\sum_q(\lambda,\alpha,\varphi_L)$, if each of the following subordination conditions holds true:

$$\frac{D^{n+1}_{q,\alpha,\lambda}f(z)}{D^{n}_{q,\alpha,\lambda}f(z)} \prec \varphi_L(z), \quad z \in D,$$

and

$$\frac{D^{n+1}_{q,\alpha,\lambda}g(w)}{D^{n}_{q,\alpha,\lambda}g(w)} \prec \varphi_L(w), \quad w \in D,$$

provided $g(w) = f^{-1}(w)$ and $\varphi_L(z)$ has the significance of (5).

Theorem 2.2. *Let* $\alpha \geq 0$, $0 < q \leq 1$ and $\lambda \geq 0$. *For every element f in the class* $\sum_q(\lambda,\alpha,\varphi_L)$, *defined by* (1), *we have*

$$|a_2| \leq \frac{2q}{\sqrt{(1-q)\left[2q(\Lambda_3^{n+1}-\Lambda_3^n)-\left(q+1+\frac{2}{1+q}\right)\left(\Lambda_2^{n+1}-\Lambda_2^n\right)^2+q\left((\Lambda_2^{n+1})^2-(\Lambda_2^n)^2\right)\right]}} \tag{7}$$

and

$$|a_3| \leq 2\left(\frac{q}{1-q}\right)^2\frac{4}{(\Lambda_2^{n+1}-\Lambda_2^n)^2}+\frac{2q}{1-q}\frac{1}{\Lambda_3^{n+1}-\Lambda_3^n}, \tag{8}$$

where Λ_k^n *is given in* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

Proof. Let $f \in \Sigma_q(\lambda,\alpha,\varphi_l), g(w) = f^{-1}(w)$ and $z \in D$. Then, in view of (3), we get

$$\frac{D^{n+1}_{q,\alpha,\lambda}f(z)}{D^{n}_{q,\alpha,\lambda}f(z)} = 1+(\Lambda_2^{n+1}-\Lambda_2^n)a_2z+\left(\left(\Lambda_3^{n+1}-\Lambda_3^n\right)a_3+\Lambda_2^n\left(\Lambda_2^{n+1}-\Lambda_2^n\right)a_2^2\right)z^2+\ldots, \tag{9}$$

and

$$\frac{D^{n+1}_{q,\alpha,\lambda}g(w)}{D^{n}_{q,\alpha,\lambda}g(w)} = 1-(\Lambda_2^{n+1}-\Lambda_2^n)a_2w+\left(\left(\Lambda_3^{n+1}-\Lambda_3^n\right)\left(2a_2^2-a_3\right)+\Lambda_2^n\left(\Lambda_2^{n+1}-\Lambda_2^n\right)a_2^2\right)w^2+\ldots. \tag{10}$$

Therefore, can derive analytic functions $u,v : D \to D, u(0) = v(0) = 0$ satisfying the conditions

$$\frac{D^{n+1}_{q,\alpha,\lambda}f(z)}{D^{n}_{q,\alpha,\lambda}f(z)} = \varphi_L(u(z)), \quad (z \in D),$$

and

$$\frac{D^{n+1}_{q,\alpha,\lambda}g(w)}{D^{n}_{q,\alpha,\lambda}g(w)} = \varphi_L(v(w)), \quad (w \in D).$$

Now, we introduce the functions r and s as follow

$$r(z) = \frac{1+u(z)}{1-u(z)} = 1+r_1z+r_2z^2+\ldots, \quad (z \in D),$$

and

$$s(z) = \frac{1+v(w)}{1-v(w)} = 1+s_1w+s_2w^2+\ldots, \quad (w \in D).$$

These functions are analytic in D and $r(0) = s(0) = 1$. Solving for u and v implies

$$u(z) = \frac{r(z)-1}{r(z)+1} = \frac{1}{2}r_1 z + \frac{1}{2}\left(r_2 - \frac{r_1^2}{2}\right)z^2 + \ldots, \quad (z \in D),$$

and

$$v(w) = \frac{s(w)-1}{s(w)+1} = \frac{1}{2}s_1 w + \frac{1}{2}\left(s_2 - \frac{s_1^2}{2}\right)w^2 + \ldots, \quad (w \in D).$$

So, for $z \in D$, we have

$$\frac{D_{q,\alpha,\lambda}^{n+1} f(z)}{D_{q,\alpha,\lambda}^{n} f(z)} = \varphi_L\left(\frac{r(z)-1}{r(z)+1}\right) = 1 + \frac{q}{1-q}r_1 z + \left[\frac{q}{1-q}\left(r_2 - \frac{r_1^2}{2}\right) + r_1^2 \frac{1}{1+q}\left(\frac{q}{1-q}\right)^2\right]z^2 + \ldots, \tag{11}$$

and

$$\frac{D_{q,\alpha,\lambda}^{n+1} g(w)}{D_{q,\alpha,\lambda}^{n} g(w)} = \varphi_L\left(\frac{s(w)-1}{s(w)+1}\right) = 1 + \frac{q}{1-q}s_1 w + \left[\frac{q}{1-q}\left(s_2 - \frac{s_1^2}{2}\right) + s_1^2 \frac{1}{1+q}\left(\frac{q}{1-q}\right)^2\right]w^2 + \ldots. \tag{12}$$

Now, equating the coefficients (9) and (11) implies

$$\left(\Lambda_2^{n+1} - \Lambda_2^n\right)a_2 = \frac{q}{1-q}r_1, \tag{13}$$

and

$$\left(\Lambda_3^{n+1} - \Lambda_3^n\right)a_3 + \Lambda_2^n\left(\Lambda_2^{n+1} - \Lambda_2^n\right)a_2^2 = \frac{q}{1-q}\left(r_2 - \frac{r_1^2}{2}\right) + \frac{r_1^2}{1+q}\left(\frac{q}{1-q}\right)^2. \tag{14}$$

Also, from (10) we have

$$-\left(\Lambda_2^{n+1} - \Lambda_2^n\right)a_2 = \frac{q}{1-q}s_1, \tag{15}$$

and

$$\left(\Lambda_3^{n+1} - \Lambda_3^n\right)(2a_2^2 - a_3) + \Lambda_2^n\left(\Lambda_2^{n+1} - \Lambda_2^n\right)a_2^2 = \frac{q}{1-q}\left(s_2 - \frac{s_1^2}{2}\right) + \frac{s_1^2}{1+q}\left(\frac{q}{1-q}\right)^2. \tag{16}$$

From (13) and (15), we derive

$$r_1 = -s_1 \tag{17}$$

and

$$2\left(\Lambda_2^{n+1} - \Lambda_2^n\right)^2 a_2^2 = \left(r_1^2 + s_1^2\right)\left(\frac{q}{1-q}\right)^2. \tag{18}$$

Now, employing equations (14) and (16) leads to

$$2\left[\Lambda_3^{n+1} - \Lambda_3^n + \Lambda_2^n\left(\Lambda_2^{n+1} - \Lambda_2^n\right)\right]a_2^2 = \frac{q}{1-q}\left(r_2 + s_2 - \frac{r_1^2 + s_1^2}{2}\right) + \frac{r_1^2 + s_1^2}{1+q}\left(\frac{q}{1-q}\right)^2. \tag{19}$$

By invoking equation (18) in equation (19), we establish that

$$2\left[\Lambda_3^{n+1} - \Lambda_3^n + \Lambda_2^n\left(\Lambda_2^{n+1} - \Lambda_2^n\right)\right]a_2^2 = \frac{q}{1-q}(r_2 + s_2) - a_2^2\left(\Lambda_2^{n+1} - \Lambda_2^n\right)^2\left[\frac{1-q}{q} - \frac{2}{1+q}\right], \tag{20}$$

or, equivalently,

$$\left[2q(1+q)\left[\Lambda_3^{n+1} - \Lambda_3^n + \Lambda_2^n\left(\Lambda_2^{n+1} - \Lambda_2^n\right)\right] + (1 - 2q - q^2)\left(\Lambda_2^{n+1} - \Lambda_2^n\right)^2\right]a_2^2 = \frac{q^2(q+1)}{1-q}\left(r_2 + s_2\right). \tag{21}$$

Using Lemma 1.1, for the coefficients r_2 and s_2 yields the bound on $|a_2|$, as asserted in (7). Next, to derive the bound on $|a_3|$, by subtracting (16) from (14) and using (17), we write

$$a_3 = a_2^2 + \frac{q}{2(1-q)} \frac{r_2 - s_2}{\Lambda_3^{n+1} - \Lambda_3^n}. \tag{22}$$

By virtue of (18) and (22), we obtain

$$a_3 = \left(\frac{q}{1-q}\right)^2 \frac{r_1^2 + s_1^2}{2(\Lambda_2^{n+1} - \Lambda_2^n)^2} + \frac{q}{1-q} \frac{r_2 - s_2}{2(\Lambda_3^{n+1} - \Lambda_3^n)}.$$

Finally, by applying Lemma 1.1 for the coefficient r_2 and s_2, we reach to the assertion (8). Hence, we our result has been proved.

The following theorem calculates the Fekete-Szego problem for the class $\sum_q(\lambda, \alpha, \varphi_L)$ of functions by using the estimate coefficients a_2 and a_3.

Theorem 2.3. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \leq 1$, $\alpha \geq 0$ *and* $\lambda \geq 0$ *and* $\eta \in \mathbb{C}$. *If* $f \in \Sigma_q(\lambda, \alpha, \varphi_L)$, *then we have*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{2q}{(1-q)(\Lambda_3^{n+1} - \Lambda_3^n)}, & 0 \leq \Gamma(\eta) < \dfrac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}, \\ \dfrac{4q}{1-q}\Gamma(\eta), & \Gamma(\eta) \geq \dfrac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}, \end{cases}$$

where Λ_k^n *is given in* (4), *for* $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$, *and*

$$\Gamma(\eta) = \frac{(1-\eta)q(1+q)}{2q(1+q)\left[\Lambda_3^{n+1} - \Lambda_3^n + \Lambda_2^n\left(\Lambda_2^{n+1} - \Lambda_2^n\right)\right] + (1-2q-q^2)\left(\Lambda_2^{n+1} - \Lambda_2^n\right)^2}. \tag{23}$$

Proof. By using equation (22), we have

$$a_3 - \eta a_2^2 = \frac{q}{2(1-q)} \frac{r_2 - s_2}{\Lambda_3^{n+1} - \Lambda_3^n} + (1-\eta)a_2^2.$$

On account of (21), we obtain

$$a_3 - \eta a_2^2 = \frac{q}{2(1-q)} \frac{r_2 - s_2}{\Lambda_3^{n+1} - \Lambda_3^n} + \frac{(1-\eta)q^2(1+q)(r_2 + s_2)}{(1-q)\left[2q(1+q)\left[\Lambda_3^{n+1} - \Lambda_3^n + \Lambda_2^n\left(\Lambda_2^{n+1} - \Lambda_2^n\right)\right] + (1-2q-q^2)\left(\Lambda_2^{n+1} - \Lambda_2^n\right)^2\right]}.$$

Therefore, we have

$$a_3 - \eta a_2^2 = \frac{q}{1-q}\left[\left(\Gamma(\eta) + \frac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}\right)r_2 + \left(\Gamma(\eta) - \frac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}\right)s_2\right],$$

where $\Gamma(\eta)$ is defined by (23). Since $0 < q \leq 1$, we derive

$$|a_3 - \eta a_2^2| = \frac{q}{1-q}\left|\left(\Gamma(\eta) + \frac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}\right)r_2 + \left(\Gamma(\eta) - \frac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}\right)s_2\right|.$$

This ends the proof of Theorem 2.3.

Remark 2.4. Let $\eta = 1$. *If* $f \in \Sigma_q(\lambda, \alpha, \varphi_L)$, then we have

$$|a_3 - a_2^2| \leq \frac{2q}{(1-q)(\Lambda_3^{n+1} - \Lambda_3^n)},$$

where Λ_k^n is given in (4), $k = 2, 3, \ldots$ and $n \in \mathbb{N}$.

2.2. Coefficient estimates for the function class $\sum_q^{\mu}(\lambda,\alpha,\varphi_L)$. *Following is a definition decribing the elements of the class* $\sum_q^{\mu}(\lambda,\alpha,\varphi_L)$.

Definition 2.5. Let $0 \le q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$ and $0 \le \mu \le 1$. A function $f \in \Sigma$ is said to belong in the class $\sum_q^{\mu}(\lambda,\alpha,\varphi_L)$, if all of the following subordination criteria are met:

$$\left(\frac{z}{f(z)}\right)^{1-\mu}\frac{D_{q,\alpha,\lambda}^{n+1}f(z)}{z} \prec \varphi_L(z), \quad z \in D,$$

and

$$\left(\frac{w}{f(w)}\right)^{1-\mu}\frac{D_{q,\alpha,\lambda}^{n+1}g(w)}{w} \prec \varphi_L(w), \quad w \in D,$$

where $g(w) = f^{-1}(w)$ and $\varphi_L(z)$ is given by (5).

Theorem 2.6. *Let* $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$ *and* $0 \le \mu \le 1$. *If the function f is a member of the class* $\Sigma_q(\lambda, \alpha, \varphi_L)$, *defined by* (1), *then we have*

$$|a_2| \le \min\left\{ \begin{array}{c} \dfrac{2}{\Lambda_2^n - 1 + \mu}\left(\dfrac{q}{1-q}\right), \\ \dfrac{2q}{\sqrt{(1-q)\left[q(2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu) - (q+1-\dfrac{2}{q+1})\left(\Lambda_2^n - 1 - \mu\right)^2\right]}} \end{array} \right\}, \tag{24}$$

and

$$|a_3| \le \frac{2q}{(1-q)\,|\Lambda_3^n - (1-\mu)|} + \min\left\{ \begin{array}{c} \dfrac{4}{(\Lambda_2^n - (1-\mu))^2}\left(\dfrac{q}{1-q}\right)^2, \\ \dfrac{4q^2}{(1-q)\left|q\left(2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu)\right) - (q+1-\dfrac{2}{q+1})(\Lambda_2^n - 1 + \mu)^2\right|} \end{array} \right\}, \tag{25}$$

where Λ_k^n *is given in* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

Proof. Let $f \in \sum_q^{\mu}(\lambda,\alpha,\varphi_L)$ and $g(w) = f^{-1}(w)$. Then, in view of (2), we obtain

$$\left(\frac{z}{f(z)}\right)^{1-\mu}\frac{D_{q,\alpha,\lambda}^{n+1}f(z)}{z} = 1 + \left(\Lambda_2^n - (1-\mu)\right)a_2 z + \left(\Lambda_3^n a_3 + \frac{1-\mu}{2}\left((4-\mu)a_2^2 - 2a_3\right) - \Lambda_2^n(1-\mu)a_2^2\right)z^2 + \ldots, \tag{26}$$

and

$$\left(\frac{w}{f(w)}\right)^{1-\mu}\frac{D_{q,\alpha,\lambda}^{n+1}g(w)}{w} = 1 + \left((1-\mu) - \Lambda_2^n\right)a_2 w + \left(\Lambda_3^n(2a_2^2 - a_3) + \frac{1-\mu}{2}(2a_3 - \mu a_2^2) - \Lambda_2^n(1-\mu)a_2^2\right)w^2 + \ldots. \tag{27}$$

Now, by equating the coefficients (26) and (11), we obtain

$$\left(\Lambda_2^n - (1-\mu)\right)a_2 = \frac{q}{1-q}r_1, \tag{28}$$

$$\Lambda_3^n a_3 + \frac{1-\mu}{2}\left((4-\mu)a_2^2 - 2a_3\right) - \Lambda_2^n(1-\mu)a_2^2 = \frac{q}{1-q}\left(r_2 - \frac{r_1^2}{2}\right) + r_1^2\frac{1}{1+q}\left(\frac{q}{1-q}\right)^2. \tag{29}$$

Once again, equating the corresponding coefficients (27) and (12) reveals

$$-\left(\Lambda_2^n-(1-\mu)\right)a_2=\frac{q}{1-q}s_1, \tag{30}$$

$$\Lambda_3^n(2a_2^2-a_3)+\frac{1-\mu}{2}(2a_3-\mu a_2^2)-\Lambda_2^n(1-\mu)a_2^2=\frac{q}{1-q}\left(s_2-\frac{s_1^2}{2}\right)+s_1^2\frac{1}{1+q}\left(\frac{q}{1-q}\right)^2. \tag{31}$$

By comparing the coefficients (28) and (29), we get

$$r_1=-s_1, \tag{32}$$

$$2\left(\Lambda_2^n-(1-\mu)\right)^2a_2^2=(r_1^2+s_1^2)\left(\frac{q}{1-q}\right)^2. \tag{33}$$

Now, by equating equations (29) and (31), we write

$$a_2^2\left[2\Lambda_3^n+(1-\mu)(2-\mu)+2\Lambda_2^n(1-\mu)\right]=\frac{q}{1-q}\left(r_2+s_2-\frac{r_1^2+s_1^2}{2}\right)+(r_1^2+s_1^2)\frac{1}{1+q}\left(\frac{q}{1-q}\right)^2. \tag{34}$$

In view of (33) and (34), we derive

$$a_2^2\left[2\Lambda_3^n+(1-\mu)(2-\mu)+2\Lambda_2^n(1-\mu)\right]=\frac{q}{1-q}(r_2+s_2)-(\Lambda_2^n-1+\mu)^2a_2^2\left[\frac{1-q}{q}-\frac{2}{1+q}\right], \tag{35}$$

or, equivalently,

$$a_2^2=\frac{q^2(r_2+s_2)}{(1-q)\left[q\left(2\Lambda_3^n+(1-\mu)(2-\mu)+2\Lambda_2^n(1-\mu)\right)-(q+1-\frac{2}{q+1})(\Lambda_2^n-1+\mu)^2\right]}. \tag{36}$$

From (33) and (36) and using Lemma 1.1 for the coefficients r_2 and s_2, we reach the bounds on $|a_2|$ as asserted in (24). The proof can be finalized by determining the limit for the absolute value of the coefficient $|a_3|$. By subtracting equation (31) from equation (29), we get

$$a_2^2-a_3=\frac{q(s_2-r_2)}{2(1-q)(\Lambda_3^n-(1-\mu))}. \tag{37}$$

Put the value a_2 given in (33) into (37). Then, we get

$$a_3=\frac{q(r_2-s_2)}{2(1-q)(\Lambda_3^n-(1-\mu))}+\frac{r_1^2+s_1^2}{2(\Lambda_2^n-(1-\mu))^2}\left(\frac{q}{1-q}\right)^2. \tag{38}$$

Insert the value of a_2 given by (33) in (36) to have

$$a_3=\frac{q(r_2-s_2)}{2(1-q)(\Lambda_3^n-(1-\mu))}+\frac{q^2(r_2+s_2)}{(1-q)\left[q\left(2\Lambda_3^n+(1-\mu)(2-\mu)+2\Lambda_2^n(1-\mu)\right)-(q+1-\frac{2}{q+1})(\Lambda_2^n-1+\mu)^2\right]}. \tag{39}$$

Therefore, from (37), we conclude that

$$|a_3|\le\frac{2q}{(1-q)|\Lambda_3^n-(1-\mu)|}+\frac{4}{(\Lambda_2^n-(1-\mu))^2}\left(\frac{q}{1-q}\right)^2. \tag{40}$$

Similarly, from (39), we obtain the following bound

$$|a_3|\le\frac{2q}{(1-q)|\Lambda_3^n-(1-\mu)|}+\frac{4q^2}{(1-q)\left|q\left(2\Lambda_3^n+(1-\mu)(2-\mu)+2\Lambda_2^n(1-\mu)\right)-(q+1-\frac{2}{q+1})(\Lambda_2^n-1+\mu)^2\right|}. \tag{41}$$

This ends the proof of our Theorem.

The following useful theorem calculates the Fekete-Szego problem for the class $\sum_q^{\mu}(\lambda,\alpha,\varphi_L)$ of functions by aid of the coefficients a_2 and a_3.

Theorem 2.7. *Let f be a function defined by* (1), $g(w)=f^{-1}(w)$, $0<q\le 1$, $\alpha\ge 0$, $\lambda\ge 0$, $0\le\mu\le 1$ *and* $\eta\in\mathbb{C}$. *If* $f\in\sum_q^{\mu}(\lambda,\alpha,\varphi_L)$, *then we have*

$$|a_3-\eta a_2^2|\le\frac{q}{1-q}\begin{cases}\dfrac{2}{\Lambda_3^n-(1-\mu)}, & 0\le\Theta(\eta)<\dfrac{1}{2(\Lambda_3^n-(1-\mu))},\\[2ex] 4\Theta(\eta), & \Theta(\eta)\ge\dfrac{1}{2(\Lambda_3^n-(1-\mu))},\end{cases},$$

where Λ_k^n *is given by* (4), $k=2,3,\ldots$ *and* $n\in\mathbb{N}$, *and*

$$\Theta(\eta)=\frac{(1-\eta)q}{q\left(2\Lambda_3^n+(1-\mu)(2-\mu)+2\Lambda_2^n(1-\mu)\right)-(q+1-\dfrac{2}{q+1})(\Lambda_2^n-1+\mu)^2}.$$

Corollary 2.8. *Let f be a function defined by* (1), $g(w)=f^{-1}(w)$, $0<q\le 1$, $\alpha\ge 0$, $\lambda\ge 0$ *and* $0\le\mu\le 1$. *If* $f\in\sum_q^{\mu}(\lambda,\alpha,\varphi_L)$, *then we have*

$$|a_3-a_2^2|\le\frac{2}{1-q}\frac{2}{\Lambda_3^n-(1-\mu)},$$

where Λ_k^n *is given by* (4), $k=2,3,\ldots$ *and* $n\in\mathbb{N}$.

3. Applications of the coefficient inequality

In this section, for $0\le\beta\le 1$, let us denote by $\psi_L(z)$ the function such that

$$\psi_L(z)=\frac{1+\beta(1-q)ln_q(1-z)}{1-(1-q)ln_q(1-z)}=1+(1+\beta)qz+2(1+\beta)\frac{q^2}{(1-q)(1+q)}z^2+\ldots. \tag{42}$$

On the other hand, for $0<\gamma\le 1$, let us denote by $\widehat{\psi}_L(z)$ the function such that

$$\widehat{\psi}_L(z)=\left(\frac{1+(1-q)ln_q(1-z)}{1-(1-q)ln_q(1-z)}\right)^{\gamma}=1+2\gamma qz+\left(4\gamma\frac{q^2}{(1-q)(1+q)}+2q^2\gamma(\gamma-1)\right)z^2+\ldots. \tag{43}$$

Therefore, we have the following useful definition.

Definition 3.1. Let $0<q\le 1$, $\alpha\ge 0$, $\lambda\ge 0$ *and* $0\le\beta\le 1$. A member $f\in\Sigma$ is in the class $\sum_q^{\beta}(\lambda,\alpha,\psi_L)$, if each of the following conditions (subordination) holds true

$$\frac{D_{q,\alpha,\lambda}^{n+1}f(z)}{D_{q,\alpha,\lambda}^{n}f(z)}\prec\psi_L(z),\quad z\in D,$$

and

$$\frac{D_{q,\alpha,\lambda}^{n+1}g(w)}{D_{q,\alpha,\lambda}^{n}g(w)}\prec\psi_L(w),\quad w\in D,$$

where $g(w)=f^{-1}(w)$ and $\psi_L(z)$ is given by (42).

Theorem 3.2. *Let* $0 < q \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$ *and* $0 \leq \beta \leq 1$. *If a function f in the class* $\sum_q^\beta(\lambda,\alpha,\psi_L)$, *defined by* (1), *then we have*

$$|a_2| \leq \frac{2q(1-\beta)}{\sqrt{4q(1-\beta)\left[\Lambda_3^{n+1}-\Lambda_3^n+\Lambda_2^n(\Lambda_2^{n+1}-\Lambda_2^n)\right]+\left(1-\frac{2q}{1-q^2}\right)(\Lambda_2^{n+1}-\Lambda_2^n)^2}},$$

and

$$|a_3| \leq \frac{q(1-\beta)}{\Lambda_3^{n+1}-\Lambda_3^n}+\frac{4q^2(1-\beta)^2}{(\Lambda_2^{n+1}-\Lambda_2^n)^2},$$

where Λ_k^n *is given by* (4), *for* $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

The following corollary describes the Fekete-Szego problem for the function class $\sum_q^\beta(\lambda,\alpha,\psi_L)$ by using the coefficients a_2 and a_3.

Corollary 3.3. *Let f be a function defined by* (4.1), $g(w) = f^{-1}(w)$, $0 < q \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$ *and* $0 \leq \beta \leq 1$ *and* $\eta \in \mathbb{C}$. *If* $f \in \sum_q^\beta(\lambda,\alpha,\psi_L)$, *then we have*

$$|a_3-\eta a_2^2| \leq \begin{cases} \frac{q(1-\beta)}{\Lambda_3^{n+1}-\Lambda_3^n}, & 0 \leq \Gamma\beta(\eta) < \frac{1}{4(\Lambda_3^{n+1}-\Lambda_3^n)}, \\ 4q(1-\beta)\Gamma\beta(\eta), & \Gamma\beta(\eta) \geq \frac{1}{4(\Lambda_3^{n+1}-\Lambda_3^n)}, \end{cases}$$

where Λ_k^n *is given by* (4), *for* $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$, and

$$\Gamma_\beta(\eta) = \frac{(1-\eta)(1-\beta)q}{4q(1-\beta)\left[\Lambda_3^{n+1}-\Lambda_3^n+\Lambda_2^n(\Lambda_2^{n+1}-\Lambda_2^n)\right]+\left(1-\frac{2q}{1-q^2}\right)(\Lambda_2^{n+1}-\Lambda_2^n)^2}.$$

Corollary 3.4. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$ *and* $0 \leq \beta \leq 1$. *If* $f \in \sum_q^\beta(\lambda,\alpha,\psi_L)$, *then we have*

$$|a_3-a_2^2| \leq \frac{q(1-\beta)}{\Lambda_3^{n+1}-\Lambda_3^n},$$

where Λ_k^n *is given by* (4), *for* $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

Definition 3.5. Let $0 < q \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$ and $0 < \gamma < 1$. A function $f \in \Sigma$ is in the class $\sum_q^\gamma(\lambda,\alpha,\widehat{\psi}_L)$, if all of the following subordination conditions are met

$$\frac{D_{q,\alpha,\lambda}^{n+1}f(z)}{D_{q,\alpha,\lambda}^{n}f(z)} \prec \widehat{\psi}_L(z), \quad z \in D,$$

and

$$\frac{D_{q,\alpha,\lambda}^{n+1}g(w)}{D_{q,\alpha,\lambda}^{n}g(w)} \prec \widehat{\psi}_L(w), \quad w \in D,$$

where $g(w) = f^{-1}(w)$ and $\widehat{\psi}_L(z)$ is given by (43).

Theorem 3.6. Let $0 < q \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$ *and* $0 < \gamma < 1$. *If a function f is a member of the class* $\sum_q^\gamma(\lambda,\alpha,\widehat{\psi}_L)$, *defined by* (1), *then we have*

$$|a_2| \le \frac{2\gamma q}{\sqrt{2\gamma q\left[\Lambda_3^{n+1} - \Lambda_3^n + \Lambda_2^n(\Lambda_2^{n+1} - \Lambda_2^n)\right] + \left[1 - q\left(\frac{2}{1-q^2} + \gamma - 1\right)\right](\Lambda_2^{n+1} - \Lambda_2^n)^2}},$$

and

$$|a_3| \le \frac{2\gamma q}{\Lambda_3^{n+1} - \Lambda_3^n} + \frac{4\gamma^2 q^2}{(\Lambda_2^{n+1} - \Lambda_2^n)^2},$$

where Λ_k^n *is given by* (4), *for* $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

The following corollary computes the Fekete-Szego problem for the class $\sum_q^{\gamma}(\lambda,\alpha,\widehat{\psi}_L)$, of functions by using the coefficients a_2 and a_3.

Corollary 3.7. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 < \gamma < 1$ *and* $\eta \in \mathbb{C}$. *If* $f \in \sum_q^{\gamma}(\lambda,\alpha,\widehat{\psi}_L)$, *then we have*

$$|a_3 - \eta a_2^2| \le \begin{cases} \dfrac{2\gamma q}{\Lambda_3^{n+1} - \Lambda_3^n}, & 0 \le \Gamma\gamma(\eta) < \dfrac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}, \\ 4\gamma q \Gamma\gamma(\eta), & \Gamma_\gamma(\eta) \ge \dfrac{1}{2(\Lambda_3^{n+1} - \Lambda_3^n)}, \end{cases}$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$., and

$$\Gamma_\gamma(\eta) = \frac{(1-\eta)\gamma q}{2\gamma q\left[\Lambda_3^{n+1} - \Lambda_3^n + \Lambda_2^n(\Lambda_2^{n+1} - \Lambda_2^n)\right] + \left[1 - q\left(\frac{2}{1-q^2} + \gamma - 1\right)\right](\Lambda_2^{n+1} - \Lambda_2^n)^2}.$$

Corollary 3.8. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$ *and* $0 < \gamma < 1$. *If* $f \in \sum_q^{\gamma}(\lambda,\alpha,\widehat{\psi}_L)$, *then we have*

$$|a_3 - a_2^2| \le \frac{2\gamma q}{\Lambda_3^{n+1} - \Lambda_3^n},$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

Definition 3.9. Let $0 \le q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$ and $0 \le \beta \le 1$. A function $f \in \Sigma$ is said to be in the class $\sum_q^{\mu,\beta}(\lambda,\alpha,\psi_L)$, if all of the next subordination conditions are met

$$\left(\frac{z}{f(z)}\right)^{1-\mu} \frac{D_{q,\alpha,\lambda}^{n+1} f(z)}{z} \prec \psi_L(z), \quad z \in D,$$

and

$$\left(\frac{w}{f(w)}\right)^{1-\mu} \frac{D_{q,\alpha,\lambda}^{n+1} g(w)}{w} \prec \psi_L(w), \quad w \in D,$$

where $g(w) = f^{-1}(w)$ and $\psi_L(z)$ is given by (42).

Theorem 3.10. *Let* $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$ and $0 \le \beta \le 1$. *If a function f belongs to the class* $\sum_q^{\mu,\beta}(\lambda,\alpha,\psi_L)$, *defined by* (1), *then we have*

$$|a_2| \le \min\left\{ \begin{array}{c} \dfrac{\sqrt{2}q(1-\beta)}{\Lambda_2^n - 1 + \mu}, \\ \dfrac{q(1-\beta)}{\sqrt{2(1-\beta)q\left[2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu)\right] + 2(1-\frac{2q}{1-q^2})\left(\Lambda_2^n - (1-\mu)\right)^2}} \end{array} \right\},$$

and

$$|a_3| \le \frac{q(1-\beta)}{|\Lambda_3^n - (1-\mu)|} + \min\left\{ \begin{array}{c} \dfrac{2q^2(1-\beta)^2}{(\Lambda_2^n - (1-\mu))^2}, \\ \dfrac{2q^2(1-\beta)^2}{q(1-\beta)\left(2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu)\right) + (1-\frac{2q}{1-q^2})(\Lambda_2^n - 1 + \mu)^2} \end{array} \right\},$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

Following is a corollary, which calculates the Fekete-Szego problem for the function class $\sum_q^{\mu,\beta}(\lambda,\alpha,\psi_L)$, by using the coefficients a_2 and a_3.

Corollary 3.11. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$, $0 \le \beta \le 1$ *and* $n \in \mathbb{C}$. *If* $f \in \sum_q^{\mu,\beta}(\lambda,\alpha,\psi_L)$, *then we have*

$$|a_3 - \eta a_2^2| \le q(1-\beta)\begin{cases} \dfrac{1}{\Lambda_3^n - (1-\mu)}, & 0 \le \Theta\beta(\eta) < \dfrac{1}{2(\Lambda_3^n - (1-\mu))}, \\ 2\Theta\beta(\eta), & \Theta\beta(\eta) \ge \dfrac{1}{2(\Lambda_3^n - (1-\mu))}, \end{cases}$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$, and

$$\Theta_\beta(\eta) = \frac{q(1-\eta)}{q(1-\beta)\left(2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu)\right) + (1-\frac{2q}{1-q^2})(\Lambda_2^n - (1-\mu))^2}.$$

Corollary 3.12. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$ *and* $0 \le \beta \le 1$. *If* $f \in \sum_q^{\mu,\beta}(\lambda,\alpha,\psi_L)$, *then we have*

$$|a_3 - a_2^2| \le \frac{q(1-\beta)}{\Lambda_3^n - (1-\mu)},$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

Definition 3.13. Let $0 \le q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$ and $0 < \gamma < 1$. A function $f \in \Sigma$ is said to be in the class $\sum_q^{\mu,\gamma}(\lambda,\alpha,\widehat{\psi}_L)$, if each of the following subordination conditions holds true:

$$\left(\frac{z}{f(z)}\right)^{1-\mu} \frac{D_{q,\alpha,\lambda}^{n+1} f(z)}{z} \prec \widehat{\psi}_L(z), \quad z \in D,$$

and

$$\left(\frac{w}{f(w)}\right)^{1-\mu} \frac{D_{q,\alpha,\lambda}^{n+1} g(w)}{w} \prec \widehat{\psi}_L(w), \quad w \in D,$$

where $g(w) = f^{-1}(w)$ and $\widehat{\psi}_L(z)$ is given by (42).

Theorem 3.14. *Let* $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$ *and* $0 < \gamma < 1$. *If a function f belongs to the class* $\sum_q^{\mu,\gamma}(\lambda,\alpha,\widehat{\psi}_L)$, *defined by* (1), *then we have*

$$|a_2| \le \min\left\{ \begin{array}{c} \dfrac{2\gamma q}{\Lambda_2^n - 1 + \mu}, \\ \dfrac{2\gamma q}{\sqrt{\gamma q\left[2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu)\right] + (1 - q(\frac{2}{1-q^2} + \gamma - 1))\left(\Lambda_2^n - (1-\mu)\right)^2}} \end{array} \right\},$$

and

$$|a_3| \le \frac{2q\gamma}{|\Lambda_3^n - (1-\mu)|} + \min\left\{ \begin{array}{c} \dfrac{4q^2\gamma^2}{(\Lambda_2^n - (1-\mu))^2}, \\ \dfrac{4q^2\gamma^2}{\gamma q\left(2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu)\right) + (1 - q(\frac{2}{1-q^2} + \mu - 1))(\Lambda_2^n - 1 + \mu)^2} \end{array} \right\},$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

Following is a corollary, which establishes the Fekete-Szego problem for the functions class $\sum_q^{\mu,\gamma}(\lambda,\alpha,\widehat{\psi}_L)$ by using the coefficients a_2 and a_3.

Corollary 3.15. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$, $0 < \gamma < 1$ $n \in \mathbb{C}$. *If* $f \in \sum_q^{\mu,\gamma}(\lambda,\alpha,\widehat{\psi}_L)$, *then we have*

$$|a_3 - \eta a_2^2| \le \gamma q \begin{cases} \dfrac{2}{\Lambda_3^n - (1-\mu)}, & 0 \le \Theta\gamma(\eta) < \dfrac{1}{2(\Lambda_3^n - (1-\mu))}, \\ 4\Theta\gamma(\eta), & \Theta\gamma(\eta) \ge \dfrac{1}{2(\Lambda_3^n - (1-\mu))}, \end{cases}$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$, and

$$\Theta_\gamma(\eta) = \frac{4(1-\eta)}{\gamma q\left[2\Lambda_3^n + (1-\mu)(2-\mu) + 2\Lambda_2^n(1-\mu)\right] + (1 - q(\frac{2}{1-q^2} + \gamma - 1))\left(\Lambda_2^n - (1-\mu)\right)^2}.$$

Corollary 3.16. *Let f be a function defined by* (1), $g(w) = f^{-1}(w)$, $0 < q \le 1$, $\alpha \ge 0$, $\lambda \ge 0$, $0 \le \mu \le 1$ *and* $0 < \gamma < 1$. *If* $f \in \sum_q^{\mu,\gamma}(\lambda,\alpha,\widehat{\psi}_L)$, *then we have*

$$|a_3 - a_2^2| \le \frac{2\gamma q}{\Lambda_3^n - (1-\mu)},$$

where Λ_k^n *is given by* (4), $k = 2, 3, \ldots$ *and* $n \in \mathbb{N}$.

4. Conclusions

This article introduces certain classes of bi-univalent functions through the use of differential subordinations and specific generalized q-Salagean differential operators. These classes of bi-univalent functions, influenced by the q-logarithm function, are formulated within an open unit disc. Additionally, two operators, referred to as $\Psi_L(z)$ and $\widehat{\Psi}_L(z)$, are examined. Furthermore, the article explores and obtains various Fekte-Szego results concerning the coefficients $|a_2|$ and $|a_3|$.

References

[1] Gasper, G. Rahman, M. (2004). Basic Hypergeometric Series, 2nd edn. Cambridge university Press, Cambridge.

[2] Annaby, M. H., & Mansour, Z. S. (2012). q-Fractional Calculus and Equations. Springer Publishing.

[3] AlOmari, S. K. Q. (2020). On a qLaplacetype integral operator and certain class of series expansion. Mathematical Methods in the Applied Sciences.

[4] Al-Omari, S. (2021). Estimates and properties of certain q-Mellin transform on generalized q-calculus theory. *Advances in Difference Equations*, 2021**(1)**.

[5] Osburn, R., & Schneider, C. (2009). Gaussian Hypergeometric series and supercongruences. *Mathematics of Computation*, **78(265)**, 275.

[6] Jackson, F. H. (1910). On q-definite integrals, *Quart. J. Pure Appl. Math*. **41**, 193–203.

[7] Srivastava, H. M. (1989). Univalent functions, fractional calculus, and associated generalized hypergeometric functions, In: H. M. Srivastava and S. Owa, Editors, Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 329354.

[8] Kanas, S., & Rducanu, D. (2014). Some class of analytic functions related to conic domains. *Mathematica Slovaca*, **64(5)**, 11831196.

[9] Srivastava, H. M., Khan, S., Ahmad, Q. Z., Khan, N., & Hussain, S. (2018). The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator. *Studia Universitatis Babes-Bolyai Matematica*, **63(4)**, 419436.

[10] Kanas, S., Altinkaya, ., & Yalin, S. (2019). Subclass of k-Uniformly Starlike Functions Defined by the Symmetric q-Derivative Operator. *Ukrainian Mathematical Journal*, **70(11)**, 17271740.

[11] Srivastava, H. M., Khan, N., Khan, S., Ahmad, Q. Z., & Khan, B. (2021). A Class of k-Symmetric Harmonic Functions Involving a Certain q-Derivative Operator. *Mathematics*, **9(15)**, 1812.

[12] Srivastava, H. M. (2020). Operators of Basic (or q-) Calculus and Fractional q-Calculus and Their Applications in Geometric Function Theory of Complex Analysis. *Iranian Journal of Science and Technology, Transactions A: Science*, **44(1)**, 327–344.

[13] Arif, M., Srivastava, H. M., & Umar, S. (2018). Some applications of a q-analogue of the Ruscheweyh type operator for multivalent functions. Revista de La Real Academia de Ciencias Exactas, Fsicas y Naturales. *Serie A. Matemticas*, **113(2)**, 12111221.

[14] Ismail, M. E. H., Merkes, E., & Styer, D. (1990). A generalization of starlike functions. *Complex Variables, Theory and Application: An International Journal*, **14(14)**, 7784.

[15] Khan, S., Hussain, S., Darus, M. (2021). Inclusion relations of q-Bessel functions associated with generalized conic domain. A*IMS Mathematics*, **6(4)**, 36243640.

[16] Zhang, X., Khan, S., Hussain, S., Tang, H., & Shareef, Z. (2020). New subclass of q-starlike functions associated with generalized conic domain, *AIMS Mathematics*, **5(5)**, 4830–4848.

[17] Seoudy, T. M. (2022). Some subclasses of univalent functions associated with -Ruscheweyh derivative operator. *Ukrainskyi Matematychnyi Zhurnal*, **74(1)**, 122136.

[18] Matala-aho, T., Vnnen, K., & Zudilin, W. (2005). New irrationality measures for q-logarithms. *Mathematics of Computation*, **75(254)**, 879889.

[19] Yamano, T. (2002). Some properties of q-logarithm and q-exponential functions in Tsallis statistics. *Physica A: Statistical Mechanics and Its Applications*, **305(34)**, 486496.

[20] Graham, I, & Kohr, G. (2003). Geometric Function Theory in One and Higher Dimensions (1ste editie). Amsterdam University Press.

[21] Amini, E., Al-Omari, S., & Rahmatan, H. (2022b). On geometric properties of cer tain subclasses of univalent functions defined by Noor integral operator. *Analysis*, **0(0)**. https://doi.org/10.1515/anly-2022-1043

[22] Amini, E., Fardi, M., Al-Omari, S., & Nonlaopon, K. (2022). Results on Univalent Functions Defined by q-Analogues of Salagean and Ruscheweh Operators. *Symmetry*, **14(8)**, 1725.

[23] Miller, S. S., & Mocanu, P. T. (2000). Differential Subordinations. Taylor & Francis.

[24] Duren, P. L. (1983). Univalent Functions, in: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo.

[25] Xu, Q. H., Xiao, H. G., & Srivastava, H. (2012). A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. *Applied Mathematics and Computation*, **218(23)**, 1146111465.

[26] Caglar, M., Orhan, H., & Yagmur, N. (2013). Coefficient bounds for new subclasses of bi-univalent functions. *Filomat*, **27(7)**, 11651171.

[27] Hussain, S., Khan, S., Zaighum, M. A., & Darus, M. (2018). On certain classes of bi-univalent functions related to m-fold symmetry. *Journal of Nonlinear Sciences and Applications*, **11(04)**, 490499.

[28] Salagean, G. S. (1983). subclass of univalent functios, Lecture Note in Math, Springer-Verlag, Berlin, **1013**(1983), 362372.

[29] Alb Lupa, A. (2011). On special differential superordinations using a generalized Slgean operator and Ruscheweyh derivative. *Computers & Mathematics with Applications*, **61(4)**, 10481058.

[30] Al-Oboudi, F. M. (2004). On univalent functions defined by a generalized Slgean operator. *International Journal of Mathematics and Mathematical Sciences*, 2004**(27)**, 14291436.

[31] Bulut, S. (2008). A New Subclass of Analytic Functions Defined by Generalized Ruscheweyh Differential Operator. *Journal of Inequalities and Applications*, 2008**(1)**, 134932.

[32] Oros, G. I., & Oros, G. (2008). On a class of univalent functions defined by a generalized Slgean operator. *Complex Variables and Elliptic Equations*, **53(9)**, 869877.

[33] Darus, M. & Ibrahim, W. (2011) On New Subclasses of Analytic Functions Involving Gener alized Differential and Integral Operators, *Eur. J. Pure Appl. Math.* **1(4)**, 59–66.