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A study of neutrosophic controlled pentagonal metric space with applications

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Abstract

In this manuscript, we present the concept of neutrosophic controlled pentagonal metric space (NCPMS), and prove some new fixed point results. Furthermore, we established many interesting outcomes for contraction maps. At last, we show the uniqueness and existence results for fractional differential and integral equations to illustrate the validity of the main outcomes.

Keywords: Neutrosophic metric space; neutrosophic controlled pentagonal metric space; fixed point theorem; fractional differential equation

Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The concept of metric spaces (MS) and the Banach contraction principle are the backbone of fixed-point theory. Several academics are drawn to the spaciousness of the axiomatic interpretation of MS. There have been several generalizations about MS. This illustrates the beauty, attraction, and scope of the notion of a MS.

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The idea of fuzzy sets (FSs) was introduced by Zadeh [1]. In modern research involving the set-theoretical underpinnings and logic of mathematics, the adjective "fuzzy" appears to be a particularly popular and common one. In our opinion, the fundamental explanation for this rapid development. There is a great deal of confusion in the world around us for the following reasons: the information we develop from the environment, the concepts we employ, and the facts arising from our observations or measurements are all unreliable and inaccurate in general. For this, any formal description of the real world or some of its characteristics is naturally idealized and approximates reality as it exists. FSs, fuzzy orderings, fuzzy languages, and other ideas enable us to approach and explore the level of uncertainty mentioned above in a rigorously formal and mathematical way.

The concept of FSs was effective at modifying various mathematical concepts present in its idea. In 1960, Schweizer and Sklar [2] defined the concept of continuous t-norms. Fuzzy MS notion was introduced by Kramosil and Michalek [3], in 1975. The authors had given the notion of fuzziness, via continuous t-norms, to classical concepts of metric and MS and compared the idea of alternative MS generalizations, specifically statistical and probabilistic types. In 1988, Garbiec [4] demonstrated the fuzzy interpretation of the Banach contraction principle in fuzzy MS. Recently, Ur-Reham et al. [5] demonstrated some α–ϕ-fuzzy cone contraction theorems with application of integral type equations, in 2021.

Furthermore, Park [6] established an intuitionistic fuzzy MS that supports both membership and non-membership maps. Konwar [7] given the notion of an intuitionistic fuzzy b-MS and proved a few fixed point theorems. The notion of intuitionistic fuzzy double-controlled metric-like spaces was introduced and fixed point results were demonstrated by Ishtiaq et al. [8]. With the use of a relation, Saleem et al. [9] developed the idea of graphical fuzzy metric spaces, a generalization of fuzzy metric spaces, and demonstrated fixed point outcomes. Saleem et al. [10] established fixed point results and presented the idea of a fuzzy triple-controlled metric-like space in the sense that the self-distance might not equal one. Saleem [11] introduced the notions of fuzzy rectangular and fuzzy b-rectangular metric-like spaces and proved fixed point results. Saleem et al. [12] established fixed point results and presented the concepts of controlled rectangular fuzzy metric-like spaces and extended b-rectangular spaces. Furqan et al. [13] introduced the notion of Fuzzy n-controlled metric space and proved fixed point theorems. As expansions of fuzzy triple-controlled metric spaces and fuzzy extended hexagonal b-metric spaces, Hussain et al. [14] introduced pentagonal-controlled fuzzy metric spaces and fuzzy controlled hexagonal metric spaces, and established fixed point results. In 2020, Kirişci et al. [15] initiated the concept of neutrosophic MS that is used to deal with naturalness, non-membership, and membership. Simsek and Kirişci [16] proved fixed point theorems in the surrounding area of neutrosophic metric space (NMS). In the year 2020, Sowndrarajan et al. [17] demonstrated some fixed point theorems in NMS. Itoh [18] explained an application respecting random differential equations in Banach spaces. Mlaiki [19] coined the notion of controlled MS and demonstrated several fixed-point theorems for the contraction map. Sezen [20] initiated the concept of controlled fuzzy MS and derived a distinct type contraction map. For related articles, see [21–28]. In 2022, Gunaseelan et al. [29] introduced neutrosophic rectangular triple-controlled MS and proved fixed point theorems (FPTs). In 2023, Gunaseelan, et al. [30] introduced orthogonal neutrosophic rectangular MS and proved FPTs. Using an iterated multifunction system that consists of a finite number of neutrosophic B-contractions and neutrosophic Edelstein contractions, Saleem et al. [31] proposed the idea of multivalued fractals in neutrosophic metric spaces. As a generalization of neutrosophic metric spaces, Uddin [32] introduced the concept of controlled neutrosophic metric-like spaces and demonstrated fixed point findings. The notions of ξ-chainable neutrosophic metric space and generalized neutrosophic cone metric spaces were presented by Riaz et al. [33], which also demonstrated fixed point findings. Neutronosophic 2-metric spaces demonstrated fixed point theorems on generalized neutrosophic cone metric spaces were first proposed by Ishtiaq et al. [34], which also showed common fixed point findings. In 2023, Gunaseelan et al. [35] introduced the concept of neutrosophic pentagonal MS and proved FPTs.

In this article, we introduce the concept of NCPMS and prove FPTs. The following are the main aspects of this paper:

- To introduce the concept of NCPMS;
- To show several FPTs for contraction functions;
- To find the uniqueness and existence of the solution of an integral, and fractional differential equation (FDE).

2. Preliminaries

In this part, we offer some definitions to help readers clarify the key findings.

Definition 2.1 *([6]) Let* $\circledast : [0,1]\times[0,1]\to[0,1]$ *be a binary operation is called a continuous triangle norm if: [label=()]*

1. $\upsilon \circledast h_\ell = h_\ell \circledast \upsilon, \forall \upsilon, h_\ell \in [0,1]$;
2. $\circledast$ is continuous;
3. $\upsilon \circledast 1 = \upsilon, \forall \upsilon \in [0,1]$;
4. $(\upsilon \circledast h_\ell)\circledast i = \upsilon \circledast (h_\ell \circledast i), \forall \upsilon, h_\ell, i \in [0,1]$;
5. If $\upsilon \le i$ and $h_\ell \le d$, with $\upsilon, h_\ell, i,\ d \in [0,1]$, then .

Definition 2.2 *([6]) A binary operation* $\odot : [0,1]\times[0,1]\to[0,1]$ *is said to be a continuous triangle co-norm if:*

1. $\upsilon \odot h_\ell = h_\ell \odot \upsilon, \forall \upsilon, h_\ell \in [0,1]$;
2. $\odot$ is continuous;
3. $\upsilon \odot 0 = 0, \forall \upsilon \in [0,1]$;
4. $(\upsilon \odot h_\ell)\odot i = \upsilon \odot (h_\ell \odot i), \forall \upsilon, h_\ell, i \in [0,1]$;
5. If $\upsilon \le i$ and $i \le d$, with $\upsilon, h_\ell, i, d \in [0,1]$, then $\upsilon \odot h_\ell \le i \odot d$.

Definition 2.3 *([7]) Let* $\mho^\star \neq \emptyset$, $\circledast$ *be a continuous t-norm,* $\odot$ *be a continuous t-co-norm,* $\mathfrak{b} \ge 1$ *and* Λ Φ *be FSs on* $\mho^\star\times\mho^\star\times(0,+\infty)$. *If* $(\mho^\star, \Lambda, \Phi, \circledast, \odot)$ *fullfils all* $\Theta, \curlyvee_2 \in \mho^\star$ *and š*$\aleph' > 0$*:*

1. $\Lambda(\Theta, \curlyvee_2, \aleph') + \Phi(\Theta, \curlyvee_2, \aleph') \le 1$;
2. $\Lambda(\Theta, \curlyvee_2, \aleph') > 0$;
3. $\Lambda(\Theta, \curlyvee_2, \aleph') = 1 \Leftrightarrow \Theta = \curlyvee_2$;
4. $\Lambda(\Theta, \curlyvee_2, \aleph') = \Lambda(\curlyvee_2, \Theta, \aleph')$;
5. $\Lambda(\Theta, \varrho, \mathfrak{b}(\aleph' + a)) \ge \Lambda(\Theta, \curlyvee_2, \aleph') \circledast \Lambda(\curlyvee_2, \varrho, a)$;
6. $\Lambda(\Theta, \curlyvee_2, \cdot)$ is a non-decreasing map of $\Re^+$ and $\lim_{\aleph'\to+\infty} \Phi(\Theta, \curlyvee_2, \aleph') = 1$;
7. $\Phi(\Theta, \curlyvee_2, \aleph') > 0$;
8. $\Phi(\Theta, \curlyvee_2, \aleph') = 0$ iff $\Theta = \curlyvee_2$;
9. $\Phi(\Theta, \curlyvee_2, \aleph') = \Phi(\curlyvee_2, \Theta, \aleph')$;
10. $\Phi(\Theta, \varrho, \mathfrak{b}(\aleph' + a)) \le \Phi(\Theta, \curlyvee_2, \aleph') \odot \Phi(\curlyvee_2, \varrho, a)$;
11. $\Phi(\Theta, \curlyvee_2, \cdot)$ is a non-increasing map of $\Re^+$ and $\lim_{\aleph'\to+\infty} \Phi(\Theta, \curlyvee_2, \aleph') = 0$,

Hence $(\mho^\star, \Lambda, \Phi, \circledast, \odot)$ is an intuitionistic fuzzy $\mathfrak{b}$-MS.

Definition 2.4 *([15]) Let* $\mho^\star \neq \emptyset, \circledast$ *is a continuous t-norm,* $\odot$ *be a continuous t-co-norm, and* Λ, Φ, S *are NS's (neutrosophic sets) on* $\mho^\star\times\mho^\star\times(0,+\infty)$ *is called a neutosophic metric on* $\mho^\star$, *if* $\forall \Theta, \curlyvee_2, \varrho \in \mho^\star$, *the given axioms are fulfilled:*

1. $\Lambda(\Theta, \curlyvee 2, \aleph') + \Phi(\Theta, \curlyvee 2, \aleph') + S(\Theta, \curlyvee 2, \aleph') \le 3$;
2. $\Lambda(\Theta, \curlyvee 2, \aleph') > 0$;

3. $\Lambda(\Theta,\Upsilon_2,\aleph) = 1, \ \forall \ \aleph > 0 \Leftrightarrow \ \Theta = \Upsilon_2$;
4. $\Lambda(\Theta,\Upsilon_2,\aleph) = \Lambda(\Upsilon_2,\Theta,\aleph)$;
5. $\Lambda(\Theta,\varrho,\aleph + a) \geq \Lambda(\Theta,\Upsilon_2,\aleph) \circledast \Lambda(\Upsilon_2,\varrho,a)$;
6. $\Lambda(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ is continuous and $\lim_{\aleph \to +\infty} \Lambda(\Theta,\Upsilon_2,\aleph) = 1$;
7. $\Phi(\Theta,\Upsilon_2,\aleph) < 1$;
8. $\Phi(\Theta,\Upsilon_2,\aleph) = 0 \ \forall \ \aleph > 0 \Leftrightarrow \Theta = \Upsilon_2$;
9. $\Phi(\Theta,\Upsilon_2,\aleph) = \Phi(\Upsilon_2,\Theta,\aleph)$;
10. $\Phi(\Theta,\varrho,\aleph + \mathrm{a}) \leq \Phi(\Theta,\Upsilon_2,\aleph) \odot \Phi(\Upsilon_2,\varrho,\mathrm{a})$;
11. $\Phi(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ is continuous and $\lim_{\aleph \to +\infty} \Phi(\Theta,\Upsilon_2,\aleph) = 0$;
12. $S(\Theta,\Upsilon_2,\aleph) < 1$;
13. $S(\Theta,\Upsilon_2,\aleph) = 0 \ \forall \ \aleph > 0$ iff $\Theta = \Upsilon_2$;
14. $S(\Theta,\Upsilon_2,\aleph) = S(\Upsilon_2,\Theta,\aleph)$;
15. $S(\Theta,\varrho,\aleph + \breve{s}) \leq S(\Theta,\Upsilon_2,\aleph) \odot S(\Upsilon_2,\varrho,\breve{s})$;
16. $S(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ is continuous and $\lim_{\aleph \to +\infty} S(\Theta,\Upsilon_2,\aleph) = 0$;
17. If $\aleph \leq 0$, then $\Lambda(\Theta,\Upsilon_2,\aleph) = 0, \Phi(\Theta,\Upsilon_2,\aleph) = 0$;

Then, $(\mho^\star,\Lambda,\Phi,S,\circledast,\odot)$ is called a neutrosophic MS.

Definition 2.5 *[35] Let* $\mho^\star \neq \emptyset, \circledast$ *be a continuous t-norm,* $\odot$ *is a continuous t-co-norm and* Λ,Φ,Ξ *be NS′s on* $\mho^\star \times \mho^\star \times (0,+\infty)$ *is called neutrosophic pentagonal metric on* $\mho^\star$*, if for every* $\Theta,\varrho \in \mho^\star$ *and all different* $\ddot{g},\Upsilon_2,\varrho \in \mho^\star$*, the following axioms are fulfilled:*

1. $\Lambda(\Theta,\Upsilon_2,\aleph) + \Phi(\Theta,\Upsilon_2,\aleph) + \Xi(\Theta,\Upsilon_2,\aleph) \leq 3$;
2. $\Lambda(\Theta,\Upsilon_2,\aleph) > 0$;
3. $\Lambda(\Theta,\Upsilon_2,\aleph) = 1, \ \forall \ \aleph > 0$ *iff* $\Theta = \Upsilon_2$;
4. $\Lambda(\Theta,\Upsilon_2,\aleph) = \Lambda(\Upsilon_2,\Theta,\aleph)$;
5. $\Lambda(\Theta,\varrho,\aleph + \breve{s} + \ddot{q} + \ddot{c}) \geq \Lambda(\Theta,\Upsilon_2,\aleph) \circledast \Lambda(\Upsilon_2,\ddot{g},a) \circledast \Lambda(\ddot{g},\ddot{h},\ddot{q}) \circledast \Lambda(\ddot{h},\varrho,\ddot{c})$;
6. $\Lambda(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ is continuous and $\lim_{\aleph \to +\infty} \Lambda(\Theta,\Upsilon_2,\aleph) = 1$;
7. $\Phi(\Theta,\Upsilon_2,\aleph) < 1$;
8. $\Phi(\Theta,\Upsilon_2,\aleph) = 0, \ \forall \ \aleph > 0 \Leftrightarrow \Theta = \Upsilon_2$;
9. $\Phi(\Theta,\Upsilon_2,\aleph) = \Phi(\Upsilon_2,\Theta,\aleph)$;
10. $\Phi(\Theta,\varrho,\aleph + \breve{s} + \ddot{q} + \ddot{c}) \leq \Phi(\Theta,\Upsilon_2,\aleph) \odot \Phi(\Upsilon_2,\ddot{g},\breve{s}) \odot \Phi(\ddot{g},\ddot{h},\ddot{q}) \odot \Phi(\ddot{h},\varrho,\ddot{c})$;
11. $\Phi(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ is continuous and $\lim_{\aleph \to +\infty} \Phi(\Theta,\Upsilon_2,\aleph) = 0$;
12. $\Xi(\Theta,\Upsilon_2,\aleph) < 1$;
13. $\Xi(\Theta,\Upsilon_2,\aleph) = 0, \ \forall \ \aleph > 0 \Leftrightarrow \Theta = \Upsilon_2$;
14. $\Xi(\Theta,\Upsilon_2,\aleph) = \Xi(\Upsilon_2,\Theta,\aleph)$;
15. $\Xi(\Theta,\varrho,\aleph + \breve{s} + \ddot{q} + \ddot{c}) \leq \Xi(\Theta,\Upsilon_2,\aleph) \odot \Xi(\Upsilon_2,\ddot{g},\breve{s}) \odot \Xi(\ddot{g},\ddot{h},\ddot{q}) \odot \Xi(\ddot{h},\varrho,\ddot{c})$;
16. $\Xi(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ is continuous and $\lim_{\aleph \to +\infty} \Xi(\Theta,\Upsilon_2,\aleph) = 0$;
17. If $\aleph \leq 0$, then $\Lambda(\Theta,\Upsilon_2,\aleph) = 0, \Phi(\Theta,\Upsilon_2,\aleph) = 1$ and $S(\Theta,\Upsilon_2,\aleph) = 1$.

Then, $(\mho^\star,\Lambda,\Phi,\Xi,\circledast,\odot)$ is said to be a neutrosophic pentagonal MS.

In this article, we propose the concept of NCPMS and show that FPTs.

3. Main results

In this part, we present few fixed point theorems in NCPMS.

Definition 3.1 *Let* $\mho^\star \neq \emptyset$ *and* $\xi : \mho^\star \times \mho^\star \to [1,+\infty)$ *be a non-comparable maps,* $\circledast, \odot$ *are the continuous t-norm and t-co-norm, and* Λ, Φ, Ξ *is a NS's on* $\mho^\star \times \mho^\star \times (0,+\infty)$ *is called neutrosophic controlled pentagonal metric on* $\mho^\star$*, if for any* $\Theta, \varrho \in \mho^\star$ *and all distinct* $\ddot{g}, \Upsilon_2, \ddot{h}, \varrho \in \mho^\star$*, the following axioms are fulfilled:*

1. $\Lambda(\Theta,\Upsilon_2,\aleph) + \Phi(\Theta,\Upsilon_2,\aleph) + \Xi(\Theta,\Upsilon_2,\aleph) \leq 3$;
2. $\Lambda(\Theta,\Upsilon_2,\aleph) > 0$;
3. $\Lambda(\Theta,\Upsilon_2,\aleph) = 1, \ \forall \ \aleph > 0$ *iff* $\Theta = \Upsilon_2$;
4. $\Lambda(\Theta,\Upsilon_2,\aleph) = \Lambda(\Upsilon_2,\Theta,\aleph)$;
5. $\Lambda(\Theta,\varrho,\aleph + \breve{s} + \ddot{q} + \ddot{c}) \geq \Lambda(\Theta,\Upsilon_2,\frac{\aleph}{\xi(\Theta,\Upsilon_2)}) \circledast \Lambda(\Upsilon_2,\ddot{g},\frac{\breve{s}}{\xi(\Upsilon_2,\ddot{g})}) \circledast \Lambda(\ddot{g},\ddot{h},\frac{\ddot{q}}{\xi(\ddot{g},\ddot{h})}) \circledast \Lambda(\ddot{h},\varrho,\frac{\ddot{c}}{\xi(\ddot{h},\varrho)})$;
6. $\Lambda(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ is continuous and $\lim_{\aleph \to +\infty} \Lambda(\Theta,\Upsilon_2,\aleph) = 1$;
7. $\Phi(\Theta,\Upsilon_2,\aleph) < 1$;
8. $\Phi(\Theta,\Upsilon_2,\aleph) = 0, \ \forall \ \aleph > 0 \Leftrightarrow \Theta = \Upsilon_2$;
9. $\Phi(\Theta,\Upsilon_2,\aleph) = \Phi(\Upsilon_2,\Theta,\aleph)$;
10. $\Phi(\Theta,\varrho,\aleph + \breve{s} + \ddot{q} + \ddot{c}) \leq \Phi(\Theta,\Upsilon_2,\frac{\aleph}{\xi(\Theta,\Upsilon_2)}) \odot \Phi(\Upsilon_2,\ddot{g},\frac{\breve{s}}{\xi(\Upsilon_2,\ddot{g})}) \odot \Phi(\ddot{g},\ddot{h},\frac{\ddot{q}}{\xi(\ddot{g},\ddot{h})}) \odot \Phi(\ddot{h},\varrho,\frac{\ddot{c}}{\xi(\ddot{h},\varrho)})$;
11. let $\Phi(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ be a continuous and $\lim_{\aleph \to +\infty} \Phi(\Theta,\Upsilon_2,\aleph) = 0$;
12. $\Xi(\Theta,\Upsilon_2,\aleph) < 1$;
13. $\Xi(\Theta,\Upsilon_2,\aleph) = 0, \ \forall \ \aleph > 0 \Leftrightarrow \Theta = \Upsilon_2$;
14. $\Xi(\Theta,\Upsilon_2,\aleph) = \Xi(\Upsilon_2,\Theta,\aleph)$;
15. $\Xi(\Theta,\varrho,\aleph + \breve{s} + \ddot{q} + \ddot{c}) \leq \Xi(\Theta,\Upsilon_2,\frac{\aleph}{\xi(\Theta,\Upsilon_2)}) \odot \Xi(\Upsilon_2,\ddot{g},\frac{\breve{s}}{\xi(\Upsilon_2,\ddot{g})}) \odot \Xi(\ddot{g},\ddot{h},\frac{\ddot{q}}{\xi(\ddot{g},\ddot{h})}) \odot \Xi(\ddot{h},\varrho,\frac{\ddot{c}}{\xi(\ddot{h},\varrho)})$;
16. let $\Xi(\Theta,\Upsilon_2,\cdot) : (0,+\infty) \to [0,1]$ be a continuous and $\lim_{\aleph \to +\infty} \Xi(\Theta,\Upsilon_2,\aleph) = 0$;
17. If $\aleph \leq 0$, then $\Lambda(\Theta,\Upsilon_2,\aleph) = 0, \Phi(\Theta,\Upsilon_2,\aleph) = 1$ and $S(\Theta,\Upsilon_2,\aleph) = 1$.

Then, $(\mho^\star, \Lambda, \Phi, \Xi, \circledast, \odot)$ is called a NCPMS.

Example 3.1 *Let* $\mho^\star = \{1,2,3,4,5\}$ *and* $\xi : \mho^\star \times \mho^\star \to [1,+\infty)$ *be a function given by* $\xi(\Theta,\Upsilon_2) = \Theta + \Upsilon_2 + 1$. *Define* $\Lambda, \Phi, \Xi : \mho^\star \times \mho^\star \times (0,+\infty) \to [0,1]$ *as*

$$\Lambda(\Theta,\Upsilon_2,\aleph) = \begin{cases} 1, & \text{if} \quad \Theta = \Upsilon_2 \\ \dfrac{\aleph}{\aleph + \max\{\Theta,\Upsilon_2\}}, & \text{if} \quad \text{otherwise,} \end{cases}$$

$$\Lambda(\Theta,\Upsilon_2,\aleph) = \begin{cases} 0, & \text{if} \quad \Theta = \Upsilon_2 \\ \dfrac{\max\{\Theta,\Upsilon_2\}}{\aleph + \max\{\Theta,\Upsilon_2\}}, & \text{if} \quad \text{otherwise,} \end{cases}$$

and

$$\Xi(\Theta,\Upsilon_2,\aleph) = \begin{cases} 0, & \text{if} \quad \Theta = \Upsilon_2 \\ \dfrac{\max\{\Theta,\Upsilon_2\}}{\aleph}, & \text{if} \quad \text{otherwise.} \end{cases}$$

Then, $(\mho^{\star}, \Lambda, \Phi, \Xi, \circledast, \odot)$ is a NCPMS with continuous t-norm $\upsilon \circledast h_{\ell} = \upsilon h_{\ell}$, and continuous t-co-norm, $\upsilon \odot \bar{a} = \max\{\upsilon, \bar{a}\}$.

Proof. Now, we prove 3.1, 3.2 and 3.3 remains are clear.
Let $\Theta = 1$, $\Upsilon_2 = 2$, $\ddot{g} = 3$, $\ddot{h} = 4$ and $\varrho = 5$. Then

$$\Lambda(1,5,\aleph + \breve{s} + \ddot{q} + \ddot{k}) = \frac{\aleph + \breve{s} + \ddot{q} + \ddot{k}}{\aleph + \breve{s} + \ddot{q} + \ddot{k} + \max\{1,5\}} = \frac{\aleph + \breve{s} + \ddot{q} + \ddot{k}}{\aleph + \breve{s} + \ddot{q} + \ddot{k} + 5}.$$

Otherwise,

$$\Lambda(1,2,\frac{\aleph}{\xi(1,2)}) = \frac{\frac{\aleph}{\xi(1,2)}}{\frac{\aleph}{\xi(1,2)} + \max\{1,2\}} = \frac{\aleph}{\aleph + 8},$$

$$\Lambda(2,3,\frac{\breve{s}}{\xi(2,3)}) = \frac{\frac{\breve{s}}{\xi(2,3)}}{\frac{\breve{s}}{\xi(2,3)} + \max\{2,3\}} = \frac{\breve{s}}{\breve{s} + 18},$$

$$\Lambda(3,4,\frac{\ddot{q}}{\xi(3,4)}) = \frac{\frac{\ddot{q}}{\xi(3,4)}}{\frac{\ddot{q}}{\xi(3,4)} + \max\{3,4\}} = \frac{\ddot{q}}{\ddot{q} + 32}$$

and

$$\Lambda(4,5,\frac{\ddot{k}}{\xi(4,5)}) = \frac{\frac{\ddot{k}}{\xi(4,5)}}{\frac{\ddot{k}}{\xi(4,5)} + \max\{4,5\}} = \frac{\ddot{k}}{\ddot{k} + 50}.$$

That is,

$$\frac{\aleph + \breve{s} + \ddot{q} + \ddot{k}}{\aleph + \breve{s} + \ddot{q} + \ddot{k} + 5} \geq \frac{\aleph}{\aleph + 8} \frac{\breve{s}}{\breve{s} + 18} \frac{\ddot{q}}{\ddot{q} + 32} \frac{\ddot{k}}{\ddot{k} + 50}.$$

Then it satisfies all $\aleph, \breve{s}, \ddot{q}, \ddot{k} > 0$. Hence,

$$\Lambda(\Theta, \epsilon, \aleph + \breve{s} + \ddot{q} + \ddot{k}) \geq \Lambda(\Theta, \Upsilon_2, \aleph) \circledast \Lambda(\Upsilon_2, \ddot{q}, \breve{s}) \circledast \Lambda(\ddot{g}, \varrho, \ddot{q}) \circledast \Lambda(\varrho, \epsilon, \ddot{k}).$$

Now,

$$\Phi(1,5,\aleph + \breve{s} + \ddot{q} + \ddot{k}) = \frac{\max\{1,5\}}{\aleph + \breve{s} + \ddot{q} + \ddot{k} + \max\{1,5\}} = \frac{5}{\aleph + \breve{s} + \ddot{q} + \ddot{k} + 5}.$$

Conversely, however,

$$\Phi(1,2,\frac{\aleph}{\xi(1,2)}) = \frac{\max\{1,2\}}{\frac{\aleph}{\xi(1,2)} + \max\{1,2\}} = \frac{8}{\aleph + 8},$$

$$\Phi(2,3,\frac{\breve{s}}{\xi(2,3)}) = \frac{\max\{2,3\}}{\frac{\breve{s}}{\xi(2,3)} + \max\{2,3\}} = \frac{18}{\breve{s} + 18},$$

$$\Phi(3,4,\frac{\ddot{q}}{\xi(3,4)}) = \frac{\max\{3,4\}}{\frac{\ddot{q}}{\xi(3,4)} + \max\{3,4\}} = \frac{32}{\ddot{q}+32},$$

and

$$\Phi(4,5,\frac{\ddot{k}}{\xi(4,5)}) = \frac{\max\{4,5\}}{\frac{\ddot{k}}{\xi(4,5)} + \max\{4,5\}} = \frac{50}{\ddot{k}+50}.$$

That is,

$$\frac{5}{\aleph + \breve{s} + \ddot{q} + \ddot{k} + 5} \leq \max\{\frac{8}{\aleph + 8}, \frac{18}{\breve{s}+18}, \frac{32}{\ddot{q}+32}, \frac{50}{\ddot{k}+50}\}.$$

Then it satisfies all $\aleph, \breve{s}, \ddot{q}, \ddot{k} > 0$. Hence,

$$\Phi(\Theta,\epsilon,\aleph + \breve{s} + \ddot{q} + \ddot{k}) \leq \Phi(\Theta,\curlyvee_2,\aleph) \odot \Phi(\ddot{g},\varrho,\hat{s}) \odot \Phi(\ddot{q},\varrho,\hat{w}) \odot \Phi(\varrho,\epsilon,\hat{y}).$$

Now,

$$\Xi(1,5,\aleph + \breve{s} + \ddot{q} + \ddot{k}) = \frac{\max\{1,5\}}{\aleph + \breve{s} + \ddot{q} + \ddot{k}} = \frac{5}{\aleph + \breve{s} + \ddot{q} + \ddot{k}}.$$

Conversely, however

$$\Xi(1,2,\frac{\aleph}{\xi(1,2)}) = \frac{\max\{1,2\}}{\frac{\aleph}{\xi(1,2)}} = \frac{8}{\aleph},$$

$$\Xi(2,3,\frac{\breve{s}}{\xi(2,3)}) = \frac{\max\{2,3\}}{\frac{\breve{s}}{\xi(2,3)}} = \frac{18}{\breve{s}},$$

$$\Xi(3,4,\frac{\ddot{q}}{\xi(3,4)}) = \frac{\max\{3,4\}}{\frac{\ddot{q}}{\xi(3,4)}} = \frac{32}{\ddot{q}},$$

and

$$\Xi(4,5,\frac{\ddot{k}}{\xi(4,5)}) = \frac{\max\{4,5\}}{\frac{\ddot{k}}{\xi(4,5)}} = \frac{50}{\ddot{k}}.$$

That is,

$$\frac{5}{\aleph + \breve{s} + \ddot{q} + \ddot{k}} \leq \max\{\frac{8}{\aleph}, \frac{18}{\breve{s}}, \frac{32}{\ddot{q}}, \frac{50}{\ddot{k}}\}.$$

Then it satisfies all $\aleph, \breve{s}, \ddot{q}, \ddot{k} > 0$. Hence,

$$\Xi(\Theta,\epsilon,\aleph + \breve{s} + \ddot{q} + \ddot{k}) \leq \Xi(\Theta,\curlyvee_2,\aleph) \odot \Xi(\curlyvee_2,\ddot{g},\breve{s}) \odot \Xi(\ddot{g},\varrho,\ddot{q}) \odot \Xi(\varrho,\epsilon,\ddot{q}).$$

Hence, $(\mho^{\star}, \Lambda, \Phi, \Xi, \circledast, \odot)$ is a NCPMS. □

Remark 3.2 *The previously given example is also true for continuous t-norm* $\upsilon \circledast \bar{a} = \min\{\upsilon, \bar{a}\}$*, and continuous t-co-norm* $\upsilon \odot \bar{a} = \max\{\upsilon, \bar{a}\}$.

Remark 3.3 *The above example is not a neutrosophic pentagonal MS if* $\xi = 1$.

Example 3.4 *Let* $\mho^{\star}=\{1,2,3,4,5\}$ *and* $\xi:\mho^{\star}\times\mho^{\star}\to[1,+\infty)$ *be a function given by* $\xi(\Theta,\Upsilon_2)=\Theta+\Upsilon_2+2$. *Define* $\Lambda,\Phi,\Xi:\mho^{\star}\times\mho^{\star}\times(0,+\infty)\to[0,1]$ *as*

$$\Lambda(\Theta,\Upsilon_2,\aleph_{\prime})=\begin{cases}1, & \text{if}\quad \Theta=\Upsilon_2\\ \dfrac{\aleph_{\prime}}{\aleph_{\prime}+|\Theta-\Upsilon_2|^2}, & \text{if}\quad \text{otherwise,}\end{cases}$$

$$\Phi(\Theta,\Upsilon_2,\aleph_{\prime})=\begin{cases}0, & \text{if}\quad \Theta=\Upsilon_2\\ \dfrac{|\Theta-\Upsilon_2|^2}{\aleph_{\prime}+|\Theta-\Upsilon_2|^2}, & \text{if}\quad \text{otherwise,}\end{cases}$$

and

$$(\Theta,\Upsilon_2,\aleph_{\prime})=\begin{cases}0, & \text{if}\quad \Theta=\Upsilon_2\\ \dfrac{|\Theta-\Upsilon_2|^2}{\aleph_{\prime}+|\Theta-\Upsilon_2|^2}, & \text{if}\quad \text{otherwise,}\end{cases}$$

Then, $(\mho^{\star},\Lambda,\Phi,\Xi,\circledast,\odot)$ is a NCPMS with continuous t-norm $\upsilon\circledast h_{\ell}=\upsilon h_{\ell}$, and continuous t-co-norm, $\upsilon\odot\bar{a}=\max\{\upsilon,\bar{a}\}$.

Proof. Now, we prove (v, x) and (xv) remains are clear.
Let $\Theta=1,\ \Upsilon_2=2,\ \ddot{g}=3,\ \ddot{h}=4$ and $\varrho=5$ and . Then

$$\Lambda(1,5,\aleph_{\prime}+\breve{s}+\ddot{q}+\ddot{k})=\frac{\aleph_{\prime}+\breve{s}+\ddot{q}+\ddot{k}}{\aleph_{\prime}+\breve{s}+\ddot{q}+\ddot{k}+|1-5|^2}=\frac{\aleph_{\prime}+\breve{s}+\ddot{q}+\ddot{k}}{\aleph_{\prime}+\breve{s}+\ddot{q}+\ddot{k}+16}.$$

Otherwise,

$$\Lambda\left(1,2,\frac{\aleph_{\prime}}{\xi(1,2)}\right)=\frac{\frac{\aleph_{\prime}}{\xi(1,2)}}{\frac{\aleph_{\prime}}{\xi(1,2)}+|1-2|^2}=\frac{\aleph_{\prime}}{\aleph_{\prime}+5},$$

$$\Lambda\left(2,3,\frac{\breve{s}}{\xi(2,3)}\right)=\frac{\frac{\breve{s}}{\xi(2,3)}}{\frac{\breve{s}}{\xi(2,3)}+|2-3|^2}=\frac{\breve{s}}{\breve{s}+7},$$

$$\Lambda\left(3,4,\frac{\ddot{q}}{\xi(3,4)}\right)=\frac{\frac{\ddot{q}}{\xi(3,4)}}{\frac{\ddot{q}}{\xi(3,4)}+|3-4|^2}=\frac{\ddot{q}}{\ddot{q}+9}$$

and

$$\Lambda\left(4,5,\frac{\ddot{k}}{\xi(4,5)}\right)=\frac{\frac{\ddot{k}}{\xi(4,5)}}{\frac{\ddot{k}}{\xi(4,5)}+|4-5|^2}=\frac{\ddot{k}}{\ddot{k}+11}.$$

That is,

$$\frac{\aleph_{\prime}+\breve{s}+\ddot{q}+\ddot{k}}{\aleph_{\prime}+\breve{s}+\ddot{q}+\ddot{k}+16}\geq\frac{\aleph_{\prime}}{\aleph_{\prime}+5}\,\frac{\breve{s}}{\breve{s}+7}\,\frac{\ddot{q}}{\ddot{q}+9}\,\frac{\ddot{k}}{\ddot{k}+11}.$$

Then it satisfies all $\aleph, \breve{s}, \ddot{q}, \ddot{k} > 0$. Hence,

$$\Lambda(\Theta,\epsilon,\aleph+\breve{s},\ddot{q},\ddot{k}) \geq \Lambda(\Theta,\Upsilon_2,\aleph) \circledast \Lambda(\Upsilon_2,\ddot{g},\breve{s}) \circledast \Lambda(\ddot{g},\varrho,\ddot{q}) \circledast \Lambda(\varrho,\epsilon,\ddot{k}).$$

Now,

$$\Phi(1,5,\aleph+\breve{s},\ddot{q},\ddot{k}) = \frac{|1-5|^2}{\aleph+\breve{s},\ddot{q},\ddot{k}+|1-5|^2} = \frac{16}{\aleph+\breve{s},\ddot{q},\ddot{k}+16}.$$

Conversely, however,

$$\Phi(1,2,\frac{\aleph}{\xi(1,2)}) = \frac{|1-2|^2}{\frac{\aleph}{\xi(1,2)}+|1-2|^2} = \frac{5}{\aleph+5},$$

$$\Phi(2,3,\frac{\breve{s}}{\xi(2,3)}) = \frac{|2-3|^2}{\frac{\breve{s}}{\xi(2,3)}+|2-3|^2} = \frac{7}{\breve{s}+7},$$

$$\Phi(3,4,\frac{\ddot{q}}{\xi(3,4)}) = \frac{|3-4|^2}{\frac{\ddot{q}}{\xi(3,4)}+|3-4|^2} = \frac{9}{\ddot{q}+9},$$

and

$$\Phi(4,5,\frac{\ddot{k}}{\xi(4,5)}) = \frac{|4-5|^2}{\frac{\ddot{k}}{\xi(4,5)}+|4-5|^2} = \frac{11}{\ddot{k}+11}.$$

That is,

$$\frac{16}{\aleph+\breve{s},\ddot{q},\ddot{k}+16} \leq \max\left\{\frac{5}{\aleph+5},\frac{7}{\breve{s}+7},\frac{9}{\ddot{q}+9},\frac{11}{\ddot{k}+11}\right\}.$$

Then it satisfies all $\aleph, \breve{s}, \ddot{q}, \ddot{k} > 0$. Hence,

$$\Phi(\Theta,\epsilon,\aleph+\breve{s},\ddot{q},\ddot{k}) \leq \Phi(\Theta,\Upsilon_2,\aleph) \odot \Phi(\ddot{g},\varrho,\widehat{s}) \odot \Phi(\ddot{g},\varrho,\widehat{w}) \odot \Phi(\varrho,\epsilon,\widehat{y}).$$

Now,

$$\Xi(1,5,\aleph+\breve{s},\ddot{q},\ddot{k}) = \frac{|1-5|^2}{\aleph+\breve{s},\ddot{q},\ddot{k}} = \frac{16}{\aleph+\breve{s},\ddot{q},\ddot{k}}.$$

Conversely, however

$$\Xi(1,2,\frac{\aleph}{\xi(1,2)}) = \frac{|1-2|^2}{\frac{\aleph}{\xi(1,2)}} = \frac{5}{\aleph},$$

$$\Xi(2,3,\frac{\breve{s}}{\xi(2,3)}) = \frac{|2-3|^2}{\frac{\breve{s}}{\xi(2,3)}} = \frac{7}{\breve{s}},$$

$$\Xi(3,4,\frac{\ddot{q}}{\xi(3,4)}) = \frac{|3-4|^2}{\frac{\ddot{q}}{\xi(3,4)}} = \frac{9}{\ddot{q}},$$

and

$$\Xi(4,5,\frac{\ddot{k}}{\xi(4,5)}) = \frac{|4-5|^2}{\frac{\ddot{k}}{\xi(4,5)}} = \frac{11}{\ddot{k}}.$$

That is,

$$\frac{16}{\aleph + \breve{s}, \ddot{q}, \ddot{k}} \le \max\left\{\frac{5}{\aleph}, \frac{7}{\breve{s}}, \frac{9}{\ddot{q}}, \frac{11}{\ddot{k}}\right\}.$$

Then it satisfies all $\aleph, \breve{s}, \ddot{q}, \ddot{k} > 0$. Hence,

$$\Xi(\Theta, \epsilon, \aleph + \breve{s}, \ddot{q}, \ddot{k}) \le \Xi(\Theta, \Upsilon_2, \aleph) \odot \Xi(\Upsilon_2, \ddot{g}, \breve{s}) \odot \Xi(\ddot{g}, \varrho, \ddot{q}) \odot \Xi(\varrho, \epsilon, \ddot{q}).$$

Hence, $(\mho^{\star}, \Lambda, \Phi, \Xi, \circledast, \odot)$ is a NCPMS. □

Remark 3.5 *The above example is not a neutrosophic pentagonal MS if* $\xi = 1$.

Definition 3.2 *Let* $(\mho^{\star}, \Lambda, \Phi, \Xi, \circledast, \odot)$ *is a NCPMS, an open ball is then defined* $\Xi(\Theta, \dot{r}, \aleph)$ *with center* Θ*, radius* $\dot{r}, 0 < \dot{r} < 1$ *and* $\aleph > 0$ *as follows:*

$$\Xi(\Theta, \dot{r}, \aleph) = \{\Upsilon_2 \in \mho^{\star} : \Lambda(\Theta, \Upsilon_2, \aleph) > 1 - \dot{r}, \Phi(\Theta, \Upsilon_2, \aleph) < \dot{r}, \Xi(\Theta, \Upsilon_2, \aleph) < \dot{r}\}.$$

Definition 3.3 *Let* $(\mho^{\star}, \Lambda, \Phi, \Xi, \circledast, \odot)$ *is a NCPMS and* $\{\Theta_\tau\}$ *be a sequence in* $\mho^{\star}$*. Then the sequence* $\{\Theta_\tau\}$ *is called:*

1. a convergent exists if there exists $\Theta \in \mho^{\star}$ s.t.
   $\lim_{\tau \to +\infty} \Lambda(\Theta_\tau, \Theta, \aleph) = 1, \lim_{\tau \to +\infty} \Phi(\Theta_\tau, \Theta, \aleph) = 0, \lim_{\tau \to +\infty} \Xi(\Theta_\tau, \Theta, \aleph) = 0, \ \forall \ \aleph > 0,$
2. a Cauchy sequence, iff for every $\bar{a} > 0, \aleph > 0$, there exists $\tau_0 \in \mathbb{N}$ s.t.
   $\Lambda(\Theta_\tau, \Theta_{\tau+\mathfrak{q}}, \aleph) \ge 1 - \bar{a}, \Phi(\Theta_\tau, \Theta_{\tau+\mathfrak{q}}, \aleph) \le \bar{a}, \Phi(\Theta_\tau, \Theta_{\tau+\mathfrak{q}}, \aleph) \le \bar{a}, \ \forall \ \tau, \mathfrak{p} \ge \tau_0.$

If every Cauchy sequence convergent in $\mho^{\star}$, then $(\mho^{\star}, \Lambda, \Phi, \Xi, \circledast, \odot)$ is said to be a complete NCPMS.

Lemma 3.1 *Suppose* $\{\Theta_\tau\}$ *be a Cauchy sequence in NCPMS* $(\mho^{\star}, \Lambda, \Phi, \Xi, \circledast, \odot)$ *s.t.* $\Theta_\tau \ne \Theta_{\mathfrak{p}}$ *furthermore* $\mathfrak{p}, \tau \in \mathbb{N}$ *with* $\tau \ne \mathfrak{p}$*. Then, the sequence* $\{\Theta_\tau\}$ *converge to at most one limit point.*

Proof. By contradiction, assume that $\Theta_\tau \to \Theta$ and $\Theta_\tau \to \Upsilon_2$, for $\Theta \ne \Upsilon_2$. We can find, $\lim_{\tau \to +\infty} \Lambda(\Theta_\tau, \Theta, \aleph) = 1$, $\lim_{\tau \to +\infty} \Phi(\Theta_\tau, \Theta, \aleph) = 0, \lim_{\tau \to +\infty} \Xi(\Theta_\tau, \Theta, \aleph) = 0$ and $\lim_{\tau \to +\infty} \Lambda(\Theta_\tau, \Upsilon_2, \aleph) = 1$, $\lim_{\tau \to +\infty} \Phi(\Theta_\tau, \Upsilon_2, \aleph) = 0$, $\lim_{\tau \to +\infty} \Xi(\Theta_\tau, \Upsilon_2, \aleph) = 0, \forall \ \aleph > 0$. Suppose

$$\begin{aligned}
\Lambda(\Theta, \Upsilon_2, \aleph) \ge\; & \Lambda\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \circledast \Lambda\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
& \circledast \Lambda\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \circledast \Lambda\left(\Theta_{\tau+2}, \Upsilon_2, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \Upsilon_2)}\right) \\
& \to 1 \circledast 1 \circledast 1 \circledast 1, \quad \text{as} \quad \tau \to +\infty, \\
\Phi(\Theta, \Upsilon_2, \aleph) \le\; & \Phi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Phi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
& \odot \Phi\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Phi\left(\Theta_{\tau+2}, \Upsilon_2, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \Upsilon_2)}\right) \\
& \to 0 \odot 0 \odot 0 \odot 0, \quad \text{as} \quad \tau \to +\infty, \\
\Xi(\Theta, \Upsilon_2, \aleph) \le\; & \Xi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Xi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
& \odot \Xi\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Xi\left(\Theta_{\tau+2}, \Upsilon_2, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \Upsilon_2)}\right) \\
& \to 0 \odot 0 \odot 0 \odot 0, \quad \text{as} \quad \tau \to +\infty.
\end{aligned}$$

That is, $\Lambda(\Theta,\Upsilon_2,\aleph)\geq 1\circledast 1\circledast 1=1, \Phi(\Theta,\Upsilon_2,\aleph)\leq 0\odot 0\odot 0=0$, and $\Xi(\Theta,\Upsilon_2,\aleph)\leq 0\odot 0\odot 0=0$. Therefore $\Theta=\Upsilon_2$, i.e., the sequence converges to at most one limit point. □

Lemma 3.2 *Let* $(\mho^\star,\Lambda,\Phi,\Xi,\circledast,\odot)$ *is a NCPMS. If for some* $0<\theta_1<1$ *and for every* Θ,Υ_2 *in* $\mho^\star$, $\aleph>0$,

$$\Lambda(\Theta,\Upsilon_2,\aleph)\geq\Lambda\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1}\right),\Phi(\Theta,\Upsilon_2,\aleph)\leq\Phi\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1}\right),$$
$$\Xi(\Theta,\Upsilon_2,\aleph)\leq\Xi\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1}\right),$$

then $\Theta=\Upsilon_2$.

Proof. We have

$$\Lambda(\Theta,\Upsilon_2,\aleph)\geq\Lambda\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1^\tau}\right),\Phi(\Theta,\Upsilon_2,\aleph)\leq\Phi\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1^\tau}\right),$$
$$\Xi(\Theta,\Upsilon_2,\aleph)\leq\Xi\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1^\tau}\right),\tau\in\mathbb{N},\aleph>0.$$

Now

$$\Lambda(\Theta,\Upsilon_2,\aleph)\geq\lim_{\tau\to+\infty}\Lambda\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1^\tau}\right)=1,$$
$$\Phi(\Theta,\Upsilon_2,\aleph)\leq\lim_{\tau\to+\infty}\Phi\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1^\tau}\right)=0,$$
$$\Xi(\Theta,\Upsilon_2,\aleph)\leq\lim_{\tau\to+\infty}\Xi\left(\Theta,\Upsilon_2,\frac{\aleph}{\theta_1^\tau}\right)=0,\aleph>0.$$

By definition of (iii), (viii), (xiii), that is, $\Theta=\Upsilon_2$. □

Theorem 3.1 *Let* $(\mho^\star,\Lambda,\Phi,\Xi,\circledast,\odot)$ *be a complete NCPMS in the company of* $\xi:\mho^\star\times\mho^\star\to[1,+\infty)$ *with* $0<\theta_1<1$ *and assume that*

$$\lim_{\aleph\to+\infty}\Lambda(\Theta,\Upsilon_2,\aleph)=1,\ \lim_{\aleph\to+\infty}\Phi(\Theta,\Upsilon_2,\aleph)=0\quad\text{and}\quad\lim_{\aleph\to+\infty}\Xi(\Theta,\Upsilon_2,\aleph)=0,\tag{3.1}$$

$\forall\,\Theta,\Upsilon_2\in\mho^\star$ and $\aleph>0$. Consider $\dot{y}:\mho^\star\to\mho^\star$ be a map fulfilling

$$\Lambda(\dot{y}\Theta,\dot{y}\Upsilon_2,\theta_1\aleph)\geq\Lambda(\Theta,\Upsilon_2,\aleph),$$
$$\Phi(\dot{y}\Theta,\dot{y}\Upsilon_2,\theta_1\aleph)\leq\Phi(\Theta,\Upsilon_2,\aleph)\quad\text{and}\quad\Xi(\dot{y}\Theta,\dot{y}\Upsilon_2,\theta_1\aleph)\leq\Xi(\Theta,\Upsilon_2,\aleph),\tag{3.2}$$

$\forall\,\Theta,\Upsilon_2\in\mho^\star$ and $\aleph>0$. Then has a unique fixed point.

Proof. Consider Θ_0 be a point of $\mho^\star$ and define a sequence Θ_τ by $\Theta_\tau=\dot{y}^\tau\Theta_0=\dot{y}\Theta_{\tau-1},\tau\in\mathbb{N}$.
By using (3.2), $\forall\,\aleph>0$, we get

$$\Lambda(\Theta_\tau,\Theta_{\tau+1},\theta_1\aleph)=\Lambda(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_\tau,\theta_1\aleph)\geq\Lambda(\Theta_{\tau-1},\Theta_\tau,\aleph)\geq\Lambda\left(\Theta_{\tau-2},\Theta_{\tau-1},\frac{\aleph}{\theta_1}\right)$$
$$\geq\Lambda\left(\Theta_{\tau-3},\Theta_{\tau-2},\frac{\aleph}{\theta_1^2}\right)\geq\cdots\geq\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

$$\Phi(\Theta_\tau,\Theta_{\tau+1},\theta_1\aleph) = \Phi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_\tau,\theta_1\aleph) \le \Phi(\Theta_{\tau-1},\Theta_\tau,\aleph) \le \Phi\left(\Theta_{\tau-2},\Theta_{\tau-1},\frac{\aleph}{\theta_1}\right)$$
$$\le \Phi\left(\Theta_{\tau-3},\Theta_{\tau-2},\frac{\aleph}{\theta_1^2}\right) \le \cdots \le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{\theta_1^{\tau-1}}\right).$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+1},\theta_1\aleph) = \Xi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_\tau,\aleph) \le \Xi(\Theta_{\tau-1},\Theta_\tau,\aleph) \le \Xi\left(\Theta_{\tau-2},\Theta_{\tau-1},\frac{\aleph}{\theta_1}\right)$$
$$\le \Xi(\Theta_{\tau-3},\Theta_{\tau-2},\frac{\aleph}{\theta_1^2}) \le \cdots \le \Xi(\Theta_0,\Theta_1,\frac{\aleph}{\theta_1^{\tau-1}}).$$

We obtain

$$\Lambda(\Theta_\tau,\Theta_{\tau+1},\theta_1\aleph) \ge \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{\theta_1^{\tau-1}}\right),$$
$$\Phi(\Theta_\tau,\Theta_{\tau+1},\theta_1\aleph) \le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{\theta_1^{\tau-1}}\right) \text{ and } \Xi(\Theta_\tau,\Theta_{\tau+1},\theta_1\aleph) \le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{\theta_1^{\tau-1}}\right). \tag{3.3}$$

Consequently,

$$\Lambda(\Theta_\tau,\Theta_{\tau+2},\theta_1\aleph) = \Lambda(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+1},\theta_1\aleph) \ge \Lambda(\Theta_{\tau-1},\Theta_{\tau+1},\aleph) \ge \Lambda\left(\Theta_{\tau-2},\Theta_\tau,\frac{\aleph}{\theta_1}\right)$$
$$\ge \Lambda(\Theta_{\tau-3},\Theta_{\tau-1},\frac{\aleph}{\theta_1^2}) \ge \cdots \ge \Lambda(\Theta_0,\Theta_2,\frac{\aleph}{\theta_1^{\tau-1}}),$$
$$\Phi(\Theta_\tau,\Theta_{\tau+2},\theta_1\aleph) = \Phi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+1},\theta_1\aleph) \le \Phi(\Theta_{\tau-1},\Theta_{\tau+1},\aleph) \le \Phi\left(\Theta_{\tau-2},\Theta_\tau,\frac{\aleph}{\theta_1}\right)$$
$$\le \Phi\left(\Theta_{\tau-3},\Theta_{\tau-1},\frac{\aleph}{\theta_1^2}\right) \le \cdots \le \Phi\left(\Theta_0,\Theta_2,\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+2},\theta_1\aleph) = \Xi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+1},\aleph) \le \Xi(\Theta_{\tau-1},\Theta_{\tau+1},\aleph) \le \Xi\left(\Theta_{\tau-2},\Theta_\tau,\frac{\aleph}{\theta_1}\right)$$
$$\le \Xi\left(\Theta_{\tau-3},\Theta_{\tau-1},\frac{\aleph}{\theta_1^2}\right) \le \cdots \le \Xi\left(\Theta_0,\Theta_2,\frac{\aleph}{\theta_1^{\tau-1}}\right).$$

We obtain

$$\Lambda(\Theta_\tau,\Theta_{\tau+2},\theta_1\aleph) \ge \Lambda\left(\Theta_0,\Theta_2,\frac{\aleph}{\theta_1^{\tau-1}}\right),$$
$$\Phi(\Theta_\tau,\Theta_{\tau+2},\theta_1\aleph) \le \Phi\left(\Theta_0,\Theta_2,\frac{\aleph}{\theta_1^{\tau-1}}\right) \text{ and } \Xi(\Theta_\tau,\Theta_{\tau+2},\theta_1\aleph) \le \Xi\left(\Theta_0,\Theta_2,\frac{\aleph}{\theta_1^{\tau-1}}\right). \tag{3.4}$$

It follows that

$$\Lambda(\Theta_\tau,\Theta_{\tau+3},\theta_1\aleph) = \Lambda(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+2},\theta_1\aleph) \ge \Lambda(\Theta_{\tau-1},\Theta_{\tau+2},\aleph) \ge \Lambda\left(\Theta_{\tau-2},\Theta_{\tau+1},\frac{\aleph}{\theta_1}\right)$$

$$\geq \Lambda\left(\Theta_{\tau-3},\Theta_{\tau},\frac{\aleph}{\theta_1^2}\right)\geq\cdots\geq\Lambda\left(\Theta_0,\Theta_3,\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3},\theta_1\aleph)=\Phi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+2},\theta_1\aleph)\leq\Phi(\Theta_{\tau-1},\Theta_{\tau+2},\aleph)\leq\Phi\left(\Theta_{\tau-2},\Theta_{\tau+1},\frac{\aleph}{\theta_1}\right)$$

$$\leq\Phi\left(\Theta_{\tau-3},\Theta_{\tau},\frac{\aleph}{\theta_1^2}\right)\leq\cdots\leq\Phi\left(\Theta_0,\Theta_3,\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+3},\theta_1\aleph)=\Xi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+2},\aleph)\leq\Xi(\Theta_{\tau-1},\Theta_{\tau+2},\aleph)\leq\Xi\left(\Theta_{\tau-2},\Theta_{\tau+1},\frac{\aleph}{\theta_1}\right)$$

$$\leq\Xi\left(\Theta_{\tau-3},\Theta_{\tau},\frac{\aleph}{\theta_1^2}\right)\leq\cdots\leq\Xi\left(\Theta_0,\Theta_3,\frac{\aleph}{\theta_1^{\tau-1}}\right).$$

We obtain

$$\Lambda(\Theta_\tau,\Theta_{\tau+3},\theta_1\aleph)\geq\Lambda\left(\Theta_0,\Theta_3,\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3},\theta_1\aleph)\leq\Phi\left((\Theta_0,\Theta_3,\frac{\aleph}{\theta_1^{\tau-1}}\right),\quad\text{and}\quad\Xi(\Theta_\tau,\Theta_{\tau+3},\theta_1\aleph)\leq\Xi\left(\Theta_0,\Theta_3,\frac{\aleph}{\theta_1^{\tau-1}}\right).\tag{3.5}$$

Similarly, for j = 1, 2, 3, ..., we have

$$\Lambda(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph)\geq\Lambda\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph)\leq\Phi\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right),\quad\text{and}\quad\Xi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph)\leq\Xi\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right),\tag{3.6}$$

$$\Lambda(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph)\geq\Lambda\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph)\leq\Phi\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right),\quad\text{and}\quad\Xi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph)\leq\Xi\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right),\tag{3.7}$$

$$\Lambda(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph)\geq\Lambda\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right),$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph)\leq\Phi\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right),\quad\text{and}\quad\Xi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph)\leq\Xi\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right).\tag{3.8}$$

Similarly, we get for each j = 1, 2, 3, ...,

$$\Lambda(\Theta_0,\Theta_{3j+1},\aleph)\geq\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\circledast\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$

$$\circledast\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right)\circledast\cdots\circledast\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+1})}\right),$$

$$\Phi(\Theta_0,\Theta_{3j+1},\aleph) \le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right) \odot \cdots \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+1})}\right),$$

and

$$\Xi(\Theta_0,\Theta_{3j+1},\aleph) \le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right) \odot \cdots \odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+1})}\right).$$

Now, from 3.6, we get

$$\Lambda(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \ge \Lambda\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\ge \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\circledast \cdots \circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+1})}\right), \tag{3.9}$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \le \Phi\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+1})}\right), \tag{3.10}$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \le \Xi\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+1})}\right). \tag{3.11}$$

In the same manner, we get for every j = 1, 2, 3, ...,

$$\Lambda(\Theta_0,\Theta_{3j+2},\aleph) \ge \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right) \circledast \cdots \circledast \Lambda\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+2})}\right),$$

$$\Phi(\Theta_0,\Theta_{3j+2},\aleph) \le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$

$$\odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right)\odot\cdots\odot \Phi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+2})}\right),$$

and

$$\Xi(\Theta_0,\Theta_{3j+2},\aleph)\le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right)\odot\cdots\odot \Xi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+2})}\right).$$

Now, from 3.6, we get

$$\begin{aligned}\Lambda(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph)&\ge \Lambda\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right)\\&\ge \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)\circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)\\&\circledast\cdots\circledast \Lambda\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+2})}\right),\end{aligned}\tag{3.12}$$

$$\begin{aligned}\Phi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph)&\le \Phi\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right)\\&\le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)\odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)\\&\odot\cdots\odot \Phi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+2})}\right),\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}\Xi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph)&\le \Xi\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right)\\&\le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)\odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)\\&\odot\cdots\odot \Xi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+2})}\right).\end{aligned}\tag{3.14}$$

In the same procedure, we have for every j = 1, 2, 3, ...,

$$\Lambda(\Theta_0,\Theta_{3j+3},\aleph)\ge \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right)\circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^2\xi(\Theta_3,\Theta_4)}\right)$$
$$\circledast\cdots\circledast \Lambda\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+3})}\right),$$

$$\Phi(\Theta_0,\Theta_{3j+3},\aleph) \le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right) \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^2\xi(\Theta_3,\Theta_4)}\right)$$
$$\odot \cdots \odot \Phi\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+3})}\right),$$

and

$$\Xi(\Theta_0,\Theta_{3j+3},\aleph) \le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1\xi(\Theta_2,\Theta_3)}\right) \odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^2\xi(\Theta_3,\Theta_4)}\right)$$
$$\odot \cdots \odot \Xi\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{3j-1}\xi(\Theta_{3j},\Theta_{3j+3})}\right).$$

Now, from 3.6, we get

$$\Lambda(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph) \ge \Lambda\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\ge \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \circledast \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\circledast \cdots \circledast \Lambda\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+3})}\right), \tag{3.15}$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph) \le \Phi\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \odot \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \Phi\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+3})}\right), \tag{3.16}$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph) \le \Xi\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \odot \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \Xi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{3j+\tau-2}\xi(\Theta_{3j},\Theta_{3j+3})}\right). \tag{3.17}$$

Using (3.9)–(3.17), for each case $\tau \to +\infty$, we deduce that

$$\lim_{\tau\to+\infty} \Lambda(\Theta_\tau,\Theta_{\tau+i},\aleph) = 1 \circledast 1 \circledast \cdots \circledast 1 = 1,$$
$$\lim_{\tau\to+\infty} \Phi(\Theta_\tau,\Theta_{\tau+i},\aleph) = 0 \odot 0 \odot \cdots \odot 0 = 0,$$

and

$$\lim_{\tau\to+\infty} \Xi(\Theta_\tau, \Theta_{\tau+i}, \aleph) = 0 \odot 0 \odot \cdots \odot 0 = 0.$$

Therefore, $\{\Theta_\tau\}$ is a Cauchy sequence. Since $(\mho^\star, \Lambda, \Phi, \Xi, \circledast, \odot)$ is a complete NCPMS, there exists

$$\lim_{\tau\to+\infty} \Theta_\tau = \Theta.$$

Let us now evaluate truth that Θ is a fixed point of $\dot{y}$, using 3.1, 3.1, 3.1, and (3.1), we obtain

$$\begin{aligned}
\Lambda(\Theta, \dot{y}\Theta, \aleph) &\geq \Lambda\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \circledast \Lambda\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\quad \circledast \Lambda\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \circledast \Lambda\left(\Theta_{\tau+2}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&= \Lambda\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \circledast \Lambda\left(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_\tau, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\quad \circledast \Lambda\left(\dot{y}\Theta_\tau, \dot{y}\Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \circledast \Lambda\left(\dot{y}\Theta_{\tau+1}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&\geq \Lambda\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \circledast \Lambda\left(\Theta_{\tau-1}, \Theta_\tau, \frac{\aleph}{4\theta_1^{\tau-2}\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\quad \circledast \Lambda\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \\
&\quad \circledast \Lambda(\Theta_{\tau+1}, \Theta, \frac{\aleph}{4\theta_1\xi(\Theta_{\tau+2}, \dot{y}\Theta)}) \to 1 \circledast 1 \circledast 1 \circledast 1 = 1 \quad \text{as} \quad \tau \to +\infty,
\end{aligned}$$

$$\begin{aligned}
\Phi(\Theta, \dot{y}\Theta, \aleph) &\leq \Phi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Phi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\quad \odot \Phi\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Phi\left(\Theta_{\tau+2}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&= \Phi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Phi\left(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_\tau, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\quad \odot \Phi\left(\dot{y}\Theta_\tau, \dot{y}\Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Phi\left(\dot{y}\Theta_{\tau+1}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta_\tau)}\right) \\
&\leq \Phi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Phi\left(\Theta_{\tau-1}, \Theta_\tau, \frac{\aleph}{4\theta_1^{\tau-2}\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\quad \odot \Phi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Phi\left(\Theta_{\tau+1}, \Theta, \frac{\aleph}{4\theta_1\xi(\Theta_{\tau+2}, \dot{y}\Theta_\tau)}\right) \\
&\to 0 \odot 0 \odot 0 \odot 0 = 0 \quad \text{as} \quad \tau \to +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\Theta, \dot{y}\Theta, \aleph) &\leq \Xi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Xi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\quad \odot \Xi\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Xi\left(\Theta_{\tau+2}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right)
\end{aligned}$$

$$= \Xi\left(\Theta,\Theta_\tau,\frac{\aleph}{4\xi(\Theta,\Theta_\tau)}\right)\odot\Xi\left(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_\tau,\frac{\aleph}{4\xi(\Theta_\tau,\Theta_{\tau+1})}\right)$$
$$\odot\Xi\left(\dot{y}\Theta_\tau,\dot{y}\Theta_{\tau+1},\frac{\aleph}{4\xi(\Theta_{\tau+1},\Theta_{\tau+2})}\right)\odot\Xi\left(\dot{y}\Theta_{\tau+1},\dot{y}\Theta,\frac{\aleph}{4\xi(\Theta_{\tau+2},\dot{y}\Theta)}\right)$$
$$\leq \Xi\left(\Theta,\Theta_\tau,\frac{\aleph}{4\xi(\Theta,\Theta_\tau)}\right)\odot\Xi\left(\Theta_{\tau-1},\Theta_\tau,\frac{\aleph}{4\theta_1^{\tau-2}\xi(\Theta_\tau,\Theta_{\tau+1})}\right)$$
$$\odot\Xi\left(\Theta_\tau,\Theta_{\tau+1},\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{\tau+1},\Theta_{\tau+2})}\right)\odot\Xi\left(\Theta_{\tau+1},\Theta,\frac{\aleph}{4\theta_1\xi(\Theta_{\tau+2},\dot{y}\Theta)}\right)$$
$$\to 0\odot 0\odot 0\odot 0 = 0 \quad \text{as} \quad \tau\to+\infty.$$

Hence, $\dot{y}\Theta=\Theta$. Let $\dot{y}i=i$ for some $i\in\mho^\star$, then

$$1\geq \Lambda(i,\Theta,\aleph)=\Lambda(\dot{y}i,\dot{y}\Theta,\aleph)\geq\Lambda\left(i,\Theta,\frac{\aleph}{\theta_1}\right)=\Lambda\left(\dot{y}i,\dot{y}\Theta,\frac{\aleph}{\theta_1}\right)$$
$$\geq\Lambda\left(i,\Theta,\frac{\aleph}{\theta_1^2}\right)\geq\cdots\geq\Lambda\left(i,\Theta,\frac{\aleph}{\theta_1^\tau}\right)\to 1 \quad \text{as} \quad \tau\to+\infty,$$
$$0\leq \Phi(i,\Theta,\aleph)=\Phi(\dot{y}i,\dot{y}\Theta,\aleph)\leq\Phi\left(i,\Theta,\frac{\aleph}{\theta_1}\right)=\Phi\left(\dot{y}i,\dot{y}\Theta,\frac{\aleph}{\theta_1}\right)$$
$$\leq\Phi\left(i,\Theta,\frac{\aleph}{\theta_1^2}\right)\leq\cdots\leq\Phi\left(i,\Theta,\frac{\aleph}{\theta_1^\tau}\right)\to 0 \quad \text{as} \quad \tau\to+\infty,$$

and

$$0\leq \Xi(i,\Theta,\aleph)=\Xi(\dot{y}i,\dot{y}\Theta,\aleph)\leq\Xi\left(i,\Theta,\frac{\aleph}{\theta_1}\right)=\Xi\left(\dot{y}i,\dot{y}\Theta,\frac{\aleph}{\theta_1}\right)$$
$$\leq\Xi\left(i,\Theta,\frac{\aleph}{\theta_1^2}\right)\leq\cdots\leq\Xi\left(i,\Theta,\frac{\aleph}{\theta_1^\tau}\right)\to 0 \quad \text{as} \quad \tau\to+\infty,$$

by using (iii), (viii) and (xiii), $\Theta=i$. □

Definition 3.4 *Let $(\mho^\star,\Lambda,\Phi,\Xi,\circledast,\odot)$ be a NCPMS. A map $\dot{y}:\mho^\star\to\mho^\star$ is an neutrosophic controlled pentagonal contraction (NCPC), if $\exists\, 0<\theta_1<1$, s.t.*

$$\frac{1}{\Lambda(\mathcal{P}\Theta,\mathcal{P}\Upsilon_2,\aleph)}-1\leq\theta_1\left[\frac{1}{\Lambda(\Theta,\Upsilon_2,\aleph)}-1\right] \tag{3.18}$$

$$\Phi(\mathcal{P}\Theta,\mathcal{P}\Upsilon_2,\aleph)\leq\theta_1\Phi(\Theta,\Upsilon_2,\aleph), \tag{3.19}$$

and

$$\Xi(\mathcal{P}\Theta,\mathcal{P}\Upsilon_2,\aleph)\leq\theta_1\Xi(\Theta,\Upsilon_2,\aleph),\ \forall\ \Theta,\Upsilon_2\in\mho^\star, \tag{3.20}$$

and $\aleph>0$.

Now, we show the theorem for NCPC(neutrosophic controlled pentagonal) contraction.

Theorem 3.2 *Let $(\mho^\star,\Lambda,\Phi,\Xi,\circledast,\odot)$ be a complete NCPMS with $\xi:\mho^\star\times\mho^\star\to[1,+\infty)$, and suppose that*

$$\lim_{\aleph\to+\infty}\Lambda(\Theta,\Upsilon_2,\aleph)=1,\ \lim_{\aleph\to+\infty}\Phi(\Theta,\Upsilon_2,\aleph)=0,\ \text{and}\ \lim_{\aleph\to+\infty}\Xi(\Theta,\Upsilon_2,\aleph)=0, \tag{3.21}$$

$\forall \Theta, \Upsilon_2 \in \mho^\star$ and $\aleph > 0$. Let $\dot{y} : \mho^\star \to \mho^\star$ be a NCPC. Moreover, assume that $\Theta_0 \in \mho^\star$ be an arbitrary, and $\tau, \mathfrak{q} \in \mathbb{N}$, where $\Theta_\tau = \dot{y}^\tau \Theta_0 = \dot{y}\Theta_{\tau-1}$. Then, $\dot{y}$ has a UFP.

Proof. Consider Θ_0 be a point of $\mho^\star$ and define a sequence Θ_τ by $\Theta_\tau = \dot{y}^\tau \Theta_0 = \dot{y}\Theta_{\tau-1}, \tau \in \mathbb{N}$. From using (3.18), (3.19) and (3.20), $\forall\ \aleph > 0, \tau > \mathfrak{q}$, we deduce

$$\begin{aligned}
\frac{1}{\Lambda(\Theta_\tau, \Theta_{\tau+1}, \aleph)} - 1 &= \frac{1}{\Lambda(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_\tau, \aleph)} - 1 \\
&\le \theta_1 \left[\frac{1}{\Lambda(\Theta_{\tau-1}, \Theta_\tau, \aleph)} \right] = \frac{\theta_1}{\Lambda(\Theta_{\tau-1}, \Theta_\tau, \aleph)} - \theta_1 \\
&\Rightarrow \frac{1}{\Lambda(\Theta_\tau, \Theta_{\tau+1}, \aleph)} \le \frac{\theta_1}{\Lambda(\Theta_{\tau-1}, \Theta_\tau, \aleph)} + (1 - \theta_1) \\
&\le \frac{\theta_1^2}{\Lambda(\Theta_{\tau-2}, \Theta_{\tau-1}, \aleph)} + \theta_1(1-\theta_1) + (1-\theta_1).
\end{aligned}$$

Proceeding in this way, we conclude

$$\begin{aligned}
\frac{1}{\Lambda(\Theta_\tau, \Theta_{\tau+1}, \aleph)} &\le \frac{\theta_1^\tau}{\Lambda(\Theta_0, \Theta_1, \aleph)} + \theta_1^{\tau-1}(1-\theta_1) + \theta_1^{\tau-2}(1-\theta_1) \\
&\quad + \cdots + \theta_1(1-\theta_1) + (1-\theta_1) \\
&\le \frac{\theta_1^\tau}{\Lambda(\Theta_0, \Theta_1, \aleph)} + (\theta_1^{\tau-1} + \theta_1^{\tau-2} + \cdots + 1)(1-\theta_1) \\
&\le \frac{\theta_1^\tau}{\Lambda(\Theta_0, \Theta_1, \aleph)} + (1-\theta_1^\tau).
\end{aligned}$$

We obtain

$$\frac{1}{\dfrac{\theta_1^\tau}{\Lambda(\Theta_0, \Theta_1, \aleph)} + (1-\theta_1^\tau)} \le \Lambda(\Theta_\tau, \Theta_{\tau+1}, \aleph),$$

$$\begin{aligned}
\Phi(\Theta_\tau, \Theta_{\tau+1}, \aleph) &= \Phi(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_\tau, \aleph) \le \theta_1 \Phi(\Theta_{\tau-1}, \Theta_\tau, \aleph) = \Phi(\dot{y}\Theta_{\tau-2}, \dot{y}\Theta_{\tau-1}, \aleph) \\
&\le \theta_1^2 \Phi(\Theta_{\tau-2}, \Theta_{\tau-1}, \aleph) \le \cdots \le \theta_1^\tau \Phi(\Theta_0, \Theta_1, \aleph),
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\Theta_\tau, \Theta_{\tau+1}, \aleph) &= \Xi(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_\tau, \aleph) \le \theta_1 \Xi(\Theta_{\tau-1}, \Theta_\tau, \aleph) = \Xi(\dot{y}\Theta_{\tau-2}, \dot{y}\Theta_{\tau-1}, \aleph) \\
&\le \theta_1^2 \Xi(\Theta_{\tau-2}, \Theta_{\tau-1}, \aleph) \le \cdots \le \theta_1^\tau \Xi(\Theta_0, \Theta_1, \aleph).
\end{aligned} \tag{3.22}$$

It again follows that

$$\begin{aligned}
\frac{1}{\Lambda(\Theta_\tau, \Theta_{\tau+2}, \aleph)} - 1 &= \frac{1}{\Lambda(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_{\tau+1}, \aleph)} - 1 \\
&\le \theta_1 [\frac{1}{\Lambda(\Theta_{\tau-1}, \Theta_{\tau+1}, \aleph)}] \\
&= \frac{\theta_1}{\Lambda(\Theta_{\tau-1}, \Theta_{\tau+1}, \aleph)} - \theta_1 \\
&\Rightarrow \frac{1}{\Lambda(\Theta_\tau, \Theta_{\tau+2}, \aleph)} \le \frac{\theta_1}{\Lambda(\Theta_{\tau-1}, \Theta_{\tau+1}, \aleph)} + (1-\theta_1) \\
&\le \frac{\theta_1^2}{\Lambda(\Theta_{\tau-2}, \Theta_\tau, \aleph)} + \theta_1(1-\theta_1) + (1-\theta_1).
\end{aligned}$$

Proceeding in this way, we conclude

$$\begin{aligned}\frac{1}{\Lambda(\Theta_\tau,\Theta_{\tau+2},\aleph)} &\le \frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_2,\aleph)}+\theta_1^{\tau-1}(1-\theta_1)+\theta_1^{\tau-2}(1-\theta_1)\\ &\quad+\cdots+\theta_1(1-\theta_1)+(1-\theta_1)\\ &\le \frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_2,\aleph)}+(\theta_1^{\tau-1}+\theta_1^{\tau-2}+\cdots+1)(1-\theta_1)\\ &\le \frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_2,\aleph)}+(1-\theta_1^\tau),\end{aligned}$$

We obtain

$$\frac{1}{\frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_2,\aleph)}+(1-\theta_1^\tau)} \le \Lambda(\Theta_\tau,\Theta_{\tau+2},\aleph), \tag{3.23}$$

$$\begin{aligned}\Phi(\Theta_\tau,\Theta_{\tau+2},\aleph) &= \Phi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+1},\aleph) \le \theta_1\Phi(\Theta_{\tau-1},\Theta_{\tau+1},\aleph) = \Phi(\dot{y}\Theta_{\tau-2},\dot{y}\Theta_\tau,\aleph)\\ &\le \theta_1^2\Phi(\Theta_{\tau-2},\Theta_\tau,\aleph) \le \cdots \le \theta_1^\tau\Phi(\Theta_0,\Theta_2,\aleph),\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\Xi(\Theta_\tau,\Theta_{\tau+2},\aleph) &= \Xi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+1},\aleph) \le \theta_1\Xi(\Theta_{\tau-1},\Theta_{\tau+1},\aleph) = \Xi(\dot{y}\Theta_{\tau-2},\dot{y}\Theta_\tau,\aleph)\\ &\le \theta_1^2\Xi(\Theta_{\tau-2},\Theta_\tau,\aleph) \le \cdots \le \theta_1^\tau\Xi(\Theta_0,\Theta_2,\aleph).\end{aligned} \tag{3.25}$$

Consequently,

$$\begin{aligned}\frac{1}{\Lambda(\Theta_\tau,\Theta_{\tau+3},\aleph)}-1 &= \frac{1}{\Lambda(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+2},\aleph)}-1\\ &\le \theta_1\left[\frac{1}{\Lambda(\Theta_{\tau-1},\Theta_{\tau+2},\aleph)}\right]\\ &= \frac{\theta_1}{\Lambda(\Theta_{\tau-1},\Theta_{\tau+2},\aleph)}-\theta_1\\ \Rightarrow \frac{1}{\Lambda(\Theta_\tau,\Theta_{\tau+3},\aleph)} &\le \frac{\theta_1}{\Lambda(\Theta_{\tau-1},\Theta_{\tau+2},\aleph)}+(1-\theta_1)\\ &\le \frac{\theta_1^2}{\Lambda(\Theta_{\tau-2},\Theta_{\tau+1},\aleph)}+\theta_1(1-\theta_1)+(1-\theta_1).\end{aligned}$$

Proceeding in this way, we conclude

$$\begin{aligned}\frac{1}{\Lambda(\Theta_\tau,\Theta_{\tau+3},\aleph)} &\le \frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_3,\aleph)}+\theta_1^{\tau-1}(1-\theta_1)+\theta_1^{\tau-2}(1-\theta_1)\\ &\quad+\cdots+\theta_1(1-\theta_1)+(1-\theta_1)\\ &\le \frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_3,\aleph)}+(\theta_1^{\tau-1}+\theta_1^{\tau-2}+\cdots+1)(1-\theta_1)\\ &\le \frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_3,\aleph)}+(1-\theta_1^\tau).\end{aligned}$$

We obtain

$$\frac{1}{\frac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_3,\aleph)}+(1-\theta_1^\tau)} \le \Lambda(\Theta_\tau,\Theta_{\tau+3},\aleph), \tag{3.26}$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3},\aleph) = \Phi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+2},\aleph) \le \theta_1\Phi(\Theta_{\tau-1},\Theta_{\tau+2},\aleph) = \Phi(\dot{y}\Theta_{\tau-2},\dot{y}\Theta_{\tau+1},\aleph)$$
$$\le \theta_1^2\Phi(\Theta_{\tau-2},\Theta_{\tau+1},\aleph) \le \cdots \le \theta_1^\tau\Phi(\Theta_0,\Theta_3,\aleph), \tag{3.27}$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+3},\aleph) = \Xi(\dot{y}\Theta_{\tau-1},\dot{y}\Theta_{\tau+2},\aleph) \le \theta_1\Xi(\Theta_{\tau-1},\Theta_{\tau+2},\aleph) = \Xi(\dot{y}\Theta_{\tau-2},\dot{y}\Theta_{\tau+1},\aleph)$$
$$\le \theta_1^2\Xi(\Theta_{\tau-2},\Theta_{\tau+1},\aleph) \le \cdots \le \theta_1^\tau\Xi(\Theta_0,\Theta_3,\aleph). \tag{3.28}$$

Similarly, for j = 1, 2, 3, ..., we get

$$\frac{1}{\dfrac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_{3j+1},\aleph)}+(1-\theta_1^\tau)} \le \Lambda(\Theta_\tau,\Theta_{\tau+3j+1},\aleph)$$
$$\Phi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \le \theta_1^\tau\Phi(\Theta_0,\Theta_{3j+1},\aleph) \quad\text{and}\quad \Xi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \le \theta_1^\tau\Xi(\Theta_0,\Theta_{3j+1},\aleph), \tag{3.29}$$

$$\frac{1}{\dfrac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_{3j+2},\aleph)}+(1-\theta_1^\tau)} \le \Lambda(\Theta_\tau,\Theta_{\tau+3j+2},\aleph)$$
$$\Phi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph) \le \theta_1^\tau\Phi(\Theta_0,\Theta_{3j+2},\aleph) \quad\text{and}\quad \Xi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph) \le \theta_1^\tau\Xi(\Theta_0,\Theta_{3j+2},\aleph), \tag{3.30}$$

$$\frac{1}{\dfrac{\theta_1^\tau}{\Lambda(\Theta_0,\Theta_{3j+3},\aleph)}+(1-\theta_1^\tau)} \le \Lambda(\Theta_\tau,\Theta_{\tau+3j+3},\aleph)$$
$$\Phi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph) \le \theta_1^\tau\Phi(\Theta_0,\Theta_{3j+3},\aleph) \quad\text{and}\quad \Xi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph) \le \theta_1^\tau\Xi(\Theta_0,\Theta_{3j+3},\aleph). \tag{3.31}$$

By using 3.22, we have

$$\Lambda(\Theta_0,\Theta_4,\aleph) \ge \Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \circledast \Lambda\left(\Theta_1,\Theta_2,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\circledast \Lambda\left(\Theta_2,\Theta_3,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right) \circledast \Lambda\left(\Theta_3,\Theta_4,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)$$
$$\ge \frac{1}{\dfrac{1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)}} \circledast \frac{1}{\dfrac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}$$
$$\circledast \frac{1}{\dfrac{\theta_1^2}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)}+(1-\theta_1^2)} \circledast \frac{1}{\dfrac{\theta_1^3}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)}+(1-\theta_1^3)},$$

$$\Phi(\Theta_0,\Theta_4,\aleph) \le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \Phi\left(\Theta_1,\Theta_2,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \Phi\left(\Theta_2,\Theta_3,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right) \odot \Phi\left(\Theta_3,\Theta_4,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)$$
$$\le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \theta_1^1\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$

$$\odot\theta_1^2\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\odot\theta_1^3\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right),$$

and

$$\begin{aligned}\Xi(\Theta_0,\Theta_4,\aleph)\le\ &\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot\Xi\left(\Theta_1,\Theta_2,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)\\ &\odot\Xi\left(\Theta_2,\Theta_3,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\odot\Xi\left(\Theta_3,\Theta_4,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)\\ \le\ &\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot\theta_1\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)\\ &\odot\theta_1^2\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\odot\theta_1^3\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\Lambda(\Theta_0,\Theta_7,\aleph)\ge\ &\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\circledast\Lambda\left(\Theta_1,\Theta_2,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)\\ &\circledast\Lambda\left(\Theta_2,\Theta_3,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\circledast\Lambda\left(\Theta_3,\Theta_4,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)\\ &\circledast\Lambda\left(\Theta_4,\Theta_5,\frac{\aleph}{4\xi(\Theta_4,\Theta_5)}\right)\circledast\Lambda\left(\Theta_5,\Theta_6,\frac{\aleph}{4\xi(\Theta_5,\Theta_6)}\right)\\ &\circledast\Lambda\left(\Theta_6,\Theta_7,\frac{\aleph}{4\xi(\Theta_6,\Theta_7)}\right)\\ \ge\ &\frac{1}{\dfrac{1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)}}\circledast\frac{1}{\dfrac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}\\ &\circledast\frac{1}{\dfrac{\theta_1^2}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)}+(1-\theta_1^2)}\circledast\frac{1}{\dfrac{\theta_1^3}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)}+(1-\theta_1^3)}\\ &\circledast\frac{1}{\dfrac{\theta_1^4}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_4,\Theta_5)}\right)}+(1-\theta_1^4)}\circledast\frac{1}{\dfrac{\theta_1^5}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_5,\Theta_6)}\right)}+(1-\theta_1^5)}\\ &\circledast\frac{1}{\dfrac{\theta_1^6}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_6,\Theta_7)}\right)}+(1-\theta_1^6)},\end{aligned}$$

$$\Phi(\Theta_0,\Theta_7,\aleph)\le\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\circledast\Phi\left(\Theta_1,\Theta_2,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$

$$
\begin{aligned}
&\circledast\Phi\left(\Theta_2,\Theta_3,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\circledast\Phi\left(\Theta_3,\Theta_4,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)\\
&\circledast\Phi\left(\Theta_4,\Theta_5,\frac{\aleph}{4\xi(\Theta_4,\Theta_5)}\right)\circledast\Phi\left(\Theta_5,\Theta_6,\frac{\aleph}{4\xi(\Theta_5,\Theta_6)}\right)\\
&\circledast\Phi\left(\Theta_6,\Theta_7,\frac{\aleph}{4\xi(\Theta_6,\Theta_7)}\right)\\
&\leq\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot\theta_1\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)\\
&\odot\theta_1^2\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\odot\theta_1^3\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)\\
&\odot\theta_1^4\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_4,\Theta_5)}\right)\odot\theta_1^5\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_5,\Theta_6)}\right)\\
&\odot\theta_1^6\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_6,\Theta_7)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi(\Theta_0,\Theta_7,\aleph)&\leq\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot\Xi\left(\Theta_1,\Theta_2,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)\\
&\odot\Xi\left(\Theta_2,\Theta_3,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\odot\Xi\left(\Theta_3,\Theta_4,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)\\
&\odot\Xi\left(\Theta_4,\Theta_5,\frac{\aleph}{4\xi(\Theta_4,\Theta_5)}\right)\odot\Xi\left(\Theta_5,\Theta_6,\frac{\aleph}{4\xi(\Theta_5,\Theta_6)}\right)\\
&\odot\Xi\left(\Theta_6,\Theta_7,\frac{\aleph}{4\xi(\Theta_6,\Theta_7)}\right)\\
&\leq\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot\theta_1\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)\\
&\odot\theta_1^2\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_2,\Theta_3)}\right)\odot\theta_1^3\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_3,\Theta_4)}\right)\\
&\odot\theta_1^4\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_4,\Theta_5)}\right)\odot\theta_1^5\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_5,\Theta_6)}\right)\\
&\odot\theta_1^6\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_6,\Theta_7)}\right).
\end{aligned}
$$

We obtain for each j = 1, 2, 3, ...,

$$
\Lambda(\Theta_0,\Theta_{3j+1},\aleph)\geq\frac{1}{\dfrac{1}{\Lambda\left(\Theta_0,\Theta_1,\dfrac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)}}\circledast\frac{1}{\dfrac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\dfrac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}
$$

$$\circledast \cdots \circledast \frac{1}{\dfrac{\theta_1^{3j}}{\Lambda\left(\Theta_0,\Theta_1,\dfrac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+1})}\right)}+(1-\theta_1^{3j})},$$

$$\Phi(\Theta_0,\Theta_{3j+1},\aleph) \le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \theta_1 \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \theta_1^{3j}\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+1})}\right)$$

and

$$\Xi(\Theta_0,\Theta_{3j+1},\aleph) \le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right) \odot \theta_1 \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \theta_1^{3j}\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+1})}\right).$$

Now, from 3.22, we get

$$\Lambda(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \ge \Lambda\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\ge \frac{1}{\dfrac{1}{\Lambda\left(\Theta_0,\Theta_1,\dfrac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)}} \circledast \frac{1}{\dfrac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\dfrac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}$$
$$\circledast \cdots \circledast \frac{1}{\dfrac{\theta_1^{3j}}{\Lambda\left(\Theta_0,\Theta_1,\dfrac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+1})}\right)}+(1-\theta_1^{3j})}, \tag{3.32}$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \le \Phi\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\le \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \odot \theta_1\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \theta_1^{3j}\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+1})}\right), \tag{3.33}$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+3j+1},\theta_1\aleph) \le \Xi\left(\Theta_0,\Theta_{3j+1},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right) \odot \theta_1\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot \cdots \odot \theta_1^{3j}\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+1})}\right). \tag{3.34}$$

Similarly, we obtain for each j = 1, 2, 3, ...,

$$\Lambda(\Theta_0,\Theta_{3j+2},\aleph) \geq \frac{1}{\frac{1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)}} \circledast \frac{1}{\frac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}$$

$$\circledast\cdots\circledast \frac{1}{\frac{\theta_1^{3j}}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+2})}\right)}+(1-\theta_1^{3j})},$$

$$\Phi(\Theta_0,\Theta_{3j+2},\aleph) \leq \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot\theta_1\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$

$$\odot\cdots\odot\theta_1^{3j}\Phi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+2})}\right),$$

and

$$\Xi(\Theta_0,\Theta_{3j+2},\aleph) \leq \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot\theta_1\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$

$$\odot\cdots\odot\theta_1^{3j}\Xi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+2})}\right).$$

Now, from 3.22, we get

$$\begin{aligned}\Lambda(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph) &\geq \Lambda\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right)\\ &\geq \frac{1}{\frac{1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)}} \circledast \frac{1}{\frac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}\\ &\circledast\cdots\circledast\frac{1}{\frac{\theta_1^{3j}}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+2})}\right)}+(1-\theta_1^{3j})},\end{aligned} \tag{3.35}$$

$$\begin{aligned}\Phi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph) &\leq \Phi\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right)\\ &\leq \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)\odot\theta_1\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)\\ &\odot\cdots\odot\theta_1^{3j}\Phi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+2})}\right),\end{aligned} \tag{3.36}$$

and

$$\Xi(\Theta_\tau,\Theta_{\tau+3j+2},\theta_1\aleph) \leq \Xi\left(\Theta_0,\Theta_{3j+2},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$

$$\leq \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)\odot \theta_1\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot\cdots\odot \theta_1^{3j}\Xi\left(\Theta_0,\Theta_2,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+2})}\right). \tag{3.37}$$

In the same manner, we obtain for each j = 1, 2, 3, ...,

$$\Lambda(\Theta_0,\Theta_{3j+3},\aleph)\geq \frac{1}{\frac{1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)}}\circledast\frac{1}{\frac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}$$
$$\circledast\cdots\circledast\frac{1}{\frac{\theta_1^{3j}}{\Lambda\left(\Theta_0,\Theta_3,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+3})}\right)}+(1-\theta_1^{3j})},$$

$$\Phi(\Theta_0,\Theta_{3j+3},\aleph)\leq \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot \theta_1\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot\cdots\odot \theta_1^{3j}\Phi\left(\Theta_0,\Theta_3,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+3})}\right),$$

and

$$\Xi(\Theta_0,\Theta_{3j+3},\aleph)\leq \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_0,\Theta_1)}\right)\odot \theta_1\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot\cdots\odot \theta_1^{3j}\Xi\left(\Theta_0,\Theta_3,\frac{\aleph}{4\xi(\Theta_{3j},\Theta_{3j+3})}\right).$$

Now, from 3.22, we get

$$\Lambda(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph)\geq \Lambda\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\geq \frac{1}{\frac{1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)}}\circledast\frac{1}{\frac{\theta_1}{\Lambda\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)}+(1-\theta_1)}$$
$$\circledast\cdots\circledast\frac{1}{\frac{\theta_1^{3j}}{\Lambda\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+3})}\right)}+(1-\theta_1^{3j})}, \tag{3.38}$$

$$\Phi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1\aleph)\leq \Phi\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right)$$
$$\leq \Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)\odot \theta_1\Phi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)$$
$$\odot\cdots\odot \theta_1^{3j}\Phi\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+3})}\right), \tag{3.39}$$

and

$$\begin{aligned}\Xi(\Theta_\tau,\Theta_{\tau+3j+3},\theta_1 \aleph) &\le \Xi\left(\Theta_0,\Theta_{3j+3},\frac{\aleph}{\theta_1^{\tau-1}}\right)\\ &\le \Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_0,\Theta_1)}\right)\odot \theta_1\Xi\left(\Theta_0,\Theta_1,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_1,\Theta_2)}\right)\\ &\odot\cdots\odot \theta_1^{3j}\Xi\left(\Theta_0,\Theta_3,\frac{\aleph}{4\theta_1^{\tau-1}\xi(\Theta_{3j},\Theta_{3j+3})}\right).\end{aligned} \tag{3.40}$$

Using (3.32)–(3.40), for each case , we deduce

$$\lim_{\tau\to+\infty}\Lambda(\Theta_\tau,\Theta_{\tau+q},\aleph)=1\circledast 1\circledast\cdots\circledast=1,$$
$$\lim_{\tau\to+\infty}\Phi(\Theta_\tau,\Theta_{\tau+q},\aleph)=0\odot 0\odot\cdots\odot 0=0,$$

and

$$\lim_{\tau\to+\infty}\Xi(\Theta_\tau,\Theta_{\tau+q},\aleph)=0\odot 0\odot\cdots\odot 0=0.$$

Therefore, $\{\Theta_\tau\}$ is a Cauchy sequence. Since $(\mho^\star,\Lambda,\Phi,\Xi,\circledast,\odot)$ be a complete NCPMS, there exists

$$\lim_{\tau\to+\infty}\Theta_\tau=\Theta.$$

Using (v), (x), and (xv), we have

$$\begin{aligned}\frac{1}{\Lambda(\dot y\Theta_\tau,\dot y\Theta,\aleph)}-1 &\le \theta_1\left[\frac{1}{\Lambda(\Theta_\tau,\Theta,\aleph)}-1\right]=\frac{\theta_1}{\Lambda(\Theta_\tau,\Theta,\aleph)}-\theta_1\\ &\Rightarrow \frac{1}{\dfrac{\theta_1}{\Lambda(\Theta_\tau,\Theta,\aleph)}+(1-\theta_1)}\le \Lambda(\dot y\Theta_\tau,\dot y\Theta,\aleph).\end{aligned}$$

Utilizing the above inequality, we get

$$\begin{aligned}\Lambda(\Theta,\dot y\Theta,\aleph) &\ge \Lambda\left(\Theta,\Theta_\tau,\frac{\aleph}{4\xi(\Theta,\Theta_\tau)}\right)\circledast\Lambda\left(\Theta_\tau,\Theta_{\tau+1},\frac{\aleph}{4\xi(\Theta_\tau,\Theta_{\tau+1})}\right)\\ &\circledast\Lambda\left(\Theta_{\tau+1},\Theta_{\tau+2},\frac{\aleph}{4\xi(\Theta_{\tau+1},\Theta_{\tau+2})}\right)\circledast\Lambda\left(\Theta_{\tau+2},\dot y\Theta,\frac{\aleph}{4\xi(\Theta_{\tau+2},\dot y\Theta)}\right)\\ &\ge \Lambda\left(\Theta,\Theta_\tau,\frac{\aleph}{4\xi(\Theta,\Theta_\tau)}\right)\circledast\Lambda\left(\dot y\Theta_{\tau-1},\dot y\Theta_\tau,\frac{\aleph}{4\xi(\Theta_\tau,\Theta_{\tau+1})}\right)\\ &\circledast\Lambda\left(\dot y\Theta_\tau,\dot y\Theta_{\tau+1},\frac{\aleph}{4\xi(\Theta_{\tau+1},\Theta_{\tau+2})}\right)\circledast\Lambda\left(\dot y\Theta_{\tau+1},\dot y\Theta,\frac{\aleph}{4\xi(\Theta_{\tau+2},\dot y\Theta)}\right)\\ &\ge \Lambda\left(\Theta,\Theta_\tau,\frac{\aleph}{4\xi(\Theta,\Theta_\tau)}\right)\circledast\frac{1}{\dfrac{\theta_1^{\tau-1}}{\Lambda\left(\Theta_{\tau-1},\Theta_\tau,\dfrac{\aleph}{4\xi(\Theta_\tau,\Theta_{\tau+1})}\right)}+(1-\theta_1^{\tau-1})}\\ &\circledast\frac{1}{\dfrac{\theta_1^{\tau}}{\Lambda\left(\Theta_\tau,\Theta_{\tau+1},\dfrac{\aleph}{4\xi(\Theta_{\tau+1},\Theta_{\tau+2})}\right)}+(1-\theta_1^{\tau})}\end{aligned}$$

$$\circledast \frac{1}{\dfrac{\theta_1}{\Lambda\left(\Theta_{\tau+1}, \Theta, \dfrac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right)} + (1-\theta_1)}$$
$$\to 1 \circledast 1 \circledast 1 \circledast 1 = 1 \text{ as } \tau \to +\infty,$$

$$\begin{aligned}
\Phi(\Theta, \dot{y}\Theta, \aleph) &\le \Phi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Phi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\odot \Phi\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Phi\left(\Theta_{\tau+2}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&\le \Phi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Phi\left(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_\tau, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\odot \Phi\left(\dot{y}\Theta_\tau, \dot{y}\Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Phi\left(\dot{y}\Theta_{\tau+1}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&\le \Phi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \theta_1^{\tau-1}\Phi\left(\Theta_{\tau-1}, \Theta_\tau, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\odot \theta_1^{\tau}\Phi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \theta_1\Phi\left(\Theta_{\tau+1}, \Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&\to 0 \odot 0 \odot 0 \odot 0 = 0 \text{ as } \tau \to +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\Theta, \dot{y}\Theta, \aleph) &\le \Xi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Xi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\odot \Xi\left(\Theta_{\tau+1}, \Theta_{\tau+2}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Xi\left(\Theta_{\tau+2}, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&\le \Xi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \Xi\left(\dot{y}\Theta_{\tau-1}, \dot{y}\Theta_\tau, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\odot \Xi\left(\dot{y}\Theta_\tau, \dot{y}\Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \Xi\left(\dot{y}\Theta_\tau, \dot{y}\Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&\le \Xi\left(\Theta, \Theta_\tau, \frac{\aleph}{4\xi(\Theta, \Theta_\tau)}\right) \odot \theta_1^{\tau-1}\Xi\left(\Theta_{\tau-1}, \Theta_\tau, \frac{\aleph}{4\xi(\Theta_\tau, \Theta_{\tau+1})}\right) \\
&\odot \theta_1^{\tau}\Xi\left(\Theta_\tau, \Theta_{\tau+1}, \frac{\aleph}{4\xi(\Theta_{\tau+1}, \Theta_{\tau+2})}\right) \odot \theta_1\Xi\left(\Theta_\tau, \Theta, \frac{\aleph}{4\xi(\Theta_{\tau+2}, \dot{y}\Theta)}\right) \\
&\to 0 \odot 0 \odot 0 \odot 0 = 0 \text{ as } \tau \to +\infty.
\end{aligned}$$

Therefore, $\dot{y}\Theta = \Theta$. Let $\dot{y}i = i$ for some $i \in \mho^\star$, then

$$\begin{aligned}
\frac{1}{\Lambda(\Theta, i, \aleph)} - 1 &= \frac{1}{\Lambda(\dot{y}\Theta, \dot{y}i, \aleph)} - 1 \\
&\le \theta_1\left[\frac{1}{\Lambda(\Theta, i, \aleph)} - 1\right] < \frac{1}{\Lambda(\Theta, i, \aleph)} - 1,
\end{aligned}$$

which is a contradiction.

$$\Phi(\Theta,i,\aleph) = \Phi(\dot{y}\Theta,\dot{y}i,\aleph) \leq \theta_1\Phi(\Theta,i,\aleph) < \Phi(\Theta,i,\aleph),$$

which is a contradiction, and

$$\Xi(\Theta,i,\aleph) = \Xi(\dot{y}\Theta,\dot{y}i,\aleph) \leq \theta_1\Xi(\Theta,i,\aleph) < \Xi(\Theta,i,\aleph),$$

which is a contradiction. Therefore, we must have $\Lambda(\Theta,i,\aleph)=1, \Phi(\Theta,i,\aleph)=0$, and $\Xi(\Theta,i,\aleph)=0$, hence, $\Theta = i$. □

Example 3.6 *Let* $\mho^\star=[0,1]$ *and* $\xi:\mho^\star\times\mho^\star\to[1,+\infty)$ *be a function given by*

$$\xi(\Theta,\Upsilon_2)=\begin{cases}1 & \text{if } \Theta=\Upsilon_2,\\ \dfrac{1+\max\{\Theta,\Upsilon_2\}}{1+\max\{\Theta,\Upsilon_2\}} & \text{if } \Theta\neq\Upsilon_2.\end{cases}$$

Define $\Lambda,\Phi,\Xi:\mho^\star\times\mho^\star\times(0,+\infty)\to[0,1]$ as

$$\Lambda(\Theta,\Upsilon_2,\aleph)=\frac{\aleph}{\aleph+|\Theta-\Upsilon_2|},$$

$$\Phi(\Theta,\Upsilon_2,\aleph)=\frac{|\Theta-\Upsilon_2|}{\aleph+|\Theta-\Upsilon_2|},$$

$$\Xi(\Theta,\Upsilon_2,\aleph)=\frac{|\Theta-\Upsilon_2|}{\aleph}.$$

Then, $(\mho^\star,\Lambda,\Phi,\Xi,\circledast,\odot)$ is a complete NCPMS with continuous t-norm $\upsilon\circledast h_\ell=\upsilon h_\ell$, and continuous t-co-norm $\upsilon\odot h_\ell=\max\{\upsilon,h_\ell\}$.

define $\dot{y}:\mho^\star\to\mho^\star$ by $\dot{y}(\Theta)=\frac{1-5^{-\Theta}}{11}$ and let $\theta_1\in[\frac{1}{2},1)$, then

$$\begin{aligned}\Lambda(\dot{y}\Theta,\dot{y}\Upsilon_2,\theta_1\aleph)&=\Lambda\left(\frac{1-5^{-\Theta}}{11},\frac{1-5^{-\Upsilon_2}}{11},\theta_1\aleph\right)\\
&=\frac{\theta_1\aleph}{\theta_1\aleph+\left|\frac{1-5^{-\Theta}}{11}-\frac{1-5^{-\Upsilon_2}}{11}\right|}=\frac{\theta_1\aleph}{\theta_1\aleph+\frac{|5^{-\Theta}-5^{-\Upsilon_2}|}{11}}\\
&\geq\frac{\theta_1\aleph}{\theta_1\aleph+\frac{|\Theta-\Upsilon_2|}{11}}=\frac{11\theta_1\aleph}{11\theta_1\aleph+|\Theta-\Upsilon_2|}\geq\frac{\aleph}{\aleph+|\Theta-\Upsilon_2|}=\Lambda(\Theta,\Upsilon_2,\aleph),\end{aligned}$$

$$\begin{aligned}\Phi(\dot{y}\Theta,\dot{y}\Upsilon_2,\theta_1\aleph)&=\Phi\left(\frac{1-3^{-\Theta}}{11},\frac{1-5^{-\Upsilon_2}}{11},\theta_1\aleph\right)\\
&=\frac{\left|\frac{1-5^{-\Theta}}{11}-\frac{1-5^{-\Upsilon_2}}{11}\right|}{\theta_1\aleph+\left|\frac{1-5^{-\Theta}}{11}-\frac{1-3^{-\Upsilon_2}}{11}\right|}=\frac{\frac{|5^{-\Theta}-5^{-\Upsilon_2}|}{11}}{\theta_1\aleph+\frac{|5^{-\Theta}-5^{-\Upsilon_2}|}{11}}\\
&=\frac{|5^{-\Theta}-5^{-\Upsilon_2}|}{11\theta_1\aleph+|5^{-\Theta}-5^{-\Upsilon_2}|}\leq\frac{|\Theta-\Upsilon_2|}{11\theta_1\aleph+|\Theta-\Upsilon_2|}\leq\frac{|\Theta-\Upsilon_2|}{\aleph+|\Theta-\Upsilon_2|}=\Phi(\Theta,\Upsilon_2,\aleph),\end{aligned}$$

and

$$\Xi(\dot{y}\Theta, \dot{y}\Upsilon_2, \theta_1 \aleph) = \Xi\left(\frac{1-5^{-\Theta}}{11}, \frac{1-5^{-\Upsilon_2}}{11}, \theta_1 \aleph\right)$$

$$= \frac{\left|\frac{1-5^{-\Theta}}{11} - \frac{1-5^{-\Upsilon_2}}{11}\right|}{\theta_1 \aleph} = \frac{\frac{|5^{-\Theta} - 5^{-\Upsilon_2}|}{11}}{\theta_1 \aleph}$$

$$= \frac{|5^{-\Theta} - 5^{-\Upsilon_2}|}{11\theta_1 \aleph} \le \frac{|\Theta - \Upsilon_2|}{11\theta_1 \aleph} \le \frac{|\Theta - \Upsilon_2|}{\aleph} = \Xi(\Theta, \Upsilon_2, \aleph).$$

Hence, all of the hypothesis of Theorem 3.1 are satisfied, and is the only fixed point for $\dot{y}$.

4. Applications

Now, we remember some elementary concepts from the theory of fractional calculus. For a function $\Theta \in \mathcal{C}[0,1]$, the Reiman-Liouville fractional derivative of order $\delta_1 > 0$ is follows as

$$\frac{1}{\Gamma(\tau - \delta_1)} \frac{d^\tau}{d\zeta^\tau} \int_0^\zeta \frac{\Theta(\mathfrak{c})d\mathfrak{c}}{(\zeta - \mathfrak{c})^{\delta_1 - \tau + 1}} = \mathcal{D}_1^\delta \Theta(\zeta),$$

proved that pointwise identified on [0,1], where $[\delta_1]$ is the integer element of the number δ_1, Γ is the Euler gamma function.
Now, let the given FDE

$$^{\mathfrak{c}}\mathcal{D}^\xi \Theta(\zeta) + \mathfrak{f}(\zeta, \Theta(\zeta)) = 0, \quad 0 \le \zeta \le 1, \quad 1 < \xi \le 2;$$
$$\Theta(0) = \Theta(1) = 0, \tag{4.1}$$

where $\mathfrak{f}$ is a continuous function from $[0,1] \times \mathfrak{R}$ to $\mathfrak{R}$, and $^{\mathfrak{c}}\mathcal{D}^\xi$ denotes the Caputo fractional derivative of order ξ and it is denoted by

$$^{\mathfrak{c}}\mathcal{D}^\xi = \frac{1}{“(\tau - \xi)} \int_0^\zeta \frac{\Theta^\tau(\mathfrak{c})d\mathfrak{c}}{(\zeta - \mathfrak{c})^{\xi - \tau + 1}}.$$

The given Equation (4.1) is equivalent

$$\Theta(\zeta) = \int_0^1 \Omega(\zeta, \mathfrak{c}) \mathfrak{f}(\zeta, \Theta(\mathfrak{c})) d\mathfrak{c},$$

for all $\Theta \in \mathcal{Y}$ and $\zeta \in [0,1]$, where

$$\Omega(\zeta, c) = \begin{cases} \dfrac{[\zeta(1-c)]^{\xi-1} - (\zeta - c)^{\xi-1}}{\Gamma(\xi)}, & 0 \le c \le \zeta \le 1, \\ \dfrac{[\zeta(1-c)]^{\xi-1}}{\Gamma(\xi)}, & 0 \le \zeta \le c \le 1. \end{cases}$$

Consider the space of all continuous functions $\mathcal{C}([0,1], \mathfrak{R}) = \mho^\star$ be identified on [0, 1]. Define Λ, Φ and Ξ as follows:

$$\Lambda(\Theta(\zeta), \Upsilon_2(\zeta), \aleph) = \sup_{\zeta \in [0,1]} \frac{\aleph}{\aleph + |\Theta(\zeta) - \Upsilon_2(\zeta)|}, \ \forall\ \Theta, \Upsilon_2 \in \mho^\star \text{ and } \aleph > 0,$$

$$\Phi(\Theta(\zeta), \Upsilon_2(\zeta), \aleph) = 1 - \sup_{\zeta \in [0,1]} \frac{\aleph}{\aleph + |\Theta(\zeta) - \Upsilon_2(\zeta)|}, \ \forall\ \Theta, \Upsilon_2 \in \mho^\star \text{ and } \aleph > 0,$$

and

$$\Xi(\Theta(\zeta),\curlyvee_2(\zeta),\aleph) = \sup_{\zeta\in[0,1]} \frac{|\Theta(\zeta)-\curlyvee_2(\zeta)|}{\aleph}, \ \forall \ \Theta,\curlyvee_2 \in \mho^\star \text{ and } \aleph > 0,$$

with continuous t-norm and continuous t-co-norm define by $\upsilon \circledast h_\ell = \upsilon \cdot h_\ell$, and $\upsilon \odot h_\ell = \max\{\upsilon, h_\ell\}$. Define $\xi : \mho^\star \times \mho^\star \to [1,+\infty)$ as

$$\xi(\Theta,\curlyvee_2,\aleph) = \frac{\aleph}{\aleph + |\Theta - \curlyvee_2|}.$$

Then, $(\mho^\star, \Lambda, \Phi, \Xi, \circledast, \odot)$ is a complete NCPMS.

Theorem 4.1 *Consider the nonlinear FDE (4.1). Suppose that the given conditions are holds: [label=()]*

1. there exists $\zeta \in [0,1]$, and $\Theta,\curlyvee_2 \in \mathcal{C}([0,1],\mathfrak{R})$ s.t.
   $|\mathfrak{f}(\zeta,\Theta) - \mathfrak{f}(\zeta,\curlyvee_2)| \le |\Theta(\zeta) - \curlyvee_2(\zeta)|$;
2. $\sup_{\zeta\in[0,1]} \int_0^1 \Omega(\zeta,\mathfrak{c})d\zeta \le \theta_1 < 1.$

Then, the Equation (4.1) has a unique solution.
Proof. Let $\dot{y} : \mho^\star \to \mho^\star$ defined as

$$\mho^\star\Theta(\zeta) = \int_0^1 \Omega(\zeta,\mathfrak{c})\mathfrak{f}(\zeta,\Theta(\mathfrak{c}))d\mathfrak{c}.$$

It is clear that if $\Theta^* \in \mho^\star$ is a fixed point of $\dot{y}$ then Θ^* is a solution of the problem (4.1).
Now, $\forall \ \Theta,\curlyvee_2 \in \mho^\star$, we deduce

$$\begin{aligned}
\Lambda(\dot{y}\Theta(\zeta),\dot{y}\curlyvee_2(\zeta),\theta_1\aleph) &= \sup_{\zeta\in[0,1]} \frac{\theta_1\aleph}{\theta_1\aleph + |\dot{y}\Theta(\zeta) - \dot{y}\curlyvee_2(\zeta)|} \\
&= \sup_{\zeta\in[0,1]} \frac{\theta_1\aleph}{\theta_1\aleph + \left|\int_0^1 \Omega(\zeta,\mathfrak{c})\mathfrak{f}(\zeta,\Theta(\mathfrak{c}))d\mathfrak{c} - \int_0^1 \Omega(\zeta,\mathfrak{c})\mathfrak{f}(\zeta,\curlyvee_2(\mathfrak{c}))d\mathfrak{c}\right|} \\
&= \sup_{\zeta\in[0,1]} \frac{\theta_1\aleph}{\theta_1\aleph + \int_0^1 \Omega(\zeta,\mathfrak{c})|\mathfrak{f}(\zeta,\Theta(\mathfrak{c})) - \mathfrak{f}(\zeta,\curlyvee_2(\mathfrak{c}))|d\mathfrak{c}} \\
&\ge \sup_{\zeta\in[0,1]} \frac{\aleph}{\aleph + |\Theta(\zeta) - \curlyvee_2(\zeta)|} \\
&\ge \Lambda(\Theta(\zeta),\curlyvee_2(\zeta),\aleph),
\end{aligned}$$

$$\begin{aligned}
\Phi(\dot{y}\Theta(\zeta),\dot{y}\curlyvee_2(\zeta),\theta_1\aleph) &= 1 - \sup_{\zeta\in[0,1]} \frac{\theta_1\aleph}{\theta_1\aleph + |\dot{y}\Theta(\zeta) - \dot{y}\curlyvee_2(\zeta)|} \\
&= 1 - \sup_{\zeta\in[0,1]} \frac{\theta_1\aleph}{\theta_1\aleph + \left|\int_0^1 \Omega(\zeta,\mathfrak{c})\mathfrak{f}(\zeta,\Theta(\mathfrak{c}))d\mathfrak{c} - \int_0^1 \Omega(\zeta,\mathfrak{c})\mathfrak{f}(\zeta,\curlyvee_2(\mathfrak{c}))d\mathfrak{c}\right|} \\
&= 1 - \sup_{\zeta\in[0,1]} \frac{\theta_1\aleph}{\theta_1\aleph + \int_0^1 \Omega(\zeta,\mathfrak{c})|\mathfrak{f}(\zeta,\Theta(\mathfrak{c})) - \mathfrak{f}(\zeta,\curlyvee_2(\mathfrak{c}))|d\mathfrak{c}} \\
&\le 1 - \sup_{\zeta\in[0,1]} \frac{\aleph}{\aleph + |\Theta(\zeta) - \curlyvee_2(\zeta)|} \\
&\le \Phi(\Theta(\zeta),\curlyvee_2(\zeta),\aleph),
\end{aligned}$$

and

$$\Xi(\dot{y}\Theta(\zeta),\dot{y}\Upsilon_2(\zeta),\theta_1\aleph) = \sup_{\zeta\in[0,1]}\frac{|\dot{y}\Theta(\zeta)-\dot{y}\Upsilon_2(\zeta)|}{\theta_1\aleph}$$
$$= \sup_{\zeta\in[0,1]}\frac{\left|\int_0^1 \Omega(\zeta,\mathfrak{c})\mathfrak{f}(\zeta,\Theta(\mathfrak{c}))d\mathfrak{c}-\int_0^1 \Omega(\zeta,\mathfrak{c})\mathfrak{f}(\zeta,\Upsilon_2(\mathfrak{c}))d\mathfrak{c}\right|}{\theta_1\aleph}$$
$$= \sup_{\zeta\in[0,1]}\frac{\int_0^1 \Omega(\zeta,\mathfrak{c})|\mathfrak{f}(\zeta,\Theta(\mathfrak{c}))-\mathfrak{f}(\zeta,\Upsilon_2(\mathfrak{c}))|d\mathfrak{c}}{\theta_1\aleph}$$
$$\leq \sup_{\zeta\in[0,1]}\frac{|\Theta(\zeta)-\Upsilon_2(\zeta)|}{\aleph}\leq \Xi(\Theta(\zeta),\Upsilon_2(\zeta),\aleph).$$

Hence, all of the hypothesis of Theorem (3.1) are satisfied and $\dot{y}$ has a unique fixed point. Therefore, an Equation (4.1) has a unique solution. □

5 Conclusion

The concept of NCPMS was proposed in this study, as well as several new forms of fixed point results that may be provided in this innovative context. We have supplemented our work with illustrative applications for checking the effectiveness of new findings that were better than existing methods in the literature. Saleem et al. [36] introduced the notion of neutrosophic rectangular extended b-metric spaces and proved fixed point results. It is an interesting open problem to neutrosophic pentagonal extended b-metric spaces and proves fixed point results.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing Interests

The authors declare that they have no competing interests.

Authors contributions

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