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Exact solution of system of multi-photograph type delay differential equations via new algorithm based on homotopy perturbation method

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Abstract

A new algorithm is proposed in this paper to explain how the modified homotopy perturbation approach can be successfully implemented based on the Pade approximants and the Laplace transform in order to acquire the accurate solutions of a nonlinear system of multi-photograph delay differential equations. The method that has been suggested has the benefit of giving exact solutions, and it is simple to implement analytically on the issues that have been presented. Examples have been provided to demonstrate that this strategy may be utilized and is successful in its application. The results show that the method that was described could be used to solve many different types of differential equations.

Key words and phrases: HPM Procedure, Series Expansion, Laplace Trans-form, Pade Approximant, Pantograph equations.

Mathematics Subject Classi ication (MSC): 34K28, 74H10, 74G10, 65P10

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1. Introduction

In applied science and engineering, there are several nonlinear phenomena that are important. Nonlinear equations are present in a wide range of physics-related phenomena, such as wave propagation in shallow water, solid mechanics, quantum field theory, fluid dynamics, and many more. Multi-photograph delay differential equations regulate the models' range of applicability. The major reason mathematicians are aware of these equations is because of how frequently they are used. Nevertheless, finding a solution to these mathematical issues is neither theoretically nor numerically simple. Finding exact or approximative solutions to these kinds of equations has received a lot of focus in recent studies.

Therefore, it is important to get knowledgeable about all analytical, numerical, and recently developed methods for solving nonlinear multi-photograph delay differential equations: for instance, the variational iteration method [1, 2], the Adomian decomposition method [3], The spectral tau technique is also used to solve systems of multi-pantograph equations with shifted Jacobi polynomials as basis functions[4], Taylor matrix method [5] and others [6–12].

The majority of the multi-photograph differential equations solved in [21–23] are covered by the system of multi-photograph equation, which is of the following form.

$$\begin{aligned} u_i^{'}(t) &= \beta_1 u_1(t) + f_1(t, u_1(\alpha_{11}t), u_2(\alpha_{12}t), \ldots, u_n(\alpha_{1n}t)), \\ u_2^{'}(t) &= \beta_2 u_2(t) + f_2(t, u_1(\alpha_{21}t), u_2(\alpha_{22}t), \ldots, u_n((\alpha_{2n}t)), \end{aligned} \tag{1}$$

subject to

$$u_i(t_0) = u_i, \quad t_0 < t \leq T, \quad i = 1, 2, \tag{2}$$

where β_i, u_i are constant, f_i denoted analytical functions in a way that $0 < \alpha_i j \leq 1$, $i, j = 1, 2, ..., n$, and $u_i(t)$, $i = 1, 2, ..., n$ are yet-to-be-determined functions. The HPM method is straightforward and effective numerical method that has been widely employed in recent years for a variety of linear and nonlinear differential equations without lineariazation, descretization, or perturbation [13–20]. It has also yielded accurate approximate solutions for various problems in the limit of infinite approximation terms. Thus, the biggest challenge for scholars is finding an accurate method that efficiently yields the closed form of the exact solution.

This paper aims to improve the HPM series solution. In order to achieve this, the truncated HPM solution is first transformed using the laplace transform; the transformed series is then transformed into a meromorphic function using Pade approximants; and last, the exact solution of the provided system is determined using the inverse laplace transformation.

The following topics will be covered in the next sections of this article: By describing the model and the process for finding accurate solutions to the given problem, we outline the essential concepts of the research methodology of this study in Section 2. And section 3 applies the suggested method to a few illustrative examples to prove its applicability, convergence, and effectiveness. Conclusions are contained in the final section.

2. Research Methods Description

We outline the fundamental concepts of the research methodologies in this part.

2.1 Review of the HPM

To highlight the fundamental concept of the HPM procedure, We utilize the nonlinear problem described below:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

where A denotes to the integral operator, B for the boundary operator, $f(r)$ is a known function, and the boundary of the domain Ω is indicate Γ is. In general, the operator A can be split into two parts: L and N. L is linear, whereas N is nonlinear. Thus, we may rewrite Eq. (1) as follows:

$$L(u) - N(u) - f(r) = 0. \tag{4}$$

He [4] formulated a homotopy $v : \Omega \times [0,1] \to R$ which performs

$$H(v;p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0. \tag{5}$$

or

$$H(v;p) = (1-p)[L(v) - L(v_0)] + p[A(v_0) - F(r)] = 0, \tag{6}$$

where $r \in \Omega$, $p \in [0,1]$ is known as the homotopy parameter, and $v_0(x)$ is an initial approximation of Eq. (1). Hence, it is obvious that

$$H(v;0) = L(u) - L(v_0) = 0, \quad H(v;1) = A(v) - F(r) = 0, \tag{7}$$

as well as the way that p changes from 0 to 1, is only that of $H(v,p)$ from $L(u)$–$L(v_0)$ to $A(v)$–$F(r)$. We refer to this as deformation in topology, where $L(u)$–$L(v_0)$ and $A(v)$–$F(r)$ are known homotopic. Using the method of perturbation [16], as a result of the fact that $0 \le p \le 1$ can be regarded as a small parameter. Assuming that the solution of Eqs. (4) or (5) is represented by a series in p, as:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots, \tag{8}$$

when p $\to$1, Eq. (4) or Eq. (5) corresponding to Eq. (3) and consequently becomes an approximation of Eq. (1). i.e.,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{9}$$

2.2 Padé technique approximation

Padé technique [18] is the ratio of two polynomials initiated derived from coefficients of a function's Taylor series expansion $y(x)$.

The $[L/M]$ Padé approximants to a function $y(x)$ are provided by

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)},$$

where $P_L(x)$ and $Q_M(x)$ are a polynomials of degree at most L and M, respectively. The power series

$$y(x) = \sum\nolimits_{i=1}^{\infty} a_i x^i,$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = \mathrm{O}\left(x^{L+M+1}\right), \tag{10}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation. Given that there is no change in $[L/M]$ when the numerator and denominator are multiplied by a constant. The need for normalization is then enforced

$$Q_M(0) = 1. \tag{11}$$

Eventually, There must be no factors in common between $P_L(x)$ and $Q_M(x)$. The coefficients of $P_L(x)$ and $Q_M(x)$ can be expressed as

$$\begin{cases} P_L(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L, \\ Q_M(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M, \end{cases} \tag{12}$$

therefore, by Eqs. (11) and (12), we multiply Eq. (10) by $Q_M(x)$, that linearizes the coefficient equations. More specifically, we can express Eq. (10) as

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0, \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0, \\ . \\ . \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0, \end{cases} \tag{13}$$

$$\begin{cases} a_0 = p_0 \\ a_0 + a_0 q_1 = p_1, \\ a_2 + a_1 q_1 + a_0 q_2 = p_2 \\ ., \\ . \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L. \end{cases} \tag{14}$$

As a starting point for solving these equations, We begin with Eq. (13), which is a set of linear equations for each unknown q. Following the determination of the $q's$, the solution is completed by Eq. (14), which provides an explicit formula for the unknown $p's$.

If Eq. (14) and Eq. (13) are non-singular. Then we may directly solve them and get the Eq. (15) [20], where Eq. (15) holds, If the lower index of a sum is greater than the higher index, the sum is substituted by zero:

$$\left[\frac{L}{M}\right] = \frac{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \dots & a_{L+1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_L & a_{L+1} & \dots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \dots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \dots & a_{L+1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_L & a_{L+1} & \dots & a_{L+M} \\ x^M & x^{M-1} & \dots & 1 \end{bmatrix}} \tag{15}$$

using computer software like Mathematica, diagonal Padé approximants of various orders, like [2/2], [4/4] or [6/6] will be obtained.

3. Numerical Examples

To illustrate how effective the new method is, consider the nonlinear system of multi-pantograph equations shown below.

3.1 Example 1

The following nonlinear system is the first tested example [23],

$$u_1'(x) = -u_1 - e^{-x}\cos(\frac{x}{2})u_2(\frac{x}{2}) - 2e^{(-\frac{3}{4}x)}\cos(\frac{x}{2})\sin(\frac{x}{4})u_1(\frac{x}{4}),$$

$$u_2'(x) = e^{x}u_1^2(\frac{x}{2}) - u_2^2(x2), \tag{16}$$

according to conditions listed below

$$u_1(0) = 1, \quad u_2(0) = 0. \tag{17}$$

In view of our HPM presented algorithm, the following family of homotopy equations will be constructed as given below

$$(1-p)\left[\frac{d}{dx}\left(\sum_{m=1}^{i} v_{1,m}(x;p)\right)\right] = h.p[\frac{d}{dx}\left(\sum_{m=0}^{i} v_{1,m}(x;p)\right)$$

$$+\left(\sum_{m=0}^{i} v_{1,m}(x;p)\right) + \left(\sum_{m=0}^{i} v_{2,m}(\frac{x}{2};p)\right)\left(\sum_{m=0}^{i} \frac{(-x)^m}{m!}\right)\left(\sum_{m=0}^{i} (-1)^m \frac{(\frac{x}{2})^{2m}}{(2m)!}\right)$$

$$+2\left(\sum_{m=0}^{i} v_{1,m}(\frac{x}{4};p)\right)\left(\sum_{m=0}^{i} \frac{(\frac{-3x^m}{4})^m}{m!}\right)\left(\sum_{m=0}^{i} (-1)^m \frac{(\frac{x}{2})^{2m}}{(2m)!}\right)\left(\sum_{m=0}^{i} (-1)^m \frac{(\frac{x}{4})^{2m+1}}{(2m+1)!}\right),$$

$$(1-p)\left[\frac{d}{dx}\left(\sum_{m=0}^{i} v_{1,m}(x;p)\right)\right] = \left(\sum_{m=0}^{i} \frac{(x)^m}{m!}\right)\left(\sum_{m=0}^{i} v_{1,m}(\frac{x}{2};p)\right)^2\left(\sum_{m=0}^{i} v_{2,m}(\frac{x}{2};p)\right)^2 \tag{18}$$

By using the same power to equalize the coefficient of p in Eq. (18) we get

$$\begin{aligned}
u_{1,1}'(x) &= \frac{1}{2}h(x+2),\\
u_{2,1}'(x) &= -h,\\
u_{1,2}'(x) &= \frac{1}{2}h\left(h\left(\frac{x^3}{64} + \frac{3x^2}{4} + 2x + 2\right) - \frac{13x^3}{96} - \frac{3x^2}{4} + x + 2\right),\\
u_{2,2}'(x) &= -\frac{1}{8}h\left(h\left(x^2 + 8x + 8\right) + 8(x+1)\right)\\
u_{1,3}'(x) &= \frac{h^3t^5}{262144} + \frac{3h^3t^4}{1024} + \frac{55h^3t^3}{384} + \frac{7h^3t^2}{8} + \frac{3h^3t}{2} + h^3 - \frac{143h^2t^5}{131072} - \frac{125h^2t^4}{3072}\\
&\quad - \frac{5h^2t^3}{24} + \frac{3h^2t^2}{4} + 2h^2t + 2h^2 + \frac{121ht^5}{61440} + \frac{13ht^4}{256} + \frac{7ht^3}{96} - \frac{3ht^2}{8}\\
u_{2,3}'(x) &= -\frac{17h^3t^4}{4096} - \frac{3h^3t^3}{32} - \frac{3h^3t^2}{8} - 2h^3t - h^3 + \frac{13h^2t^4}{6144} - \frac{3h^2t^3}{32} - \frac{5h^2t^2}{4}\\
&\quad -3h^2t - 2h^2 - \frac{ht^2}{2} - ht - h.
\end{aligned} \tag{19}$$

Consequently, the solution of problem (16) in a series form given by

$$u_1(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{30} - \frac{x^7}{630} + \frac{x^8}{2520} - \frac{250858063x^9}{4058744094720} + \frac{43097613335687x^{10}}{7979815589747097600} - \frac{2917177597567x^{11}}{1595963117949419520}$$
$$+ \frac{154423055301507277997x^{12}}{631144280435528082810470400} + \frac{207480935848103698763x^{13}}{17672039852194786318693171200}$$
$$+ \frac{259603894696995551618568131x^{14}}{981104520299639286435380929316782080}$$
$$+ \frac{6035368287615086090721482 89x^{15}}{215842994465920643015783804449692057600}$$
$$+ \frac{30758445716011609010477739730935 9311x^{16}}{27811158069055144448522467557536599939888447488000}. \tag{20}$$

$$u_2(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{3523005637x^9}{1558557732372480} - \frac{78357979363309x^{11}}{6420331628779379097600}$$
$$+ \frac{1168222939427785212 11x^{13}}{32902125106631747618803286 01600}$$
$$- \frac{1094354526412209224568131341 79x^{15}}{2411162469088393510343592171888923639808000}. \tag{21}$$

within in the limit of infinity terms, leads to exact solution. It can be seen from results parented in Tables 1 and 2 and Fig. 1 that the solution obtained HPM algorithm of eight-order is quasi-identical with exact solution, demonstrating very good accuracy with low remarkable errors based on order of the HPM approximation, which means that more efforts need to get high degree of accuracy, from this point, we will modify the HPM algorithm to get accurate results closed to the exact form based on the HPM series solution using alternative technique based on the laplace transform and Padé approximants throughout utilizing the laplace transform to the first three terms of the HPM series solutions, yields

$$L\{u_1(x)\} = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6},$$
$$L\{u_2(x)\} = \frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^4}, \tag{22}$$

therefore, for the sake of simplicity, we let $z = \frac{1}{z}$,

$$L\{u_1(x)\} = z - z^2 + 2z^4,$$
$$L\{u_2(x)\} = z^2 - z^4 + z^6, \tag{23}$$

and by using Pade approximants, respectively, $[\frac{2}{2}] = \frac{z^2 + z}{2z^2 + 2z + 1}$ and $[\frac{2}{2}] = \frac{z}{z-1}$, Recalling $z = \frac{1}{s}$, we have the $[\frac{2}{2}] = \frac{1+s}{2+2s+s^2}$ and $[\frac{2}{2}] = \frac{1}{(1+\frac{1}{s^2})s}$ and then using laplace inverse to $[\frac{2}{2}]$ Pade approximants, we have the exact solutions

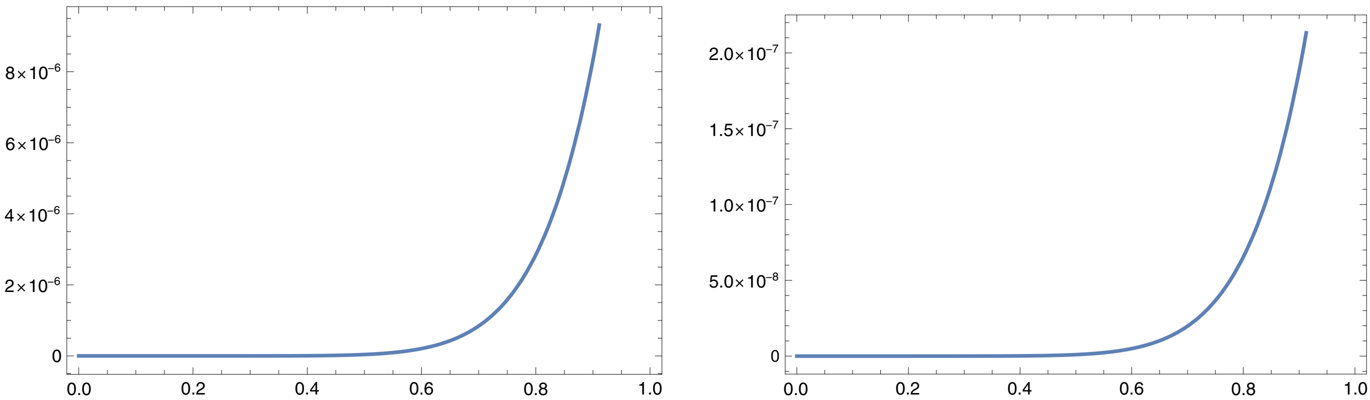
$$u_1(x) = e^{-x}\cos(x).$$
$$u_2(x) = \sin(x). \tag{24}$$

Table 1: Comparison of the exact solution $u(x)=e^{-x}\cos x$ for example 1 with the Eq. (20) (HPM solution).

x	Exact Solution	HPM Solution	Absolute Error
0.0	1.0000000000	1.0000000000	0.00
0.2	0.8024106473	0.8024106473	9.60×10^{-12}
0.4	0.6174056479	0.6174056427	5.16×10^{-9}
0.6	0.4529537891	0.4529535825	2.07×10^{-7}
0.8	0.3130505040	0.3130476607	2.84×10^{-6}
1.0	0.1987661103	0.1987444007	2.17×10^{-5}

Table 2: Comparison of the exact solution $u(x) = \sin x$ for example 1 with the Eq. (21) (HPM solution).

x	Exact Solution	HPM Solution	Absolute Error
0.0	0.0000000000	0.0000000000	0.00
0.2	0.1986693308	0.1986693308	2.53×10^{-13}
0.4	0.3894183423	0.3894183422	1.29×10^{-10}
0.6	0.5646424734	0.5646424684	4.95×10^{-9}
0.8	0.7173560909	0.7173560255	6.54×10^{-8}
1.0	0.8414709848	0.8414705022	4.83×10^{-7}



(a) (b)

Figure 1: Plot of the absolute errors resulted from the HPM approximate solution Eqs. (20) and (21) of eight order and the exact solution Eq. (24) for example 1.

3.2 Example 2

The nonlinear system of multi-pantograph equations shown below [23],

$$u_1'(x) = 2u_2(\frac{x}{2}) + u_3(x) - x\cos(\frac{x}{2}),$$

$$u_2'(x) = 1 - x\sin(x) - 2u_3^2(\frac{x}{2}),$$

$$u_3'(x) = u_2(x) - u_1(x) - x\cos(x), \tag{25}$$

subjects to

$$u_1(0) = -1, \quad u_2(0) = 0, \quad u_3(0) = 0. \tag{26}$$

In view of our HPM presented algorithm, the following family of homotopy equations will be constructed as given below

$$(1-p)\left(\frac{d\left(v_{1,i}(x;p)\right)}{dx}\right) = (h;p)\left(x\sum_{m=0}^{i}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m}}{2m!} + \frac{d\left(v_{1,i}(x;p)\right)}{dx} - 2v_{2,m}(\frac{x}{2}) - v_{3,m}(x)\right),$$
$$(1-p)\left(\frac{dv_{2,i}(x;p)}{dx}\right) = (h;p)\left(x\sum_{m=0}^{i}\frac{(-1)^m x^{2m+1}}{((2m+1)!)!} + \frac{dv_{2,i}(x;p)}{dx} + 2v_{3,m}^2\left(\frac{x}{2}\right) - 1\right)$$
$$(1-p)\left(\frac{dv_{3,i}(x;p)}{dx}\right) = (h;p)\left(x\sum_{m=0}^{i}\frac{(-1)^m x^{2m}}{2m!} + \frac{dv_{3,i}(x;p)}{dx} + v_{1,m}(x) - v_{2,m}(x)\right) \tag{27}$$

By using the same power to equalize the coefficient of p in Eq.(27) we get

$$\begin{aligned}
u_{1,1}'(x) &= hx,\\
u_{2,1}'(x) &= h(x^2-1),\\
u_{3,1}'(x) &= h(x-1),\\
u_{1,2}'(x) &= \frac{1}{8}(-16hu_{2,1}(\frac{x}{2}) - hx(4h(x-4) + x^2 - 8))\\
u_{2,2}'(x) &= \frac{1}{6}h\left(6h\left(x^2-1\right) - x^4 + 6x^2 - 6\right)\\
u_{3,2}'(x) &= -\frac{1}{6}h\left(h\left(2x^3 - 3x^2 - 12x + 6\right) + 3\left(x^3 - 2x + 2\right)\right).
\end{aligned} \tag{28}$$

Consequently, the solution of problem (25) in a series form given by

$$\begin{aligned}
u_1^{\sim}(x) = -1 &+ \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \frac{13x^7}{18432} + \frac{229x^8}{9175040} + \frac{9599x^9}{1189085184}\\
&- \frac{383t^{10}}{59454259200} - \frac{x^{11}}{39916800} + \frac{x^{12}}{490497638400}.
\end{aligned} \tag{29}$$

$$\begin{aligned}
u_2^{\sim}(x) = x &- \frac{x^3}{2} + \frac{x^5}{24} - \frac{13x^7}{10752} + \frac{17x^8}{737280} + \frac{743x^9}{46448640} + \frac{29x^{10}}{123863040}\\
&- \frac{461x^{11}}{2043740160} + \frac{x^{13}}{518918400}.
\end{aligned} \tag{30}$$

$$\begin{aligned}
u_3^{\sim}(x) = x &- \frac{x^3}{6} + \frac{x^5}{120} - \frac{251x^7}{322560} + \frac{353x^8}{3440640} - \frac{1159x^9}{371589120} - \frac{191x^{10}}{232243200}\\
&+ \frac{x^{11}}{10218700800} + \frac{x^{12}}{479001600}.
\end{aligned} \tag{31}$$

It results in the exact solutions in the terms that approach infinity. The numerical results of the HPM approximate solution are shown in the tables 3. Also, the absolute error are plotted in Fig. 2. To improve the HPM procedure's accuracy and effectiveness, The Laplace transform will be applied to the first five terms of the HPM approximation series solution (25), resulting in

$$Lu_1^{\sim}(x) = -\frac{1}{s^9} + \frac{1}{s^7} - \frac{1}{s^5} + \frac{1}{s^3} - \frac{1}{s}. \quad (32)$$

therefore, for the sake of simplicity, we let $s = \frac{1}{z}$, we have

$$Lu^{\sim}(x) = -z + z^3 - z^5 + z^7 - z^9. \quad (33)$$

We now employ the Pad approximants of $[\frac{4}{4}] = -(z/(1+z^2))$ and then using $z = \frac{1}{s}$, we get the exact solution by using the inverse Laplace transform

$$u(x) = -\cos x. \quad (34)$$

and by using the same process, we are able to get the exact solution for the system mentioned above.

$$u_1(x) = -\cos x.$$
$$u_2(x) = x\cos x.$$
$$u_3(x) = \sin x. \quad (35)$$

Table 3: Comparison of the exact solution $u(x) = -\cos x$, for example 2 with the Eq. (29) (HPM solution).

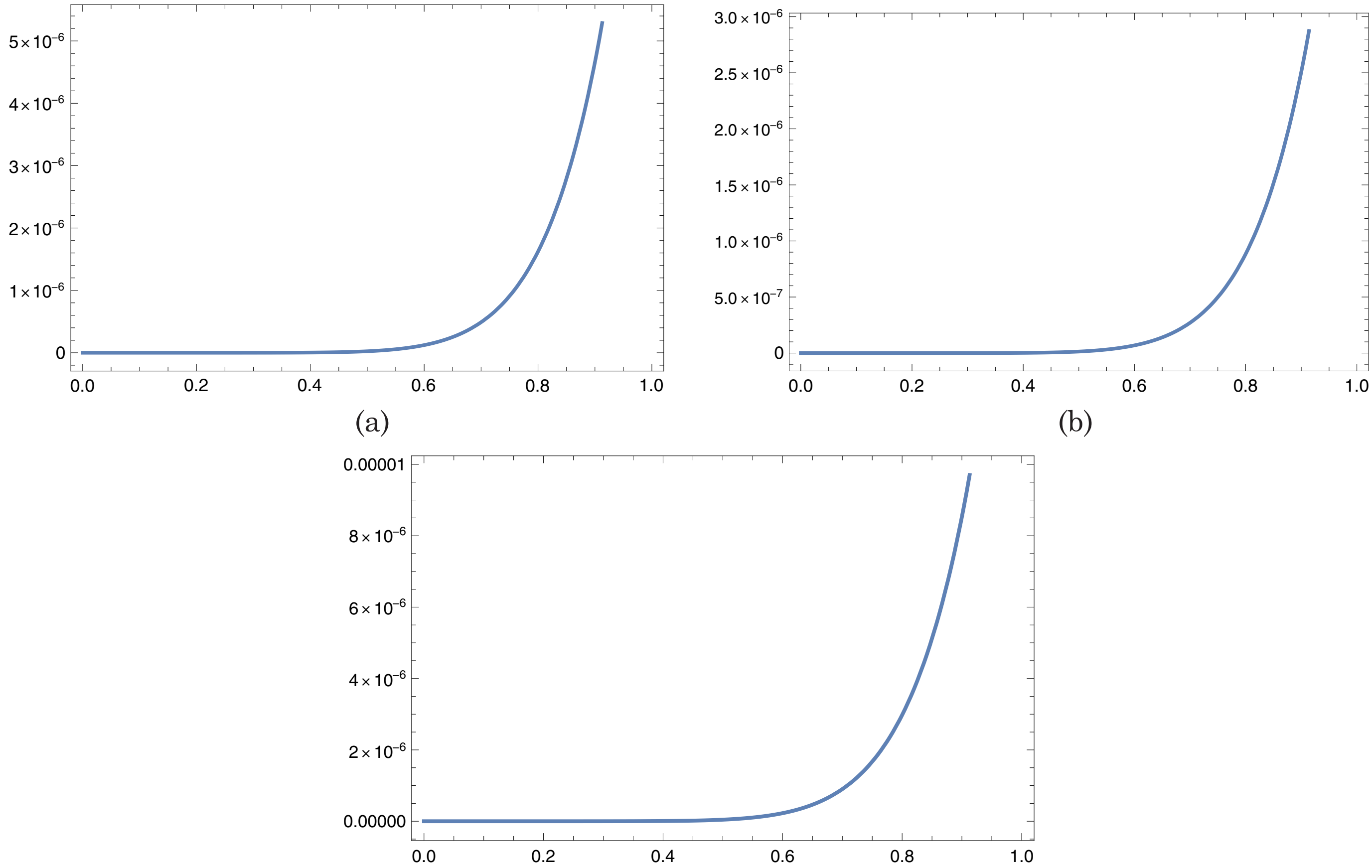
x	Exact Solution	HPM Solution	Absolute Error
0.0	0.0000000000	0.0000000000	0.00
0.2	–0.9800665778	–0.9800665778	6.32×10^{-12}
0.4	–0.9210609940	–0.9210609908	3.22×10^{-9}
0.6	–0.8253356149	–0.8253354922	1.23×10^{-7}
0.8	–0.6967067093	–0.6967050881	1.62×10^{-6}
1.0	–0.5403023058	–0.5402903397	1.20×10^{-5}

Table 4: Comparison of the exact solution $u(x)=x\cos x$ for example 2 with the Eq. (30) (HPM solution).

x	Exact Solution	HPM Solution	Absolute Error
0.0	0.0000000000	0.0000000000	0.00
0.2	0.1960133155	0.1960133156	3.74×10^{-12}
0.4	0.3684243976	0.3684243995	1.85×10^{-9}
0.6	0.4952013689	0.4952014376	6.87×10^{-8}
0.8	0.5573653675	0.5573662512	8.84×10^{-7}
1.0	0.5403023058	0.5403086663	6.36×10^{-6}

Table 5: Comparison of the exact solution $u(x) = \sin x$ for example 2 with the Eq. (31) (HPM solution).

x	Exact Solution	HPM Solution	Absolute Error
0.0	0.0000000000	0.0000000000	0.00
0.2	0.1986693308	0.1986693308	1.86×10^{-11}
0.4	0.3894183423	0.3894183483	5.98×10^{-9}
0.6	0.5646424734	0.5646427001	2.27×10^{-7}
0.8	0.7173560909	0.7173590674	2.98×10^{-6}
1.0	0.8414709848	0.8414928497	2.19×10^{-5}



(a) (b) (c)

Figure 2: Plot of the absolute errors resulted from the HPM solutions Eqs. (29), (30) and (31) of eight order and the exact solution (35) for example 2.

4. Conclusion

This research focused on a new approach based basically on the HPM procedure to obtain exact solutions for a system of multi-pantograph type DDEs. Throughout the illustrated cases and comparison with numerical results provided in the literature, the MHPM approach has been shown to be sufficient to generate the exact analytical solution by utilizing only a few terms of the HPM truncated series solution. As a result of our study, we were able to say that the proposed procedure gives very useful analytical results with less processing work strongly nonlinear problems can be solved with this strategy, which gives it an advantage over other methods. It is also dependable and effective and

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