Results in Nonlinear Analysis 7 (2024) No. 3, 29–34
https://doi.org/10.31838/rna/2024.07.03.005
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Some generalized singular value and norm inequalities for sums and products of matrices

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This work presents a generalized singular value and norm inequalities associated with 2×2 positive semidefinite block matrices.

Keywords and phrases: Singular value, Unitarily, Invariant norm, Positive operator

Mathematics Subject Classification (2010): Primary 47A30, Secondary 15A18, 47A63, 47TB10.

1. Introduction and Preliminaries

Let $\mathcal{M}_n(\mathbb{C})$ stand for the space of $n \times n$ complex matrices. We consider unitarily invariant norm $|||A||| = |||UAV|||$ for all A, U and $V \in \mathcal{M}_n(\mathbb{C})$ where U, V are unitary matrices. For two Hermitian matrices $A, B \in \mathcal{M}_n(\mathbb{C})$, one can write $A \le B$ to mean $B - A$ is positive semidefinite. Especially $A \geq 0$ indicates that A is positive semidefinite. Also, $A > 0$ is named positive definite. The singular value of A is restricted by $s_1(A) \geq s_2(A) \geq . . . s_n(A) \geq 0$, that is the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$ (the absolute value of A), arranged in decreasing order are repeated according to multiplicity. Note that $s_j(A) = s_j(A^*) = s_j(A)$ for $j = 1,2,...,n$. The operator norm of A is represent by $\|A\| = s_1(A)$. We use the direct sum notation $A \oplus B$ for the block-diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

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Received April 4, 2024; Accepted May 29, 2024; Online June 22, 2024

For the general theory of unitarily invariant norms and more additional results about 2×2 positive semidefinite block matrices and related inequalities, we refer the reader to [4], [7] and [8]. It is evident that if A, B and $X\in\mathcal{M}_n(\mathbb{C})$ such that $\begin{bmatrix} A & 0 \\ B^* & X \end{bmatrix}\geq 0,$ then A and X are positive semidefinite.

In [11] the first author showed that for $A,B,X\in\mathcal{M}_n(\mathbb{C})$ and f and g be non-negative continuous functions on $[0,\infty)$ that satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$, then

$$\begin{bmatrix} A^* f\left(\left|X^*\right|\right)^2 A & A^* X^* B \\ B^* X A & B^* g\left(\left|X\right|\right)^2 B \end{bmatrix} \tag{1.1}$$

is positive semidefinite.

As an immediate consequence of the min-max principle (see, e.g., [2, p.75]), if $A,B,X\in\mathcal{M}_n(\mathbb{C})$, then

$$s_j(AXB)\leqslant\|A\|\|B\|s_j(X) \tag{1.2}$$

for $j = 1,2,...,n$. For $1\leqslant p<\infty,$ the Schatten p-norm of A is described by $\|A\|_p=(tr\,|A|^p)^{\frac{1}{p}},$ where tr is the usual trace functional. One can see that

$$\|A\oplus B\|=\max(\|A\|,\|B\|)$$

and

$$\|A\oplus B\|_p=(\|A\|_p^p+\|B\|_p^p)^{\frac{1}{p}}. \tag{1.3}$$

It has been established by Bhatia and Kittaneh in [4] that if $A,B\in\mathcal{M}_n(\mathbb{C})$, then

$$2|||A^*B|||\leqslant|||AA^*+BB^*||| \tag{1.4}$$

and

$$|||A^*B+B^*A|||\leqslant|||AA^*+BB^*||| \tag{1.5}$$

for every unitarily invariant norm.

A singular value inequality due to [5] states that if $A,B,C,S,K\in\mathcal{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\geq 0,$ then

$$s_j\left(S^*BK+K^*B^*S\right)\leq s_j\left(\left(S^*AS+K^*CK\right)\oplus\left(S^*AS+K^*CK\right)\right) \tag{1.6}$$

for $j = 1,2,...,n$.

In [8, Theorem 2.1] it has been shown that if $A_i,B_i,X_i\in\mathcal{M}_n(\mathbb{C})$ for $j = 1,2,...,n$, then

$$s_j\left(\sum_{i=1}^n A_i^*X_iB_i\right)\leqslant\left\|\sum_{i=1}^n\left|A_i^*\right|^2\right\|^{\frac{1}{2}}\left\|\sum_{i=1}^n\left|B_i^*\right|^2\right\|^{\frac{1}{2}}s_j\left(\bigoplus_{i=n}^n X\right) \tag{1.7}$$

for $j = 1,2,...,n$.

The above inequalities have attracted the attention of several mathematicians. Different proofs and stronger versions of the inequalities mentioned above have been given; see [3,8,10,12].

In this paper, we generalize some inequalities dealing with 2×2 positive semidefinite block matrices in a different perspective.

2. Main Results

This section gives generalized singular value inequalities associated with 2×2 positive semidefinite block matrices. We should recall the following lemmas that are essential roles in our by-products. The first and second lemmas are presented in [5]. The third one has been proved in [1]. Finally, the fourth lemma is given in [6].

Lemma 2.1. *Let* $A,B,C \in \mathcal{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0,$ *then*

$$B^*B \leq C^{\frac{1}{2}}U^*AUC^{\frac{1}{2}}$$

for some unitary matrix U.

Lemma 2.2. *Let* $A,B,C,X,Y \in \mathcal{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0,$ *then*

$$\pm(X^*BY + Y^*B^*X) \leq X^*AX + Y^*CY.$$

Lemma 2.3. *Let* $A,B \in \mathcal{M}_n(\mathbb{C})$ *be such that A is Hermitian,* $A \geq 0$ *and* $\pm B \leq A,$ *then*

$$2s_j(B) \leq s_j\left((A+B) \oplus (A-B)\right)$$

for $j = 1,2,...,n$.

Lemma 2.4. *Let X be Hermitian, then*

$$\prod_{j=1}^{k} s_j(X) = \max\left|\det\left(U^*XU\right)\right|$$

where maximum is taken over $n \times k$ *matrices U for which* $U^*U = I_k$ (I_k *is identity matrix of order k*) $1 \leq k \leq n$.

In the following, we aim to provide a generalized singular value and norm inequalities associated with 2×2 block matrices.

Theorem 2.1. *Let* $A,B \in \mathcal{M}_n(\mathbb{C})$. *Then*

$$\left|\det\left(B^*|A|^2 B\right)\right| \leq \det|B|^2 \prod_{j=1}^{k} s_j^2(A) \tag{2.1}$$

for $j = 1,2,...,n$ *and* $1 \leq k \leq n$.

Proof. By Lemma 2.2 in [13] we showed that for *A*, *B* and *X* in $\mathcal{M}_n(\mathbb{C})$ and *f* and *g* be non-negative continuous functions on $[0,\infty)$ that satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$, the matrix

$$\begin{bmatrix} A^*f\left(|X^*|\right)^2 A & A^*X^*B \\ B^*XA & B^*g\left(|X|\right)^2 B \end{bmatrix} \geqslant 0. \tag{2.2}$$

Now by Lemma 2.1, we have

$$B^*X|A|^2X^*B \leq \left(\left(Bg\left(|X|\right)^2 B\right)^{\frac{1}{2}} U^*Af\left(|X^*|\right)^2 AU\left(Bg\left(|X|\right)^2 B\right)^{\frac{1}{2}}\right).$$

Put $X = I_n$. It follows that

$$B^*|A|B \leq \left(\left(B^*B\right)^{\frac{1}{2}} U^*|A|^2 U\left(B^*B\right)^{\frac{1}{2}}\right).$$

Finally, from lemma 2.4 and some property of determine function, we obtain

$$\left|\det\left(B^*|A|^2 B\right)\right| \leq \left|\det\left(|B|U^*|A|^2U|B|\right)\right| = \left|\det|B|\right|\left|\det U^*|A|^2U\right|\left|\det|B|\right|$$
$$= \det|B|^2 \prod_{j=1}^{k} s_j^2(A)$$

for $j = 1,2,...,n$ *and* $1 \leq k \leq n$

Theorem 2.2. *Let* $A,B,X,S,K \in \mathcal{M}_n(\mathbb{C})$, *f and g be non-negative continuous functions on* $[0,\infty)$ *that satisfy the relation f(t)g(t) = t for all t* $\in [0,\infty)$. *Then*

$$2s_j\left(S^*A^*X^*BK + K^*B^*XAS\right) \leqslant s_j\Bigg(\left(S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right) + \left(S^*A^*X^*BK + K^*B^*XAS\right) \oplus \left(S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right) - \left(S^*A^*X^*BK + K^*B^*XAS\right)\Bigg).$$

Proof. We can obtain the result quickly from the positivity of [1.1], Lemmas 2.2 and 2.3.

We get the following corollary as an application of Theorem 2.2.

Corollary 1. *Let* $A,B,X,S,K \in \mathcal{M}_n(\mathbb{C})$. *Then,*

$$2\left|\left|\left|S^*A^*X^*BK + K^*B^*XAS\right|\right|\right| \leqslant \left|\left|\left|\left(S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right) + \left(S^*A^*X^*BK + K^*B^*XAS\right) \oplus \left(S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right) - \left(S^*A^*X^*BK + K^*B^*XAS\right)\right|\right|\right|$$

for every unitarily invariant norm. In particular,

$$\left\|S^*A^*X^*BK + K^*B^*XAS\right\| \leq max\left(\left\|\left(S^*A^*X^*BK + K^*B^*XAS\right) + \left(S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right)\right\|\right) \left(\left\|\left(S^*A^*X^*BK + KB^*XAS\right) - \left(S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right)\right\|\right)$$

and

$$\left\|S^*A^*X^*BK + K^*B^*XAS\right\|_p \leq \left(\left\|\left(S^*A^*X^*BK + KB^*XAS\right) + \left(S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right)\right\|_p^p + \left\|\left(S^*A^*X^*BK + KB^*XAS\right) - (S^*A^*f\left(|X^*|\right)^2 AS + K^*B^*g\left(|X|\right)^2 BK\right\|_p^p\right)^{\frac{1}{p}} \tag{2.3}$$

for $1 \leq p < \infty$.

Remark 2.1. *Let A and B be in* $M_n(\mathbb{C})$. *Putting* $f(t) = g(t) = t^{\frac{1}{2}}$ *and* $X = S = K = I_k$ *in Theorem* 2.2, *we get*

$$2s_j(A^*B + B^*A) \leq s_j\left(A^*A + B^*B) + ((A^*B + B^*A) \oplus (A^*A + B^*B) - (A^*B + B^*A)\right)$$
$$= s_j\left(|A+B|^2 \oplus |A-B|^2\right)$$

for j = 1,2,...,*n*, *which was given in* [1, Theorem 2.7] *and* [9].

Remark 2.2. *The inequality in Theorem 2.2 is sharper than* [1.6]. *In fact, for* $A,B,X,S,K \in \mathcal{M}_n(\mathbb{C})$. *and* $f(t) = g(t) = t^{\frac{1}{2}}$ *in Theorem 2.2 and by the Weyls monotonicity principle, we have*

$$sj\left(\left(S^*A^*AS + K^*B^*BK\right) + \left(S^*A^*X^*BK + K^*B^*XAS\right) \oplus \left(S^*A^*AS + K^*B^*BK\right) - \left(S^*A^*X^*BK + KB^*XAS\right)\right)$$

$$= s_j\begin{bmatrix} \left(S^*A^*AS + K^*B^*BK\right) + \left(S^*A^*X^*BK + K^*B^*XAS\right) & 0 \\ 0 & \left(S^*A^*AS + K^*B^*BK\right) - \left(S^*A^*X^*BK + K^*B^*XAS\right) \end{bmatrix}$$

$$\leq s_j \begin{bmatrix} 2\left(S^*A^*AS+K^*B^*BK\right) & 0 \\ 0 & 2\left(S^*A^*AS+K^*B^*BK\right) \end{bmatrix}$$
$$= 2sj\left(S^*A^*AS+K^*B^*BK \oplus S^*A^*AS+K^*B^*BK\right)$$

Garg and Aujla gave the following lemma in [6].

Lemma 5. *Let X be Hermitian and Y be positive definite with* $\pm X < Y$, *then* $|\det(X)| < \det Y$.

Theorem 3. *Let* $A, B, X, S, K \in \mathcal{M}_n(\mathbb{C})$, *f and g be non-negative continuous functions on* $[0,\infty)$ *that satisfy the relation* $f(t)g(t) = t$ for all $t \in [0,\infty)$. *Then*

$$\prod_{j=1}^{k} s_j\left(S^*A^*X^*BK+K^*B^*XAS\right) \leqslant \prod_{j=1}^{k} s_j\left(\left(S^*A^*f\left(\left|X^*\right|\right)^2 AS+K^*B^*g\left(\left|X\right|\right)^2 BK\right) + \left(S^*A^*X^*BK+K^*B^*XAS\right)\right)$$

For $1 \leq k \leq n$.

Proof. From the positivity [1.1 and Lemma 2.2 for $n \times k$ matrices U for which $U^*U = I_k$ (I_k is identity matrix of order k) $1 \leq k \leq n$, we have

$$\pm U^*\left(S^*A^*X^*BK+K^*B^*XAS\right)U \leqslant U^*\left(\left(S^*A^*f\left(\left|X^*\right|\right)^2 AS+K^*B^*g\left(\left|X\right|\right)^2 BK\right) + \left(S^*A^*X^*BK+K^*B^*XAS\right)\right)U.$$

Now, from Lemmas 2.4 and 2.5, we get the result as follows:

$$\prod_{j=1}^{k} s_j\left(S^*A^*X^*BK+K^*B^*XAS\right) \leqslant \det\left[U^*\left(\left(S^*A^*f\left(\left|X^*\right|\right)^2 AS+K^*B^*g\left(\left|X\right|\right)^2 BK\right) + \left(S^*A^*X^*BK+K^*B^*XAS\right)\right)U\right]$$
$$\leqslant \max\left|\det\left[V^*\left(\left(S^*A^*f\left(\left|X^*\right|\right)^2 AS+K^*B^*g\left(\left|X\right|\right)^2 BK\right) + \left(S^*A^*X^*BK+K^*B^*XAS\right)\right)V\right]\right|$$
$$= \prod_{j=1}^{k} s_j\left(\left(S^*A^*f\left(\left|X^*\right|\right)^2 AS+K^*B^*g\left(\left|X\right|\right)^2 BK\right) + \left(S^*A^*X^*BK+K^*B^*XAS\right)\right).$$

for $n \times k$ matrices V for which $V^*V = I_k$ (I_k is identity matrix of order k) $1 \leq k \leq n$

Remark 2.3. *Let A and B be in* $\mathcal{M}_n(\mathbb{C})$. *Putting* $f(t) = g(t) = t^{\frac{1}{2}}$ *and* $X = S = K = I_k$ *in Theorem 2.3, we obtain*

$$\prod_{j=1}^{k} s_j\left(A^*B+B^*A\right) \leqslant \prod_{j=1}^{k} s_j\left(A^*A+B^*B+A^*B+B^*A\right) = \prod_{j=1}^{k} s_j\left(\left|A+B\right|^2\right).$$

This inequality is a refinement of inequality [6, Theorem 2.8].

3. Acknowledgments

Authors thank the referees for their valuable suggestions to improve the paper. Author[1] acknowledges Mashhad Branch, Islamic Azad University, Mashhad, Iran for their encouragement. The corresponding author[2] thank the Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, for the kind support. The author[2] acknowledges SERB, Govt. of India for the Teachers Associateship for Research Excellence (TARE) fellowship TAR/2022/000219.

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