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ρ-Einstein solitons in Lorentzian para-Kenmotsu manifolds

Mobin Ahmad[a]*, Mohd Bilal[b], Gazala[c]

[a,c]*Department of Mathematics and Statistics, Integral University, Kursi Road, Lucknow-226026, India; [b]Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Al Qura University, Makkah 21955, Saudi Arabia.

Abstract

The main purpose of the current paper is to study certain curvature conditions in Lorentzian para-Kenmotsu n-manifolds (briefly, $(LPK)_n$) admitting ρ-Einstein solitons (ρ-ES).

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Keywords: ρ-Einstein soliton; Einstein manifolds; Lorentzian para-Kenmotsu manifolds

1. Introduction

In the past two decennaries, the geometric flows are too fascinating mathematical tools for describing geometric structures in Riemannian geometry. On a Riemannian manifold $(\mathcal{M}, g)$, the Ricci flow [1] is described by an equation of the from $\frac{\partial g}{\partial t} = -2S$, where S is the Ricci curvature tensor. The metric g on $\mathcal{M}$ satisfies the Ricci soliton equation $\mathcal{L}_V g + 2S + 2\Lambda g = 0$, where $\mathcal{L}_V$ represents the Lie derivative in the direction of a vector field V on $\mathcal{M}$ and Λ is a constant. The manifolds admitting such structure are called Ricci soliton. A Ricci soliton is called shrinking (steady or expanding) if $\Lambda > 0$ ($\Lambda = 0$ or $\Lambda < 0$).

In 1980's, as a generalization of Ricci flow, Bourguignon introduced the notion of Ricci-Bourguignon flow [2]. The Ricci-Bourguignon flow is an equation on a manifold $(\mathcal{M}, g)$ given as follows

$$\frac{\partial g}{\partial t} = -2(S - \rho r g), \quad g(0) = g_0, \tag{1}$$

Email addresses: mobinahmad68@gmail.com (Mobin Ahmad)*; mohd7bilal@gmail.com (Mohd Bilal); gazala.math@gmail.com (Gazala)

*Corresponding Author

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where S is the Ricci curvature tensor, r is the scalar curvature and $\rho(\neq 0)$ is a real constant. It should be noticed that for specific values of ρ we obtain the following circumtances for the tensor $S-\rho rg$ appearing in equation (1). The evolution equation (1) is of special interest, in particular [3]

1. $\rho=\frac{1}{2}$, the Einstein tensor $S-\frac{r}{2}g$, (for Einstein soliton)
2. $\rho=\frac{1}{n}$, the traceless Ricci tensor $S-\frac{r}{n}g$,
3. $\rho=\frac{1}{2(n-1)}$, the Schouten tensor $S-\frac{r}{2(n-1)}g$, (for Schouten soliton),
4. $\rho=0$, the Ricci tensor S (for Ricci soliton).

For $n=2$, the tensors (1)-(3) are zero, hence the flow is static and in higher dimension the value of ρ are strictly ordered as above in descending order. Short time existence and uniqueness for the solution of (1) has been proved in [4]. In actual, for sufficiently small t the equation (1) has a unique solution for $\rho<\frac{1}{2(n-1)}$.

A more general type of Ricci soliton, i.e., "Ricci-Bourguignon soliton" is the soluion of Ricci-Bourguignon flow. An $(\mathcal{M},g)$ of dimension $n\geq 3$ is named as a Ricci-Bourguignon soliton or ρ-Einstein soliton (ρ-ES) if

$$\mathcal{L}_V g+2S+2(\Lambda-\rho r)g=0. \tag{2}$$

A ρ-ES is called shrinking if $\Lambda<0$, steady if $\Lambda=0$ and expanding if $\Lambda>0$. We refer the papers [5–13] for more details about the concerned studies on different types solitons.

We present our study as follows: In section 2, we give some basic definitions and results of $(LPK)_n$. In section 3, we investigate $(LPK)_n$ admitting ρ-ES. In section 4, ρ-ES on $(LPK)_n$ admitting cyclic η-recurrent Ricci tensor have been studied. Sections 5 deals with the study of ρ-ES in $(LPK)_n$ with torse forming vector field. In section 6, the curvature condition $R(\xi,X).S=0$ in $(LPK)_n$ admitting ρ-ES have been studied. In section 7, we discuss ρ-ES in conharmonically flat, φ-conharmonically flat and conharmonically φ-semisymmetric flat conditions in $(LPK)_n$.

2. Preliminaries

A differentiable manifold $\mathcal{M}$ of dimension of n with the structure (φ,ζ,η) is termed a Lorentzian almost paracontact manifold, where φ, ζ and η refer to a (1,1) type tensor field, a contravariant vector field, and a 1-form, respectivey such that [14, 15]

$$\eta(\zeta)=-1 \text{ and } \varphi^2=\eta\otimes\zeta+I, \tag{3}$$

which infer that

$$\varphi\zeta=0, \quad \eta\circ\varphi=0, \text{rank}(\varphi)=n-1. \tag{4}$$

Let g be a Lorentzian metric of $\mathcal{M}$ fulfilling

$$g(\cdot,\zeta)=\eta(\cdot) \text{ and } g(\varphi\cdot,\varphi\cdot)=g(\cdot,\cdot)+\eta(\cdot)\eta(\cdot). \tag{5}$$

Then the structure (φ,ζ,η,g) is called an almost paracontact structure and $\mathcal{M}$ is termed as an almost paracontact metric manifold.

Define Φ, the second fundamental form as:

$$\Phi(\mathcal{X}_1,\mathcal{X}_2)=\Phi(\mathcal{X}_2,\mathcal{X}_1)=g(\mathcal{X}_1,\varphi\mathcal{X}_2) \tag{6}$$

for any vector fields $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}(\mathcal{M})$, where $\mathfrak{X}(\mathcal{M})$ refers to the Lie algebra of vector fields on $\mathcal{M}$. If $d\eta(\mathcal{X}_1, \mathcal{X}_2) = \Phi(\mathcal{X}_1, \mathcal{X}_2)$, d is an exterior derivative, then $(\mathcal{M}, \varphi, \zeta, \eta, g)$ is named as a paracontact metric manifold [16].

Definition 2.1: *A Lorentzian almost paracontact manifold $\mathcal{M}$ is called an $(LPK)_n$ if* [17]

$$(\nabla_{\mathcal{X}_1}\varphi)\mathcal{X}_2 = -g(\varphi\mathcal{X}_1, \mathcal{E}_2)\zeta - \eta(\mathcal{X}_2)\varphi\mathcal{X}_1 \tag{7}$$

for any $\mathcal{X}_1, \mathcal{X}_2$ on $(LPK)_n$.

In an $(LPK)_n$, we have

$$\nabla_{\mathcal{X}_1}\zeta + \mathcal{X}_1 + \eta(\mathcal{X}_1)\zeta = 0, \tag{8}$$

$$(\nabla_{\mathcal{X}_1}\eta)\mathcal{E}_2 + g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2) = 0, \tag{9}$$

where ∇ is called the Levi-Civita connection with respect to g.

Moreover, in an $(LPK)_n$ we have [17]:

$$g(\mathcal{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3, \zeta) = \eta(\mathcal{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3) = g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_1) - g(\mathcal{X}_1, \mathcal{X}_3)\eta(\mathcal{X}_2), \tag{10}$$

$$\mathcal{R}(\zeta, \mathcal{X}_1)\mathcal{X}_2 = -\mathcal{R}(\mathcal{X}_1, \zeta)\mathcal{X}_2 = g(\mathcal{X}_1, \mathcal{X}_2)\zeta - \eta(\mathcal{X}_2)\mathcal{X}_1, \tag{11}$$

$$\mathcal{R}(\mathcal{X}_1, \mathcal{X}_2)\zeta = \eta(\mathcal{X}_2)\mathcal{X}_1 - \eta(\mathcal{X}_1)\mathcal{X}_2, \tag{12}$$

$$\mathcal{R}(\zeta, \mathcal{X}_1)\zeta = \mathcal{X}_1 + \eta(\mathcal{X}_1)\zeta, \tag{13}$$

$$S(\mathcal{X}_1, \zeta) = (n-1)\eta(\mathcal{X}_1), \; S(\zeta, \zeta) = -(n-1), \tag{14}$$

$$\mathcal{Q}\zeta = (n-1)\zeta, \tag{15}$$

for any $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ on $(LPK)_n$, where $\mathcal{R}$ and $\mathcal{Q}$ denote the curvature tensor and the Ricci operator, respectively.

Definition 2.2: *An $(LPK)_n$ is said to be η-Einstein manifold if its Ricci tensor $S(\neq 0)$ satisfies the following relation*

$$S(\mathcal{X}_1, \mathcal{X}_2) = \sigma_1 g(\mathcal{X}_1, \mathcal{X}_2) + \sigma_2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2), \tag{16}$$

for smooth functions σ_1 and σ_2. If $\sigma_2 = 0$, then $(LPK)_n$ reduces to an Einstein manifold.

Remark 2.3: *In an $(LPK)_n$, we have* [18]

$$\zeta(r) = 2(r - n(n-1)). \tag{17}$$

Remark 2.4: *From the relation* (17)*, it is observed that if an $(LPK)_n$ is of constant scalar curvature, then $r = n(n-1)$.*

3. ρ-Einstein solitons on $(LPK)_n$

Let an $(LPK)_n$ admit a ρ-ES, then (2) holds. Thus we have

$$(\mathcal{L}_\zeta g)(\mathcal{X}_1, \mathcal{X}_2) + 2S(\mathcal{X}_1, \mathcal{X}_2) + 2(\Lambda - \rho r)g(\mathcal{X}_1, \mathcal{X}_2) = 0. \tag{18}$$

As we know that

$$(\mathcal{L}_\zeta g)(\mathcal{X}_1,\mathcal{X}_2) = g(\nabla_{\mathcal{X}_1}\zeta,\mathcal{X}_2) + g(\mathcal{X}_1,\nabla_{\mathcal{X}_2}\zeta) = -2g(\mathcal{X}_1,\mathcal{X}_2) - 2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{19}$$

Thus (18) leads to

$$S(\mathcal{X}_1,\mathcal{X}_2) = -(\Lambda - \rho r - 1)g(\mathcal{X}_1,\mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{20}$$

Putting $\mathcal{X}_2 = \zeta$ in (19) then using (3) and (5) we have

$$S(\mathcal{X}_1,\zeta) = -(\Lambda - \rho r)\eta(\mathcal{X}_1). \tag{21}$$

This implies that

$$Q\zeta = -(\Lambda - \rho r)\zeta. \tag{22}$$

From (14) and (21), we get the following relation

$$\Lambda = \rho r - (n-1). \tag{23}$$

Now, if we acknowledge that r is constant, then in view of Remark 2.4, (23) turns to

$$\Lambda = (n-1)(\rho n - 1). \tag{24}$$

Thus, we have the following result:

Theorem 3.1: *An $(LPK)_n$ admitting a ρ-ES is an η-Einstein manifold and the soliton constant is given by $\Lambda = (n-1)(\rho n-1)$.*

Now we have the following corollary:

Corollary 3.2 *Let an $(LPK)_n$ admit a ρ-ES. Then we have*

Values of ρ	Soliton type	Soliton constant	Conditions for $(g, V=\zeta, \Lambda, \rho)$ to be expanding, shrinking or steady
$\rho = \frac{1}{2}$	Einstein soliton	$\Lambda = \frac{(n-1)(n-2)}{2}$	$(g, V=\zeta, \Lambda, \rho)$ is expanding.
$\rho = \frac{1}{n}$	traceless Ricci soliton	$\Lambda = 0$	$(g, V=\zeta, \Lambda, \rho)$ is steady.
$\rho = \frac{1}{2(n-1)}$	Schouten soliton	$\Lambda = -\frac{(n-2)}{2}$	$(g, V=\zeta, \Lambda, \rho)$ is shrinking.
$\rho = 0$	Ricci soliton	$\Lambda = -(n-1)$	$(g, V=\zeta, \Lambda, \rho)$ is shrinking.

Lemma 3.3: [19] *Let an $(LPK)_n$ admit a ρ-ES $(g, V=\zeta, \Lambda, \rho)$ such that $V = b\zeta$, where b is a function. Then*

(i) V is a constant multiple of ζ and $(LPK)_n$ is an η-Einstein manifold of the type

$$S(\mathcal{X}_1,\mathcal{X}_2) = (b-\Lambda)g(\mathcal{X}_1,\mathcal{X}_2) + b\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{25}$$

(ii) Moreover, ρ-ES (g, V, Λ, ρ) reduces to the Ricci soliton.

Proof. (i) This part of the lemma can be esaily proved in similar way as in [19].

(ii) Now, putting $\mathcal{X}_2 = \zeta$ in (25), we have

$$S(\mathcal{X}_1,\zeta) = -\Lambda\eta(\mathcal{X}_1). \tag{26}$$

From the relations (14), (24) and (25), we obtain $\rho = 0$. This implies that ρ-ES reduces to the Ricci solitons.

4. ρ-Einstein solitons on $(LPK)_n$ admitting cyclic η-recurrent Ricci tensor

Definition 4.1: *An $(LPK)_n$ is said to have cyclic η-recurrent Ricci tensor, if*

$$(\nabla_{\mathcal{X}_1} S)(\mathcal{X}_2, \mathcal{X}_3) + (\nabla_{\mathcal{X}_2} S)(\mathcal{X}_3, \mathcal{X}_1) + (\nabla_{\mathcal{X}_3} S)(\mathcal{X}_1, \mathcal{X}_2) = \eta(\mathcal{X}_1) S(\mathcal{X}_2, \mathcal{X}_3) + \eta(\mathcal{X}_2) S(\mathcal{X}_3, \mathcal{X}_1) + \eta(\mathcal{X}_3) S(\mathcal{X}_1, \mathcal{X}_2) \quad (27)$$

for any $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ on $(LPK)_n$.

Let an $(LPK)_n$ admitting ρ-ES has cyclic η-recurrent Ricci tensor then (27) holds. The covariant differentiation of (20) with respect to $\mathcal{X}_1$ leads to

$$(\nabla_{\mathcal{X}_1} S)(\mathcal{X}_2, \mathcal{X}_3) = \rho(\mathcal{X}_1 r) g(\mathcal{X}_2, \mathcal{X}_3) - g(\mathcal{X}_1, \mathcal{X}_2)\eta(\mathcal{X}_3) - g(\mathcal{X}_1, \mathcal{X}_3)\eta(\mathcal{X}_2) - 2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3). \quad (28)$$

Similarly, we have

$$(\nabla_{\mathcal{X}_2} S)(\mathcal{X}_3, \mathcal{X}_1) = \rho(\mathcal{X}_2 r) g(\mathcal{X}_3, \mathcal{X}_1) - g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_1) - g(\mathcal{X}_1, \mathcal{X}_2)\eta(\mathcal{X}_3) - 2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3). \quad (29)$$

and

$$(\nabla_{\mathcal{X}_3} S)(\mathcal{X}_1, \mathcal{X}_2) = \rho(\mathcal{X}_3 r) g(\mathcal{X}_1, \mathcal{X}_2) - g(\mathcal{X}_3, \mathcal{X}_1)\eta(\mathcal{X}_2) - g(\mathcal{X}_3, \mathcal{X}_2)\eta(\mathcal{X}_1) - 2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3). \quad (30)$$

By using (28)-(30) in (27), we arrive at

$$\rho[(\mathcal{X}_1 r) g(\mathcal{X}_2, \mathcal{X}_3) + (\mathcal{X}_2 r) g(\mathcal{X}_3, \mathcal{X}_1) + (\mathcal{X}_3 r) g(\mathcal{X}_1, \mathcal{X}_2)] = 9\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)$$
$$-(\Lambda - \rho r - 3)[g(\mathcal{X}_2, \mathcal{X}_3)\eta(\mathcal{X}_1) + g(\mathcal{X}_1, \mathcal{X}_3)\eta(\mathcal{X}_2) + g(\mathcal{X}_1, \mathcal{X}_2)\eta(\mathcal{X}_3)],$$

which by putting $\mathcal{X}_2 = \mathcal{X}_3 = \zeta$ and using (3) and (4), we have

$$\rho[-(Xr) + 2(\zeta r)\eta(\mathcal{X}_1)] = 3(\Lambda - \rho r)\eta(\mathcal{X}_1). \quad (31)$$

Now putting $\mathcal{X}_1 = \zeta$ in (31) and using (3), we infer

$$\Lambda = \rho[r + (\zeta r)]. \quad (32)$$

Let r is constant, then $\zeta r = 0$. Thus in view of (17), (32) gives

$$\Lambda = \rho n(n-1). \quad (33)$$

Thus, we have the following result:

Theorem 4.2: *If an $(LPK)_n$ with the constant scalar curvature admitting ρ-Einstein solitons has cyclic η-recurrent Ricci tensor, then the soliton constant is given by $\Lambda = \rho n(n-1)$.*

Now we have the following corollary:

Corollary 4.3: *Let the metric of an $(LPK)_n$ with constant scalar curvature be a ρ-Einstein soliton. Then we have*

Values of ρ	Soliton type	Soliton constant	Conditions for $(g, V = \zeta, \Lambda, \rho)$ to be expanding, shrinking or steady
$\rho = \frac{1}{2}$	Einstein soliton	$\Lambda = \frac{n(n-1)}{2}$	$(g, V = \zeta, \Lambda, \rho)$ is expanding.
$\rho = \frac{1}{n}$	traceless Ricci soliton	$\Lambda = n-1$	$(g, V = \zeta, \Lambda, \rho)$ is steady.
$\rho = \frac{1}{2(n-1)}$	Schouten soliton	$\Lambda = \frac{n}{2}$	$(g, V = \zeta, \Lambda, \rho)$ is shrinking.
$\rho = 0$	Ricci soliton	$\Lambda = 0$	$(g, V = \zeta, \Lambda, \rho)$ is shrinking.

5. ρ-Einstein Solitons on $(LPK)_n$ with Torse-forming Vector Field

Definition 5.1: *A vector field V on a (pseudo)-Riemannian manifold (M,g) is called torse-forming* [20] *if*

$$\nabla_{\mathcal{X}_1} V = f\mathcal{X}_1 + \omega(\mathcal{X}_1)V \tag{34}$$

where f: a smooth function, ω: a 1-form and ∇: the Levi-Civita connection of g.

Let us consider an $(LPK)_n$ admitting a ρ-ES $(g, V=\zeta, \Lambda, \rho)$, and also considering ζ, the Reeb vector field as a torse-forming vector field. Thus, from (34) we have

$$\nabla_{\mathcal{X}_1} \zeta = f\mathcal{X}_1 + \omega(\mathcal{X}_1)\zeta \tag{35}$$

for any $\mathcal{X}_1$ on $(LPK)_n$.

The inner product of (35) with ζ gives

$$g(\nabla_{\mathcal{X}_1}\zeta, \zeta) = f\eta(\mathcal{X}_1) - \omega(\mathcal{X}_1). \tag{36}$$

Also from (8), we obtain

$$g(\nabla_{\mathcal{X}_1}\zeta, \zeta) = 0. \tag{37}$$

Thus, from (36) and (37) we find $\omega = f\eta$, and hence (35) becomes

$$\nabla_{\mathcal{X}_1}\zeta = f(\mathcal{X}_1 + \eta(\mathcal{X}_1)\zeta). \tag{38}$$

Now, in view of (38), we have

$$(\pounds_\zeta g)(\mathcal{X}_1, \mathcal{X}_2) = 2f\{g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\}. \tag{39}$$

By virtue of (39), (18) turns to

$$S(\mathcal{X}_1, \mathcal{X}_2) = -(f + \Lambda - \rho r)g(\mathcal{X}_1, \mathcal{X}_2) - f\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{40}$$

By putting $\mathcal{X}_1 = \mathcal{X}_2 = \zeta$ in (40) then using (3) and (14), we obtain

$$\Lambda = (n-1)(\rho n - 1).$$

Thus, we have:

Theorem 5.2: *Let an $(LPK)_n$ of constant scalar curvature admit a ρ-ES $(g, V=\zeta, \Lambda, \rho)$ with a torse-forming vector field ζ, then $(LPK)_n$ is an η-Einstein. Moreover, for the particular values of ρ, the nature of solitons can be discussed as in Corollary* 4.3.

6. ρ-Einstein Solitons on $(LPK)_n$ Satisfying $R(\zeta, \mathcal{X}_1)\cdot S = 0$

Let an $(LPK)_n$ admitting ρ-ES satisfies the condition $R(\zeta, \mathcal{X}_1)\cdot S = 0$. Then we have

$$S(R(\zeta, \mathcal{X}_1)\mathcal{X}_2, \mathcal{X}_3) + S(\mathcal{X}_2, R(\zeta, \mathcal{X}_1)\mathcal{X}_3) = 0,$$

which by using (11) yields

$$g(\mathcal{X}_1, \mathcal{X}_2)S(\zeta, \mathcal{X}_3) - \eta(\mathcal{X}_2)S(\mathcal{X}_1, \mathcal{X}_3) + g(\mathcal{X}_1, \mathcal{X}_3)S(\mathcal{X}_2, \zeta) - \eta(\mathcal{X}_3)S(\mathcal{X}_1, \mathcal{X}_2) = 0.$$

Putting $\mathcal{X}_3 = \zeta$ in the foregoing equation then using (3) and (21), we infer

$$S(\mathcal{X}_1, \mathcal{X}_2) = -(\Lambda - \rho r)g(\mathcal{X}_1, \mathcal{X}_2). \tag{41}$$

Now putting $\mathcal{X}_2 = \zeta$ in (41) and using (21), we obtain

$$\Lambda = (n-1)(\rho n - 1). \tag{42}$$

Now we state:

Theorem 6.1: *Let an* $(LPK)_n$ *of constant scalar curvature tensor admit a* ρ*-ES* $(g, V = \zeta, \Lambda, \rho)$ *and satisfies* $R(\zeta, \mathcal{X}_1)\cdot S = 0$*, then* $(LPK)_n$ *is an Einstein. Moreover, for the particular values of* ρ*, the nature of solitons can be discussed as in Corollary* 4.3.

7. Conharmonic Curvature Tensor on $(LPK)_n$ Admitting ρ-ES

In 1950's, Ishii [21] introduced the idea of conharmonic transformation under which a harmonic function transform into a harmonic function. The conharmonic curvature tensor C of type (1,3) on a (pseudo)-Riemannian manifold $\mathcal{M}$ of dimension n is defined by [22, 23]

$$C(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3 = R(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3 + \frac{1}{(n-2)}[S(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2 - S(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1 + g(\mathcal{X}_1,\mathcal{X}_3)Q\mathcal{X}_2 - g(\mathcal{X}_2,\mathcal{X}_3)Q\mathcal{X}_1] \tag{43}$$

for all $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ on $\mathcal{M}$.

In this section, first we study conharmonically flat $(LPK)_n$ admitting ρ-ES, i.e., $C(\mathcal{X}_1,\mathcal{X}_1)\mathcal{X}_3 = 0$. Then from (43), we have

$$R(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3 = -\frac{1}{(n-2)}[S(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2 - S(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1 + g(\mathcal{X}_1,\mathcal{X}_3)Q\mathcal{X}_2 - g(\mathcal{X}_2,\mathcal{X}_3)Q\mathcal{X}_1],$$

which by putting $\mathcal{X}_3 = \zeta$ and using (12), (21) and (22) reduces to

$$(\Lambda - \rho r + n - 2)(\eta(\mathcal{X}_2)\mathcal{X}_1 - \eta(\mathcal{X}_1)\mathcal{X}_2) = \eta(\mathcal{X}_2)Q\mathcal{X}_1 - \eta(\mathcal{X}_1)Q\mathcal{X}_2. \tag{44}$$

By putting $\mathcal{X}_2 = \zeta$, (44) leads to

$$Q\mathcal{X}_1 = (\Lambda - \rho r + n - 2)\mathcal{X}_1 + (2\Lambda - 2\rho r + n - 2)\eta(\mathcal{X}_1)\zeta. \tag{45}$$

The inner product of (45) with $\mathcal{X}_2$ gives

$$S(\mathcal{X}_1,\mathcal{X}_2) = (\Lambda - \rho r + n - 2)g(\mathcal{X}_1,\mathcal{X}_2) + (2\Lambda - 2\rho r + n - 2)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{46}$$

Now taking $\mathcal{X}_2 = \zeta$ in (46) then using (3), (5) and (14), we obtain

$$\Lambda = \rho r - (n-1). \tag{47}$$

Now, let r is constant, then in view of Remark 2.4, (47) turns to

$$\Lambda = (n-1)(\rho n - 1). \tag{48}$$

Thus, we have the following result:

Theorem 7.1: *If the metric of a conharmonically flat* $(LPK)_n$ *whose scalar curvature* r *is constant be* ρ*-ES* $(g, \zeta, \Lambda, \rho)$*, then* $(LPK)_n$ *is an* η*-Einstein and the soliton constant is given by* $\Lambda = (n-1)(\rho n - 1)$.

Next, we consider a φ-conharmonically flat $(LPK)_n$ that admits a ρ-ES, i.e., $\varphi^2 C(\varphi\mathcal{X}_1,\varphi\mathcal{X}_2)\varphi\mathcal{X}_3 = 0 g(C(\varphi\mathcal{X}_1,\varphi\mathcal{X}_1)\varphi\mathcal{X}_3,\varphi\mathcal{X}_4) = 0$. Then from (43), it follows that

$$\begin{aligned} g(R(\varphi\mathcal{X}_1,\varphi\mathcal{X}_2)\varphi\mathcal{X}_3,\varphi\mathcal{X}_4) = \frac{1}{(n-2)}[&g(\varphi\mathcal{X}_1,\varphi\mathcal{X}_4)S(\varphi\mathcal{X}_2,\varphi\mathcal{X}_3) - g(\varphi\mathcal{X}_2,\varphi\mathcal{X}_4)S(\varphi\mathcal{X}_1,\varphi\mathcal{X}_3) \\ &+ S(\varphi\mathcal{X}_1,\varphi\mathcal{X}_4)g(\varphi\mathcal{X}_2,\varphi\mathcal{X}_3) - S(\varphi\mathcal{X}_2,\varphi\mathcal{X}_4)g(\varphi\mathcal{X}_1,\varphi\mathcal{X}_3)]. \end{aligned} \tag{49}$$

Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, \zeta\}$ be a local orthonormal basis of the vector fields in $(LPK)_n$. Using that $\{\varphi\varepsilon_1, \varphi\varepsilon_2, \ldots, \varphi\varepsilon_{n-1}, \zeta\}$ is also a local orthonormal basis of $(LPK)_n$. If we set $\mathcal{X}_1 = \mathcal{X}_4 = \varepsilon_i$ in (??) and sum up with respect to i, then

$$\begin{aligned}\sum_{i=1}^{n-1} g(R(\varphi\varepsilon_i, \varphi\mathcal{X}_2)\varphi\mathcal{X}_3, \varphi\varepsilon_i) = \frac{1}{(n-2)}[&S(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3)\sum_{i=1}^{n-1} g(\varphi\varepsilon_i, \varphi\varepsilon_i) \\ &- \sum_{i=1}^{n-1} S(\varphi\varepsilon_i, \varphi\mathcal{X}_3) g(\varphi\mathcal{X}_2, \varphi\varepsilon_i) + g(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3)\sum_{i=1}^{n-1} S(\varphi\varepsilon_i, \varphi\varepsilon_i) \\ &- \sum_{i=1}^{n-1} g(\varphi\varepsilon_i, \varphi\mathcal{X}_3) S(\varphi\mathcal{X}_2, \varphi\varepsilon_i)].\end{aligned} \tag{50}$$

As we know that

$$\sum_{i=1}^{n-1} g(R(\varphi\varepsilon_i, \varphi\mathcal{X}_2)\varphi\mathcal{X}_3, \varphi\varepsilon_i) = S(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3) - g(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3), \tag{51}$$

$$\sum_{i=1}^{n-1} S(\varphi\varepsilon_i, \varphi\mathcal{X}_3) g(\varphi\mathcal{X}_2, \varphi\varepsilon_i) = S(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3), \tag{52}$$

$$\sum_{i=1}^{n-1} S(\varphi\varepsilon_i, \varphi\varepsilon_i) = r - (n-1), \tag{53}$$

$$\sum_{i=1}^{n-1} g(\varphi\varepsilon_i, \varphi\varepsilon_i) = n-1. \tag{54}$$

Now by using (51)-(54) in (??), we lead to

$$S(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3) = (r-1) g(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3). \tag{55}$$

By putting $\mathcal{X}_2 = \varphi\mathcal{X}_2$ and $\mathcal{X}_3 = \varphi\mathcal{X}_3$ in (55) and using (3), (21), we find

$$S(\mathcal{X}_2, \mathcal{X}_3) = (r-1) g(\mathcal{X}_2, \mathcal{X}_3) + (\Lambda - \rho r + r - 1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3). \tag{56}$$

From (20), we find

$$S(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3) = -(\Lambda - \rho r - 1) g(\varphi\mathcal{X}_2, \varphi\mathcal{X}_3). \tag{57}$$

Thus from the equations (55) and (57), we obtain

$$\Lambda = (\rho - 1) r + 2. \tag{58}$$

If we assume that r of $(LPK)_n$ is constant, then $\zeta r = 0$. Thus in view of Remark 2.4, (57) takes the form

$$\Lambda = n(n-1)(\rho - 1) + 2. \tag{59}$$

Thus we state the following:

Theorem 7.2: *If a φ-conharmonically flat $(LPK)_n$ with the constant scalar curvature r admits a ρ-ES $(g, \zeta, \Lambda, \rho)$, then $(LPK)_n$ is an η-Einstein and the soliton constant is given by $\Lambda = n(n-1)(\rho-1)+2$.*

Now we have the following corollary:

Corollary 7.3: *Let the metric of a φ-conharmonically flat $(LPK)_n$ with constant scalar curvature be a ρ-ES. Then we have*

Values of ρ	Soliton type	Soliton constant	Conditions for $(g, V=\zeta, \Lambda, \rho)$ to be expanding, shrinking or steady
$\rho=\frac{1}{2}$	Einstein soliton	$\Lambda=-\frac{n(n-1)}{2}+2$	$(g, V=\zeta, \Lambda, \rho)$ is shrinking.
$\rho=\frac{1}{n}$	traceless Ricci soliton	$\Lambda=-(n-1)^2+2$	$(g, V=\zeta, \Lambda, \rho)$ is shrinking.
$\rho=\frac{1}{2(n-1)}$	Schouten soliton	$\Lambda=-n^2+\frac{3n}{2}+2$	$(g, V=\zeta, \Lambda, \rho)$ is shrinking.
$\rho=0$	Ricci soliton	$\Lambda=-n(n-1)+2$	$(g, V=\zeta, \Lambda, \rho)$ is shrinking.

Lastly, we consider a conharmonically φ-semisymmetric $(LPK)_n$ that admits a ρ-ES, i.e., $C(\mathcal{X}_1,\mathcal{X}_2)\cdot\varphi=0$ [24]. This implies that

$$C(\mathcal{X}_1,\mathcal{X}_2)\varphi\mathcal{X}_3-\varphi C(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3=0,$$

which by putting $\mathcal{X}_1=\zeta$ takes the form

$$C(\zeta,\mathcal{X}_2)\varphi\mathcal{X}_3-\varphi C(\zeta,\mathcal{X}_2)\mathcal{X}_3=0. \tag{60}$$

From (43), we have

$$C(\zeta,\mathcal{X}_2)\varphi\mathcal{X}_3=R(\zeta,\mathcal{X}_2)\varphi\mathcal{X}_3-\frac{1}{(n-2)}[S(\mathcal{X}_2,\varphi\mathcal{X}_3)\zeta-S(\zeta,\varphi\mathcal{X}_3)\mathcal{X}_2 +g(\mathcal{X}_2,\varphi\mathcal{X}_3)Q\zeta-g(\zeta,\varphi\mathcal{X}_3)Q\mathcal{X}_2]. \tag{61}$$

By using (4), (11), (20)–(23), (61) reduces to

$$C(\zeta,\mathcal{X}_2)\varphi\mathcal{X}_3=0. \tag{62}$$

Also from (43), we have

$$C(\zeta,\mathcal{X}_2)\mathcal{X}_3=-\frac{2\Lambda-2\rho r+n-2}{n-2}(\eta(\mathcal{X}_3)\mathcal{X}_2+\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)\zeta),$$

from which we infer

$$\varphi C(\zeta,\mathcal{X}_2)\mathcal{X}_3=-\frac{2\Lambda-2\rho r+n-2}{n-2}\eta(\mathcal{X}_3)\varphi\mathcal{X}_2. \tag{63}$$

From the equations (60), (62) and (63), we lead to

$$2\Lambda-2\rho r+n-2=0. \tag{64}$$

If we assume that r of $(LPK)_n$ is constant, then $\zeta r=0$. Thus in view of Remark 2.4, from (64) we obtain

$$\Lambda=n(n-1)\rho-\frac{n-2}{2}. \tag{65}$$

Thus we state the following:

Theorem 7.4: *If a conharmonically φ-semisymmetric $(LPK)_n$ with constant scalar curvature r admits a ρ-ES (g,ζ,Λ,ρ), then the soliton constant is given by $\Lambda=n(n-1)\rho-\frac{n-2}{2}$.*

Now we have the following corollary:

Corollary 7.5: *Let the metric of a conharmonically φ-semisymmetric $(LPK)_n$ with constant scalar curvature be a ρ-Einstein soliton. Then we have*

Values of ρ	Soliton type	Soliton constant	Conditions for $(g, V=\zeta, \Lambda, \rho)$ to be expanding, shrinking or steady
$\rho=\frac{1}{2}$	Einstein soliton	$\Lambda=\frac{n^2-2n+2}{2}$	$(g, V=\zeta, \Lambda, \rho)$ is expanding.
$\rho=\frac{1}{n}$	traceless Ricci soliton	$\Lambda=\frac{n}{2}$	$(g, V=\zeta, \Lambda, \rho)$ is expanding.
$\rho=\frac{1}{2(n-1)}$	Schouten soliton	$\Lambda=1$	$(g, V=\zeta, \Lambda, \rho)$ is expanding.
$\rho=0$	Ricci soliton	$\Lambda=-\frac{n-2}{2}$	$(g, V=\zeta, \Lambda, \rho)$ is shrinking.

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