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Fractals as Julia sets of $a\exp[d\sin(z^n)] - bz + c$ via Jungck four-step iterative method with s-convexity as well as four-step iterative method

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In this manuscript, we explore some new stunning fractals of Julia sets by developing the escape criteria for novel type of complex function $p(z) = a\exp[d\sin(z^n)] - bz + c$, where $n, |d| \geq 2$ and $a, b, c, d \in \mathbb{C}$ and furnish some graphical illustrations of the generated amazing fractals, utilizing the Jungck four-step iteration scheme equipped with s-convexity as well as four-step iterative method. Moreover, we conclude this work by examining variation in images and the impact of parameters on the deviation of dynamics, color, and appearance of fractals. At some fixed input parameters, we observe the engrossing behavior of Julia sets for different n via the considered algorithms.

Keywords: Algorithms; Escape criteria; Julia sets, Fractals, Iterative methods; Convexity.

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1. Introduction

The captivating field of fractal mathematics has enticed scientists, mathematicians, and artists for decades, providing profound insights into the complexity and order found in the natural world. Among the vast assortment of mathematical shapes and patterns, Julia sets have emerged as a key area of study, presenting mesmerizing visual representations and intriguing mathematical relationships. Julia sets, named after the French mathematician Gaston Julia, sits at the heart of the broader realm

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of complex dynamics. These intricate sets are constructed through iterations of complex functions, focusing on specific values within the complex plane. P. Fatou [6], another prominent mathematician, furthered the study of Julia sets, establishing the Fatou set as the complement to the Julia set within the domain. One of the most captivating aspects of Julia's sets lies. The word fractal originates from the Latin language that means divide or break. This is tantamount to self-similar patterns in complex graphics. Fractals are infinitely complex identical patterns with many real-life applications and are common in nature because they adequately describe: leaf patterns, tree branches, electricity, clouds, lightning, rivers, crystals, and so on. Fractals play an important role in surveying or examining various natural or living frameworks, such as microorganism culture. In addition, cryptography, image compression, encryption as well as radar frameworks, computational architectural design, and engineering models fall into the areas in which fractal theory is widely used, see e.g. [3, 6, 12–14, 16, 18, 23].

Julia sets are fascinating mathematical objects that can be generated through a process called iteration. By repeatedly applying a simple function to a complex number, we can create mesmerizing fractal patterns. To generate a Julia set, we must choose a complex number, usually denoted as "*c*". This constant value is crucial in defining the unique characteristics of each Julia set. The iterations start with an initial value, typically denoted as "*z*", and then repeatedly apply a function, usually $f(z) = z^2 + c.$ The resulting Julia set is the collection of complex numbers that do not diverge to infinity within a given set of iterations. The contrasting colors and intricate shapes within the Julia set correspond to the different convergence properties of these complex numbers. The escaping criterion determines when to stop iterating and consider a point as part of the Julia set. One common escape criterion is to set a maximum number of iterations. If a point reaches this maximum without diverging to infinity, it is considered as part of the set.

In 2004, Rani et al. [19] studied the chaotic behavior of a complex function $f(z) = z^2 + c,$ for some complex constant c through the iteration schemes known in the fixed point theory. Later on, numerous mathematicians used different iterative processes like Mann iteration, Picard iteration, Ishikawa iteration, Noor iteration, S-iteration, Junkck-Ishikawa iteration, Junkck-SP iteration with s-convexity, Junkck-CR iteration, implicit iterative scheme, and obtained variants of these sets to study their behavior and pattern for different polynomials, complex sine function, complex cosine function and transcendental functions because it is known that shape, color, and other characteristics vary with the iterative procedures for the same functions, see [1–6, 9–11, 17, 20, 22, 24]. Iteration schemes are not only used in the generation of Julia sets, but we can find their applications in the generation of other types of fractals, e.g., biomorphs, iterated function system fractals, inversion fractals, root-finding fractals etc., see e.g. [3–5, 7, 8, 12–14, 18].

The present work, inspired by Antal et al. [1] and Shatanawi et al. [21], studied Julia sets of complex cosine function using four-step iteration scheme extended by s-convexity to develop the escape criterion. We first extend the existing four-step iteration scheme with s-convexity to develop the escape criteria for new complex function $p(z) = a\exp[d\sin(z^n)] - bz + c,$ where $n, |d| \geq 2$ and $a, b, c, d \in \mathbb{C},$ and then furnish some graphical examples using the proven escape criteria, developed an escape time algorithm, color map, and MATLAB software.

The rest of the paper is organized as follows, Section 2 contains some basic definitions and results needed to achieve the goal of this paper. In Section 3, we introduce a more generalized, new complex function $p(z) = a\exp[d\sin(z^n)] - bz + c,$ where $n, |d| \geq 2$ and $a, b, c, d \in \mathbb{C},$ and study the escape criteria for the four-step iterations scheme and Jungck-four step iteration with s-convexity for the new considered complex function. In Section 4, we presents some graphical examples of Julia sets obtained with the proposed approach and showing the dependence between the size of the generated Julia sets and the values of the parameters. Finally, in Section 5, we conclude our work.

2. Preliminaries

Definition 2.1. (Julia set [6]) *Let* $p : \mathbb{C} \to \mathbb{C}.$ *The filled Julia set of p is denote by* J_p *and is defined as*

$$J_p = \{z \in \mathbb{C} : \{|\, p^k(z)\,|\}_{k=0}^{\infty} \text{ is bounded}\}.$$

Noticeably, it is a set of complex numbers for which the orbits do not converge to a point at infinity. The Julia set of p is the boundary of J_p, *that is,* $J_p = \partial J_p$.

Definition 2.2. (s-convex combination [14]) *Let* $z_1, z_2, z_3, \cdots, z_n \in \mathbb{C}$ and $s \in (0,1]$. *The s-convex combination is described as*

$$\lambda_1^s z_1 + \lambda_2^s z_2 + \lambda_3^s z_3 + \cdots + \lambda_n^s z_n.$$

where $\lambda_k \geq 0$ *and* $\sum_{k=1}^{n} \lambda_k = 1$, *for* $k \in \{1,2,3,\cdots,n\}$.

For $s = 1$, the s-convex combination diminishes to the standard convex combination.

Consider the sequence $\{z_k\}$ of iterates for the initial point $z_0 \in \mathbb{C}$ *and* $S, T : \mathbb{C} \to \mathbb{C}$ be a complex-valued mappings so that S is injective. Then sequence $\{z_k\}$ of iterates for any initial point $z_0 \in \mathbb{C}, \mu, \nu, \xi, \eta \in (0,1]$, *and* $k \in \{1,2,3,\ldots\}$, is known as the Jungck four-step iterative method with s-convexity and written as:

$$\left.\begin{aligned} Sz_k &= (1-\mu)^s Sz_{k-1} + \mu^s Ty_{k-1}, \\ Sy_{k-1} &= (1-\nu)^s Sz_{k-1} + \nu^s Tx_{k-1}, \\ Sx_{k-1} &= (1-\xi)^s Sz_{k-1} + \xi^s Tt_{k-1}, \\ St_{k-1} &= (1-\eta)^s Sz_{k-1} + \eta^s Tz_{k-1}. \end{aligned}\right\} \tag{2.1}$$

Consider the sequence $\{z_k\}$ of iterates for the initial point $z_0 \in \mathbb{C}$ *and* $p : \mathbb{C} \to \mathbb{C}$ be a complex-valued mapping. Then, the sequence $\{z_k\}$ of iterates for any initial point $z_0 \in \mathbb{C}, \mu, \nu, \xi, \eta, s \in (0,1]$, and $k \in \{1,2,3,\ldots\}$, is known as the four-step iterative method and written as:

$$\left.\begin{aligned} z_k &= (1-\mu) z_{k-1} + \mu p(y_{k-1}), \\ y_{k-1} &= (1-\nu) z_{k-1} + \nu p(x_{k-1}), \\ x_{k-1} &= (1-\xi) z_{k-1} + \xi p(t_{k-1}), \\ t_{k-1} &= (1-\eta) z_{k-1} + \eta p(z_{k-1}). \end{aligned}\right\} \tag{2.2}$$

Remark 2.1. *The Jungck-four step iterative method with s-convexity reduces to: Jungck-Noor iteration with s-convexity* [11] *when* $\eta = 0$, *Noor iteration* [16] *when* $S(z) = z, \eta = 0$ *and* $s = 1$, *Ishikawa iteration with s-convexity* [18] *when* $S(z) = z, \eta = 1, \xi = 1$ *and* $s = 1$; *Jungck-Mann iteration with s-convexity* [21] *when* $S(z) = z, \nu = 1, \eta = 1, \xi = 1$ *and* $s = 1$.

To generate fractals and escape limitations are the basic key to run the algorithms. Since it is well known that $|\, sin(z^n)\,| \leq 1$ for some $z \in \mathbb{C}$ and the Maclaurin expansion for sine and exponential functions are

$$\left|\sin(z^n)\right| = \left|\sum_{k=0}^{\infty} \frac{(-1)^k z^{n(2k+1)}}{(2k+1)!}\right| = \left|z^n\right| \left|\sum_{k=0}^{\infty} \frac{(-1)^k z^{2kn}}{(2k+1)!}\right| \geq \left|\omega_1\right| \left|z^n\right|, \tag{2.3}$$

where $0 < |\omega|_1 \leq 1$ except the values of $z \in \mathbb{C}$ for which $|\omega_1| = 0$ and satisfying the bound $\left|\sum_{k=0}^{\infty} \frac{(-1)^k z^{2kn}}{(2k+1)!}\right| \geq |\omega_1|$

and

$$\left|e^{z^n}\right| = \left|\sum_{k=0}^{\infty} \frac{z^{nk}}{k!}\right| > \left|\sum_{k=1}^{\infty} \frac{z^{nk}}{k!}\right| = \left|z^n\right| \left|\sum_{k=1}^{\infty} \frac{z^{n(k-1)}}{k!}\right| > |\alpha| \left|z^n\right| \tag{2.4}$$

where $0<|\alpha|\le 1$ and satisfying the bound $\left|\sum_{k=1}^{\infty}\frac{z^{n(k-1)}}{k!}\right|>|\alpha|$, (See, e.g. [1, 21]).

3. Escape criteria for a new considered complex functions

Motivated by numerous applications of transcendental function in science and engineering, we establish the escape time algorithm via Jungck four-step iterative method with s-convex combination as well as four-step iterative method for novel complex function of the type $p(z)=a\exp(d\sin(z^n))-bz+c$, where $n,|d|\ge 2$ and a, b, c and d are complex numbers. Consequently, we establish a novel threshold escape radii and utilize these to visualize some non-classical variants of classical fractals in the following results.

3.1. Escape criterion for Jungck-four step iterative method with s-convex combination

We establish the escape criteria for Jungck-four step iterative method with s-convex combination for novel complex function of the type $p(z)=a\exp(d\sin(z^n))-bz+c$, where $n,|d|\ge 2$ and a, b, c and d are complex numbers. We break the complex function $p(z)$ into two maps S and T so that $p(z)=Tz-Sz$, where Sz is injective.

Theorem 3.1. *Let $p(z)=a\exp[d\sin(z^n)]-bz+c$ be a complex function, where $n,|d|\ge 2$ and $a,b,c,d\in\mathbb{C}$. Assume that $z_0\in\mathbb{C},|z_0|\ge|c|>\left(\frac{2(1+|b|)}{s\eta|a||\alpha_1|}\right)^{\frac{1}{n-1}}$, $|z_0|\ge|c|>\left(\frac{2(1+|b|)}{s\xi|a||\alpha_2|}\right)^{\frac{1}{n-1}}$, $|z_0|\ge|c|>\left(\frac{2(1+|b|)}{s\nu|a||\alpha_3|}\right)^{\frac{1}{n-1}}$ and $|z_0|\ge|c|>\left(\frac{2(1+|b|)}{s\mu|a||\alpha_4|}\right)^{\frac{1}{n-1}}$, where $|\alpha_1|,|\alpha_2|,|\alpha_3|,|\alpha_4|,\eta,\xi,\nu,\mu,s\in(0,1]$, and $|\omega_1|,|\omega_2|,|\omega_3|,|\omega_4|\in[0.5,1]$. If the sequence $\{z_k\}$ is a Jungck four-step iterative method with s-convexity defined by (2.1), where $Sz = bz$ is injective, $Tz=a\exp(d\sin(z^n))+c$ is a complex function, and $k\in\mathbb{N}$. Then $|z_k|\to\infty$, as $k\to\infty$.*

Proof. For k = 1, here

$$St_{k-1}=(1-\eta)^s Sz_{k-1}+\eta^s Tz_{k-1}$$

implies

$$\begin{aligned}|St_0|&=\left|(1-\eta)^s Sz_0+\eta^s Tz_0\right|\\ &\left|(1-\eta)^s bz_0+\eta^s(a\exp(d\sin(z_0^n))+c)\right|.\end{aligned}$$

Since $\eta,s\in(0,1]$, *so* $\eta^s\ge s\eta$, and utilizing binomial expansion of $(1-\eta)^s$ up to linear terms of η, we attain

$$\begin{aligned}|St_0|&\ge\left|(1-\eta)^s bz_0+s\eta(a\exp(d\sin(z_0^n))+c)\right|\\ &\ge\left|s\eta(a\exp(d\sin(z_0^n)))+(1-\eta)^s bz_0\right|-s\eta|c|\\ &\ge\left|s\eta(a\exp(d\sin(z_0^n)))+(1-\eta s)bz_0\right|-s\eta|z_0|,\ |z_0|\ge|c|\\ &\ge\left|s\eta(a\exp(d\sin(z_0^n)))\right|-(1-\eta s)|bz_0|-s\eta|z_0|.\end{aligned}$$

Since $|d|\ge 2$ and $|\omega_1|\in[0.5,1]$, by (2.3) and (2.4), we have $\left|\sin(z^n)\right|\ge|\omega_1||z^n|$, which implies that $\exp(|d||\sin(z_0^n)|)\ge\exp(|d||\omega_1||z_0^n|)\ge\exp(|z_0^n|)>|\alpha_1||z_0^n|$, we attain

$$
\begin{aligned}
|St_0| &\geq |s\eta a \exp(d\sin(z_0^n))| - (1-s\eta)|b||z_0| - s\eta|z_0| \\
&= s\eta|a|\exp(|(d\sin(z_0^n))|) - |b||z_0| + s\eta|b||z_0| - s\eta|z_0| \\
&\geq s\eta|a|\exp(|z_0^n|) - |b||z_0| - s\eta|z_0|, \quad |b| \geq |0| \\
&> s\eta|a||\alpha_1||z_0^n| - |b||z_0| - |z_0|, \quad s\eta < 1 \\
&\geq s\eta|a||\alpha_1||z_0^n| - |b||z_0| - |z_0|, \\
&= s\eta|a||\alpha_1||z_0^n| - (1+|b|)|z_0|,
\end{aligned}
$$

which gives us

$$
\begin{aligned}
|bt_0| &\geq |z_0|(s\eta|a||\alpha_1||z_0^{n-1}| - (1+|b|)), \\
&= |z_0|(1+|b|)\left(\frac{s\eta|a||\alpha_1||z_0^{n-1}|}{(1+|b|)} - 1\right).
\end{aligned}
$$

Thus

$$
|t_0| \geq \frac{|bt_0|}{(1+|b|)} \geq |z_0|\left(\frac{s\eta|a||\alpha_1||z_0^{n-1}|}{(1+|b|)} - 1\right).
$$

Now

$$
|z_0| \geq |c| > \left(\frac{2(1+|b|)}{s\eta|a||\alpha_1|}\right)^{\frac{1}{n-1}} \quad \text{implies that} \quad \frac{s\eta|a||\alpha_1||z_0^{n-1}|}{(1+|b|)} - 1 > 1.
$$

Hence $|t_0| > |z_0|$.

For $k = 1$, here

$$
Sx_{k-1} = (1-\xi)^s Sz_{k-1} + \xi^s Tt_{k-1}
$$

implies

$$
\begin{aligned}
|Sx_0| &= |(1-\xi)^s Sz_0 + \xi^s Tt_0| \\
&= |(1-\xi)^s bz_0 + \xi^s(a\exp(d\sin(t_0^n)) + c)|.
\end{aligned}
$$

Since $\eta, s \in (0,1]$, *so* $\eta^s \geq s\eta$, and utilizing binomial expansion of $(1-\xi)^s$ up to linear terms of ξ, we attain

$$
\begin{aligned}
|Sx_0| &\geq |(1-\xi)^s bz_0 + s\xi(a\exp(d\sin(t_0^n)) + c)| \\
&\geq |s\xi(a\exp(d\sin(t_0^n))) + (1-\xi)^s bz_0| - s\xi|c| \\
&\geq || s\xi(a\exp(d\sin(t_0^n))) + (1-\xi s)bz_0| - s\xi|z_0|, \ |z_0| \geq |c| \\
&\geq |s\xi(a\exp(d\sin(t_0^n)))| - |(1-\xi s)bz_0| - s\xi|z_0|.
\end{aligned}
$$

Since $|d| \geq 2$ *and* $|\omega_2| \in [0.5,1]$, by (2.3) and (2.4), we have $|\sin(t_0^n)| \geq |\omega_2||t_0^n|$, which implies that $\exp(|d||\sin(t_0^n)|) \geq \exp(|d||\omega_2||t_0^n|) \geq \exp(|t_0^n|) > |\alpha_2||t_0^n|$, we attain

$$
\begin{aligned}
|Sx_0| &\geq |s\xi a\exp(d\sin(t_0^n))| - |(1-s\xi)bz_0| - s\xi|z_0| \\
&= s\xi|a||\exp[d\sin(t_0^n)]| - |b||z_0| + |s\xi bz_0| - s\xi|z_0| \\
&\geq s\xi|a||\exp(t_0^n)| - |b||z_0| - s\xi|z_0|, \quad |b| \geq |0| \\
&> s\xi|a||\alpha_2||t_0^n| - |b||z_0| - |z_0|, \quad s\xi < 1 \\
&\geq s\xi|a||\alpha_2||t_0^n| - (1+|b|)|z_0|, \; |t_0| > |z_0| \\
&> s\xi|a||\alpha_2||z_0^n| - (1+|b|)|z_0|, \\
&= |z_0|(s\xi|a||\alpha_2||z_0^{n-1}| - (1+|b|)),
\end{aligned}
$$

which gives us

$$
\begin{aligned}
|bx_0| &\geq |z_0|(s\xi|a||\alpha_2||z_0^{n-1}|) - (1+|b|)|z_0|, \\
&= |z_0|(1+|b|)\left(\frac{s\xi|a||\alpha_2||z_0^{n-1}|}{(1+|b|)} - 1\right).
\end{aligned}
$$

Thus

$$
|x_0| \geq \frac{|b|}{(1+|b|)}|x_0| \geq |z_0|\left(\frac{s\xi|a||\beta||d||\omega_2||z_0^{n-1}|}{(1+|b|)} - 1\right).
$$

Now

$$
|z_0| \geq |c| > \left(\frac{2(1+|b|)}{s\xi|a||\alpha_2|}\right)^{\frac{1}{n-1}} \quad \text{implies that} \quad \frac{s\xi|a||\alpha_2||z_0^{n-1}|}{(1+|b|)} - 1 > 1.
$$

Hence $|x_0| > |z_0|$.

For $k = 1$

$$
Sy_{k-1} = (1-\nu)^s Sz_{k-1} + \nu^s Tx_{k-1}
$$

implies

$$
\begin{aligned}
|Sy_0| &= |(1-\nu)^s Sz_0 + \nu^s Tx_0| \\
&= |(1-\nu)^s bz_0 + \nu^s(a\exp(d\sin(x_0^n)) + c)|.
\end{aligned}
$$

Since $\nu, s \in (0,1]$, *so* $\nu^s \geq s\nu$, and utilizing binomial expansion of $(1-\nu)^s$ up to linear terms of υ, we attain

$$
\begin{aligned}
|Sy_0| &\geq |\,|(1-\nu)^s bz_0 + s\nu(a\exp(d\sin(x_0^n)) + c)| \\
&\geq |s\nu(a\exp(d\sin(x_0^n))) + (1-\nu)^s bz_0| - s\nu|c| \\
&\geq |s\nu(a\exp(d\sin(x_0^n))) + (1-\nu s)bz_0| - s\nu|z_0|, \; |z_0| \geq |c| \\
&\geq |s\nu(a\exp(d\sin(x_0^n)))| - |(1-\nu s)bz_0| - s\nu|z_0|.
\end{aligned}
$$

Since $|d| \geq 2$ *and* $|\omega_3| \in [0.5,1]$, by (2.3) and (2.4), we have $|\sin(x_0^n)| \geq |\omega_3||x_0^n|$, which implies that $\exp(|d||\sin(x_0^n)|) \geq \exp(|d||\omega_3||x_0^n|) \geq \exp(|x_0^n|) > |\alpha_3||x_0^n|$, we attain

$$\begin{aligned}
|Sy_0| &\geq |sv\,a\exp(d\sin(x_0^n))| - (1-sv)|b||z_0| - sv|z_0| \\
&= sv|a||\exp[d\sin(x_0^n)]| - |b||z_0| + sv|b||z_0| - sv|z_0| \ \ ||b| \geq |0| \\
&\geq sv|a||\exp(x_0^n)| - |b||z_0| - sv|z_0|, \quad sv < 1 \\
&> sv|a||\alpha_3||x_0^n| - |b||z_0| - |z_0|, \ |x_0| > |z_0| \\
&> sv|a||\alpha_3||z_0^n| - (1+|b|)|z_0| \\
&= |z_0|(sv|a||\alpha_3||z_0^{n-1}| - (1+|b|)),
\end{aligned}$$

which gives us

$$\begin{aligned}
|by_0| &\geq |z_0|(sv|a||\alpha_3||z_0^{n-1}|) - (1+|b|)|z_0| \\
&= |z_0|(1+|b|)\left(\frac{sv|a||\alpha_3||z_0^{n-1}|}{(1+|b|)} - 1\right).
\end{aligned}$$

Thus

$$|y_0| \geq \frac{|by_0|}{(1+|b|)} \geq |z_0|\left(\frac{sv|a||\alpha_3||z_0^{n-1}|}{(1+|b|)} - 1\right).$$

Now

$$|z_0| \geq |c| > \left(\frac{2(1+|b|)}{sv|a||\alpha_3|}\right)^{\frac{1}{n-1}} \text{ implies that } \frac{sv|a||\alpha_3||z_0^{n-1}|}{(1+|b|)} - 1 > 1.$$

Hence $|y_0| > |z_0|$.

Now

$$Sz_k = (1-\mu)^s Sz_{k-1} + \mu^s Ty_{k-1}$$

implies

$$\begin{aligned}
|Sz_1| &= |(1-\mu)^s Sz_0 + \mu^s Ty_0| \\
&= (1-\mu)^s bz_0 + \mu^s(a\exp(d\sin(y_0^n)) + c).
\end{aligned}$$

Since $\mu, s \in (0,1]$, *so* $\mu^s \geq s\mu$, and utilizing binomial expansion of $(1-\mu)^s$ up to linear terms of μ, we attain

$$\begin{aligned}
|Sz_1| &\geq |(1-\mu)^s bz_0 + s\mu(a\exp(d\sin(y_0^n)) + c)| \\
&\geq |s\mu(a\exp(d\sin(y_0^n))) + (1-\mu)^s bz_0| - s\mu|c| \\
&\geq |s\mu(a\exp(d\sin(y_0^n))) + (1-\mu s)bz_0| - s\mu|z_0|, \ |z_0| \geq |c| \\
&\geq |s\mu a\exp(d\sin(y_0^n))| - (1-\mu s)|bz_0| - s\mu|z_0|.
\end{aligned}$$

Since $|d| \geq 2$ *and* $|\omega_4| \in [0.5,1]$, by (2.3) and (2.4), we have $|\sin(y_0^n)| \geq |\omega_4||y_0^n|$, which implies that $\exp(|d||\sin(y_0^n)|) \geq \exp(|d||\omega_4||y_0^n|) \geq \exp(|y_0^n|) > |\alpha_4||y_0^n|$, we attain

$$\begin{aligned}
|Sz_1| &\geq |s\mu a \exp(d\sin(y_0^n))| - (1-s\mu)|b||z_0| - s\mu|z_0| \\
&= s\mu|a||\exp[d\sin(y_0^n)]| - |bz_0| + s\mu|b||z_0| - s\mu|z_0|, \ \ |b| \geq |0| \\
&\geq s\mu|a||\exp(y_0^n)| - |bz_0| - s\mu|z_0|, \quad s\mu < 1 \\
&> s\mu|a||\alpha_4||y_0^n| - |b||z_0| - |z_0|, \ |y_0| > |z_0| \\
&> s\mu|a||\alpha_4||z_0^n| - (1+|b|)|z_0| \\
&= |z_0|(s\mu|a||\alpha_4||z_0^{n-1}| - (1+|b|)),
\end{aligned}$$

which gives us

$$\begin{aligned}
|bz_1| &\geq |z_0|(s\mu|a||\alpha_4||z_0^{n-1}| - (1+|b|)), \\
&= |z_0|(1+|b|)\left(\frac{(s\mu|a||\alpha_4||z_0^{n-1}|)}{(1+|b|)} - 1\right).
\end{aligned}$$

Thus

$$|z_1| \geq \frac{|bz_1|}{(1+|b|)} \geq |z_0|\left(\frac{(s\mu|a||\alpha_4||z_0^{n-1}|)}{(1+|b|)} - 1\right).$$

Since, $|z_0| \geq |c| > (\frac{2(1+|b|)}{s\mu|a||\alpha_4|})^{\frac{1}{n-1}}$, there exists a $\lambda > 0$ so that $\frac{(s\mu|a||\alpha_4||z_0^{n-1}|)}{(1+|b|)} - 1 > 1 + \lambda$.

As a result,

$$|z_1| \geq (1+\lambda)|z_0|.$$

Following the same pattern repeatedly, we obtain $|z_k| > (1+\lambda)^k|z_0|$.

Consequently, $|z_k| \to \infty$ as $k \to \infty$, that is, the orbit of z_0 tends to infinity.

Corollary 3.1. *If we consider* $|z_m| > \max\left\{|c|, \left(\frac{2(1+|b|)}{s\eta|a||\alpha_1|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|b|)}{s\xi|a||\alpha_2|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|b|)}{s\nu|a||\alpha_3|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|b|)}{s\mu|a||\alpha_4|}\right)^{\frac{1}{n-1}}\right\}$, *where* $m \geq 0$, *then* $z_{m+k} > (1+\lambda)^k |z_k|$ *and the Jungck four-step iterative method with s-convexity of sequence* $\{z_k\}$ *of iterates for any initial point* z_0 *tends ton* ∞ *as k tends to* ∞.

3.2. Escape Criterion for four step iterative method

First, we derive the escape criterion for the complex function of type $p(z) = a\exp[d\sin(z^n)] - bz + c$, where $n, |d| \geq 2$ and $a,b,c,d \in \mathbb{C}$ utilizing the four-step iteration (2.2).

Theorem 3.2. *Let* $p(z) = a\exp[d\sin(z^n)] - bz + c$, *be a complex function where* $n, |d| \geq 2$ *and* $a,b,c,d \in \mathbb{C}$. *Assume that* $z_0 \in \mathbb{C}, |z_0| \geq |c| > \left(\frac{2(1+\eta|b|)}{\eta|a||\alpha_1|}\right)^{\frac{1}{n-1}}, |z_0| \geq |c| > \left(\frac{2(1+\xi|b|)}{\xi|a||\alpha_2|}\right)^{\frac{1}{n-1}}, |z_0| \geq |c| > \left(\frac{2(1+\nu|b|)}{\nu|a||\alpha_3|}\right)^{\frac{1}{n-1}}$ *and* $|z_0| \geq |c| > \left(\frac{2(1+\mu|b|)}{\mu|a||\alpha_4|}\right)^{\frac{1}{n-1}}$, *where* $\eta, \xi, \nu, \mu, |\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4| \in (0,1]$, *and* $|\omega_1|, |\omega_2|, |\omega_3|, |\omega_4| \in [0.5,1]$. *If the sequence* $\{z_k\}$ *is a four-step iterative method defined by (2.2). Then* $|z_k| \to \infty$, *as* $k \to \infty$.

Proof. For $k = 1$, here

$$t_{k-1} = (1-\eta)z_{k-1} + \eta p(z_{k-1})$$

implies

$$\begin{aligned}|t_0| &= |(1-\eta)z_0 + \eta p(z_0)| \\ &= |(1-\eta)z_0 + \eta(a\exp(d\sin(z_0^n)) - bz_0 + c)| \\ &\geq |\eta a \exp(d\sin(z_0^n))| - |(1-\eta)z_0| - |\eta(c - bz_0)|.\end{aligned}$$

Since $|d| \geq 2$ *and* $|\omega_1| \in [0.5,1]$, by (2.3) and (2.4), we have $|\sin(z^n)| \geq |\omega_1||z^n|$, which implies that $\exp(|d|\,|\sin(z_0^n)|) \geq \exp(|d|\,|\omega_1|\,|z_0^n|) \geq \exp(|z_0^n|) > |\alpha_1|\,|z_0^n|$, we attain

$$\begin{aligned}|t_0| &> \eta|a|\alpha_1|z_0^n| - (1-\eta)|z_0| - \eta|c - bz_0| \\ &\geq \eta|a|\alpha_1|z_0^n| - |z_0| + \eta|z_0| - \eta|c| - \eta|b||z_0|,\ |z_0| \geq |c| \\ &\geq \eta|a|\alpha_1|z_0^n| - |z_0| + \eta|z_0| - \eta|z_0| - \eta|b||z_0| \\ &\geq \eta|a|\alpha_1|z_0^n| - (1+\eta|b|)|z_0|\end{aligned}$$

which gives us

$$|t_0| \geq |z_0|(1+\eta|b|)\left(\frac{\eta|a||\alpha_1||z_0^{n-1}|}{(1+\eta|b|)} - 1\right).$$

Thus

$$|t_0| \geq \frac{|t_0|}{(1+\eta|b|)} \geq |z_0|\left(\frac{\eta|a||\alpha_1||z_0^{n-1}|}{(1+\eta|b|)} - 1\right). \tag{3.1}$$

Now

$$|z_0| \geq |c| > (\frac{2(1+\eta|b|)}{\eta|a||\alpha_1|})^{\frac{1}{n-1}} \text{ implies that } \frac{\eta|a||\alpha_1||z_0^{n-1}|}{(1+\eta|b|)} - 1 > 1.$$

Hence $|t_0| > |z_0|$.

For $k = 1$, here

$$x_{k-1} = (1-\xi)z_{k-1} + \xi p(t_{k-1}) \tag{3.2}$$

implies

$$\begin{aligned}|x_0| &= |(1-\xi)z_0 + \xi p(t_0)| \\ &= |(1-\xi)z_0 + \xi(a\exp(d\sin(t_0^n)) - bt_0 + c)|.\end{aligned}$$

Using the same argument as for (3.1), we calculate

$$|x_0| \geq \frac{|x_0|}{(1+\xi|b|)} \geq |z_0|(\frac{\xi|a||\alpha_2||z_0^{n-1}|}{(1+\xi|b|)} - 1).$$

Now

$$|z_0| \geq |c| > (\frac{2(1+\xi|b|)}{\xi|a||\alpha_2|})^{\frac{1}{n-1}} \text{ implies that } \frac{\xi|a||\alpha_2||z_0^{n-1}|}{(1+\xi|b|)} - 1 > 1.$$

Hence $|x_0| > |z_0|$.

For $k = 1$, here

$$y_{k-1} = (1-\nu)z_{k-1} + \nu p(x_{k-1}) \tag{3.3}$$

implies

$$\begin{aligned}|y_0| &= |(1-\nu)z_0 + \nu p(x_0)|\\ &= \left|(1-\nu)z_0 + \nu(a\exp(d\sin(x_0^n)) - bx_0 + c)\right|.\end{aligned}$$

Using the same argument as for (3.1), we calculate

$$|y_0| \ge \frac{|y_0|}{(1+\nu|b|)} \ge |z_0|\left(\frac{\nu|a||\alpha_3||z_0^{n-1}|}{(1+\nu|b|)} - 1\right).$$

Now

$$|z_0| \ge |c| > \left(\frac{2(1+\nu|b|)}{\nu|a||\alpha_3|}\right)^{\frac{1}{n-1}} \text{ implies that } \frac{\nu|a||\alpha_3||z_0^{n-1}|}{(1+\nu|b|)} - 1 > 1.$$

Hence $|y_0| > |z_0|$.

For $k = 1$, here

$$z_k = (1-\mu)z_{k-1} + \mu p(y_{k-1}) \text{ implies } z_1 = \left|(1-\mu)z_0 + \mu(a\exp(d\sin(y_0^n)) - by_0 + c)\right|.$$

Using the same argument as for (3.1), we calculate

$$|z_1| \ge \frac{|z_1|}{(1+\mu|b|)} \ge |z_0|\left(\frac{\mu|a||\alpha_4||z_0^{n-1}|}{(1+\mu|b|)} - 1\right).$$

Since, $|z_0| \ge |c| > \left(\frac{2(1+\mu|b|)}{\mu|a||\alpha_4|}\right)^{\frac{1}{n-1}}$, there exists a $\lambda > 0$ so that $\frac{\mu|a||\alpha_4||z_0^{n-1}|}{(1+\mu|b|)} - 1 > 1 + \lambda$.

As a result,

$$|z_1| \ge (1+\lambda)|z_0|.$$

Following the same pattern repeatedly, we obtain $|z_k| > (1+\lambda)^k|z_0|$. Consequently, $|z_k| \to \infty$ as $k \to \infty$, that is, the orbit of z_0 tends to infinity.

Corollary 3.2. *If we consider* $|z_m| > \max\left\{|c|, \left(\frac{2(1+\eta|b|)}{\eta|a||\alpha_1|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+\xi|b|)}{\xi|a||\alpha_2|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+\nu|b|)}{\nu|a||\alpha_3|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+\mu|b|)}{\mu|a||\alpha_4|}\right)^{\frac{1}{n-1}}\right\}$ *where* $m \ge 0$, *then* $z_{m+k} > (1+\lambda)^k |z_k|$ *and the Jungck four-step iterative method of sequence* $\{z_k\}$ *of iterates for any initial point* z_0 *tends ton* ∞ *as* k *tends to* ∞.

4. Application of Fractals

To visualize the fractals, some convergence conditions are required, and actually, these are the main tools to execute the algorithm properly and sketch the desired type of fractals. In this section, we adjust two algorithms: one for Julia set via Jungck four-step iterative method and other for the Julia set via four-step iteration method. Finally, we visualize some Julia sets for different involve parameters, and the different value of n.

4.1. Julia Sets

In this subsection, we sketch some graphs of Julia set at different input parameters. We generate Julia sets for Jungck four-step iteration method by using Algorithm 1 and compare the images of Julia set for proposed methods. Throughout the paper, we are using a maximum number of iterations $k = 100$.

Algorithm 1 Geometry of Julia set

Input: $Tz = a\exp(d\sin(z^n)) + c, Sz = bz,$ where $a,b,c,d \in \mathbb{C}$ and $n,|d| \geq 2; A \subset \mathbb{C}$-area; K-a maximum number of iterations, $0 < \eta, \xi, \nu, \mu, |\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4|, s \leq 1,$ and $0.5 \leq |\omega_1|, |\omega_2|, |\omega_3|, |\omega_4| \leq 1$-parameters of the Jungck-four step iteration with s-convexity; colourmap [0..C-1]-color with C colors

Output: Julia set for area A

for $z_0 \in A$ *do*

$$R_1 = \left(\frac{2(1+|b|)}{s\eta|a||\alpha_1|}\right)^{\frac{1}{n-1}}$$

$$R_2 = \left(\frac{2(1+|b|)}{s\xi|a||\alpha_2|}\right)^{\frac{1}{n-1}}$$

$$R_3 = \left(\frac{2(1+|b|)}{s\nu|a||\alpha_3|}\right)^{\frac{1}{n-1}}$$

$$R_4 = \left(\frac{2(1+|b|)}{s\mu|a||\alpha_4|}\right)^{\frac{1}{n-1}}$$

$$R = \max\{|c|, R_1, R_2, R_3, R_4\}$$

k=0

while $n \leq K$ do

$$t_k = \frac{(1-\eta)^s bz_k + \eta^s Tz_k}{b}$$

$$x_k = \frac{(1-\xi)^s bz_k + \xi^s Tt_k}{b}$$

$$y_k = \frac{(1-\nu)^s bz_k + \nu^s Tx_k}{b}$$

$$z_{k+1} = \frac{(1-\mu)^s bz_k + \mu^s Ty_k}{b}$$

if $|z_{k+1}| > R$ *then*

break

end if

k = k + 1

end while

$$i = [(C-1)\frac{k}{K}]$$

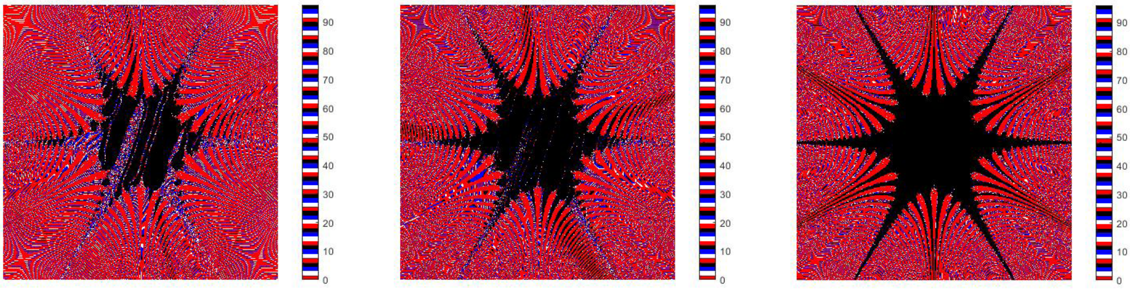
colour z_0 with colourmap [i]

end for

For $n = 3$, the parameter values as given in Table 1, we get amazing fractal objects, which are visible in Figure 1 [(*i*) to (*iii*)]. We notice that all Julia sets for $n = 3$ have six bunches of lashes and one of them have symmetry about x-axis. The size of lashes gradually decrease from the center of the bunch, and the angle between every two bunches is $\frac{\pi}{3}$. As the value of s increases, the amount of black colour in the Julia set increases from the center of the bunches, and the red colour decreases.

Table 1: Effect of parameter s on Julia sets

	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(*i*)	i	$-3+i$	$3i$	$2i$	0.002	0.003	0.001	0.004	0.1	0.04	0.07	0.08	0.03	0.7	0.8	0.7	0.6
(*ii*)	i	$-3+i$	$3i$	$2i$	0.002	0.003	0.001	0.004	0.5	0.04	0.07	0.08	0.03	0.7	0.8	0.7	0.6
(*iii*)	i	$-3+i$	$3i$	$2i$	0.002	0.003	0.001	0.004	0.9	0.04	0.07	0.08	0.03	0.7	0.8	0.7	0.6



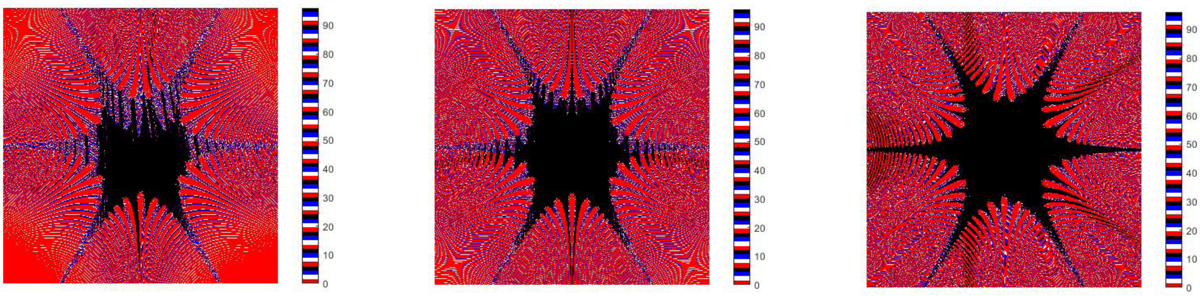
(i) (ii) (iii)

Figure 1: Julia sets for $n = 3$ via Jungck four-step iteration with s-convexity.

As s and the absolute value of b increase, the amount of black colour in the Julia set increases from the center of the bunches, and the red colour decreases. The size of lashes gradually increase from the center of the bunches, and the angle between every two bunches is $\frac{\pi}{3}$.

Table 2: Effect of parameters s and b on Julia sets

	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(*i*)	i	$1+i$	$11i$	$2i$	0.002	0.003	0.001	0.004	0.2	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(*ii*)	i	$75i$	$11i$	$2i$	0:002	0.003	0.001	0.004	0.5	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(*iii*)	i	231	$11i$	$2i$	0:002	0.003	0.001	0.004	0.85	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6



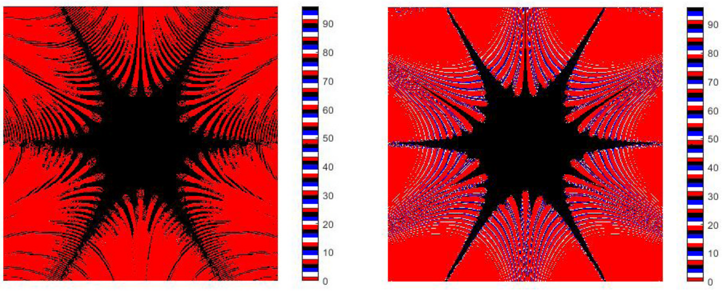
(i) (ii) (iii)

Figure 2: Julia sets for $n = 3$ via Jungck four-step iteration with s-convexity.

For $n = 3$ and the values of the parameter as given in Table 3, we get amazing fractal objects, which are visible in Figure 3 [(*i*) to (*iii*)]. As the different values of the parameters μ, ν, ξ and η decreases, the amount of black colour in the Julia set increases from the center of the bunches and the red colour decreases.

Table 3: Effect of parameters μ, ν, ξ and η on Julia sets

	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(*i*)	i	$-30+i$	$11i$	$2i$	0.8	0.7	0.8	0.9	0.7	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(*ii*)	i	$-30+i$	$11i$	$2i$	0.008	0.007	0.008	0.009	0.7	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(*iii*)	i	$-30+i$	$11i$	$2i$	0.00008	0.00007	0.00008	0.00009	0.5	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6



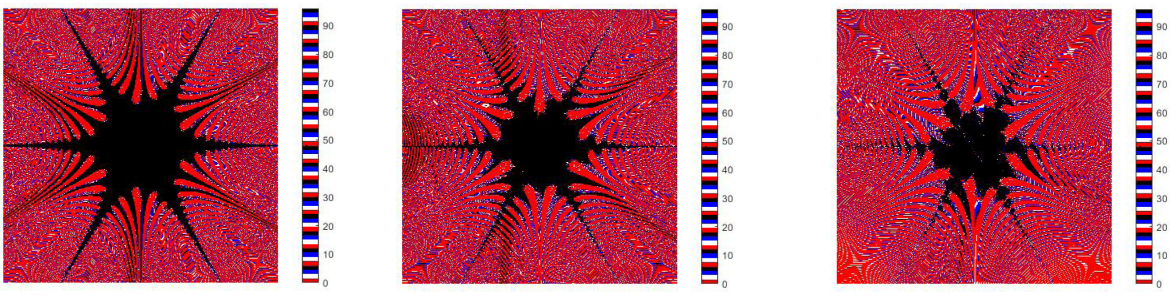
(i) (ii) (iii)

Figure 3: Julia sets for $n = 3$ via Jungck four-step iteration with s-convexity

For values of the parameter as given in Table 4, we get amazing fractal objects, which are visible in Figure 4 [(*i*) to (*iii*)]. As the different values of the parameters a, b, c and d as real, pure imaginary and complex, the amount of black colour in the Julia set slightly decreases from the center of the bunches.

Table 4: Effect of parameters a, b, c and d on Julia sets

	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(*i*)	1	85	13	2.5	0.0004	0.0006	0.0008	0.0009	0.75	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(*ii*)	i	$11i$	$31i$	$2i$	0.0004	0.0006	0.0008	0.0009	0.75	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(*iii*)	$4+2i$	$2-i$	$3+2i$	$2+i$	0.0004	0.0006	0.0008	0.0009	0.75	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6



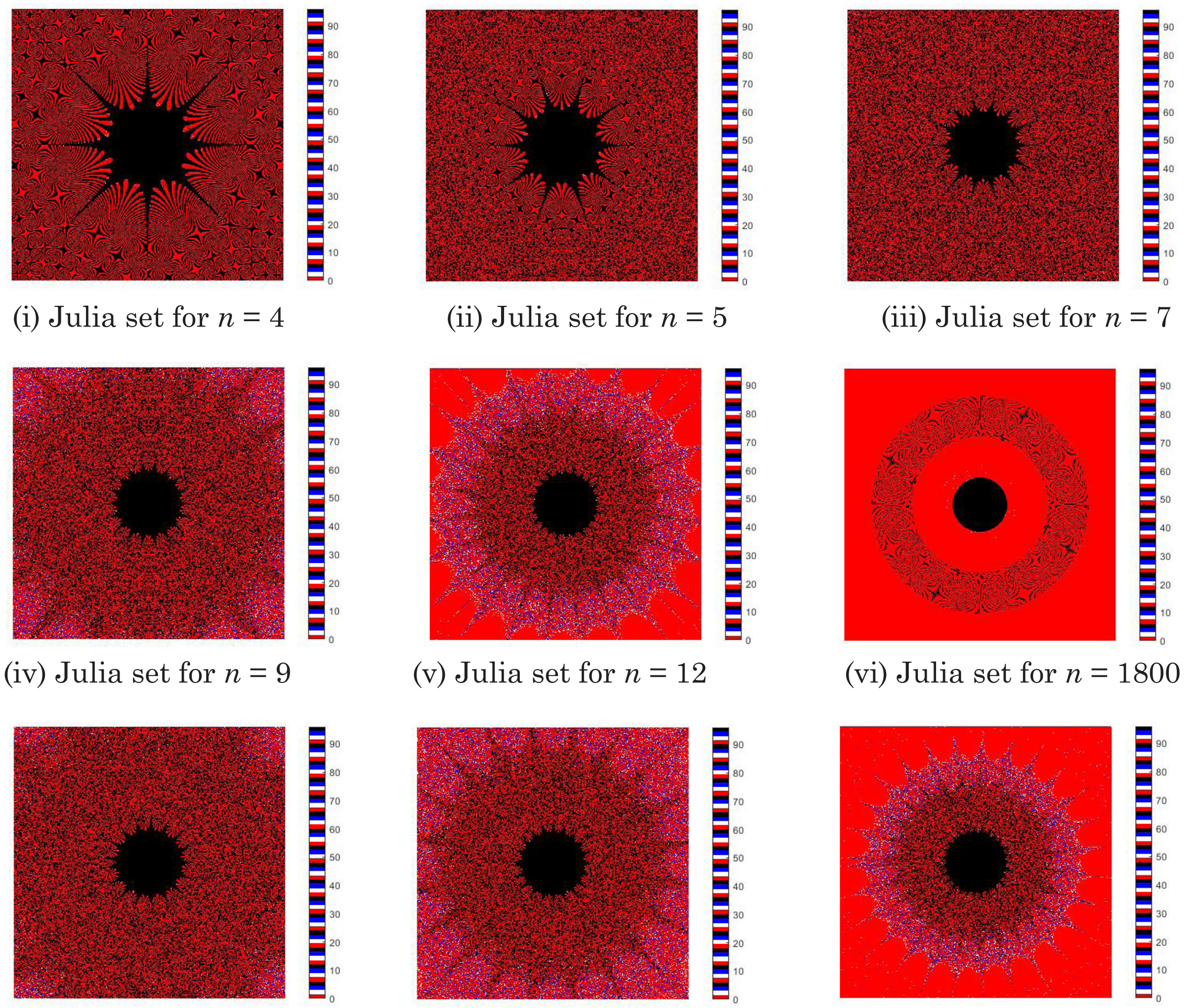
(i) (ii) (iii)

Figure 4: Julia sets for $n = 3$ via Jungck four-step iteration with s-convexity

For values of the parameter as given in Table 5, we get amazing fractal objects, which are visible in Figure 5 ((*i*) to (*ix*)). We notice that all Julia sets have bunches of lashes, and one of them have symmetry about x-axis. For integer and non-integer values of n, the amount of black colour in the Julia set increases from the center of the bunches, and the red colour decreases.

Table 5: Effect of change in the value of n on Julia sets

	n	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(i)	4	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(ii)	5	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(iii)	7	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(iv)	9	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(v)	12	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(vi)	1800	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(vii)	8.41	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
$(viii)$	10.15	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(ix)	13.65	$2i$	$31i$	$91i$	$3i$	0.0004	0.0006	0.0002	0.0007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6



(i) Julia set for n = 4 (ii) Julia set for n = 5 (iii) Julia set for n = 7

(iv) Julia set for n = 9 (v) Julia set for n = 12 (vi) Julia set for n = 1800

(vii) Julia set for n = 8.31 (viii) Julia set for n = 10.15 (ix) Julia set for n = 13.65

Figure 5: Julia sets for integer and non-integer values of n via Jungck-four step iteration with s-convexity

Algorithm 2 Geometry of Julia set

Input: $p(z) = a\exp(d\sin(z^n)) - bz + c$, where $a,b,c,d \in \mathbb{C}$ and $n, |d| \geq 2$; $A \subset \mathbb{C}$-area; K-a maximum number of iterations, $0 < \eta, \xi, \nu, \mu, |\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4|, s \leq 1$, and $0.5 \leq |\omega_1|, |\omega_2|, |\omega_3|, |\omega_4| \leq 1$-parameters of the Jungck-four step iteration with s-convexity; colourmap [0..C-1]-color with C colors

Output: Julia set for area A

for $z_0 \in A$ do

$$R_1 = \left(\frac{2(1+\eta|b|)}{\eta|a||\alpha_1|} \right)^{\frac{1}{n-1}}$$

$$R_2 = \left(\frac{2(1+\xi|b|)}{\xi|a||\alpha_2|} \right)^{\frac{1}{n-1}}$$

$$R_3 = \left(\frac{2(1+\nu|b|)}{\nu|a||\alpha_3|} \right)^{\frac{1}{n-1}}$$

$$R_4 = \left(\frac{2(1+\mu|b|)}{\mu|a||\alpha_4|} \right)^{\frac{1}{n-1}}$$

$$R = \max\left\{|c|, R_1, R_2, R_3, R_4\right\}$$

k=0

while $n \leq K$ do

$$t_k = (1-\eta)z_k + \eta p(z_k)$$

$$x_k = (1-\xi)z_k + \xi p(t_k)$$

$$y_k = (1-\nu)z_k + \nu p(x_k)$$

$$z_{k+1} = (1-\mu)z_k + \mu p(y_k)$$

if $|z_{k+1}| > R$ then

break

end if

k=k+1

end while

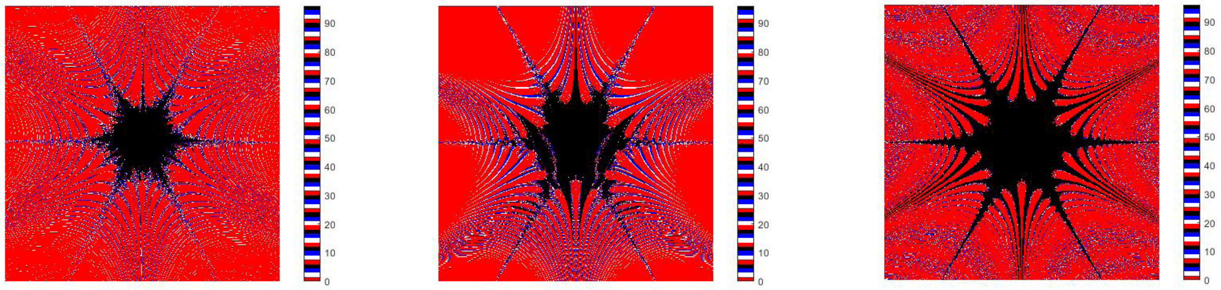
$i = [(C-1)\frac{k}{K}]$

colour z_0 with colourmap [i] end for

For values of the parameter as given in Table 6, we get amazing fractal objects, which are visible in Figure 6 [(i) to (iii)]. We notice that all Julia sets for $n = 3$ have six bunches of lashes and one of them have symmetry about x-axis, and the angle between every two bunches is $\frac{\pi}{3}$. As the different values of the parameters μ, ν, ξ and η decreases, the amount of black colour in the Julia set increases from the center of the bunches and the red colour decreases.

Table 6 : Effect of parameters μ, ν, ξ and η on Julia sets

	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(i)	i	$-30+i$	$11i$	$2i$	0.8	0.7	0.8	0.9	0.7	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(ii)	i	$-30+i$	$11i$	$2i$	0.008	0.007	0.008	0.009	0.7	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(iii)	i	$-30+i$	$11i$	$2i$	0.00008	0.00007	0.00008	0.00009	0.5	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6



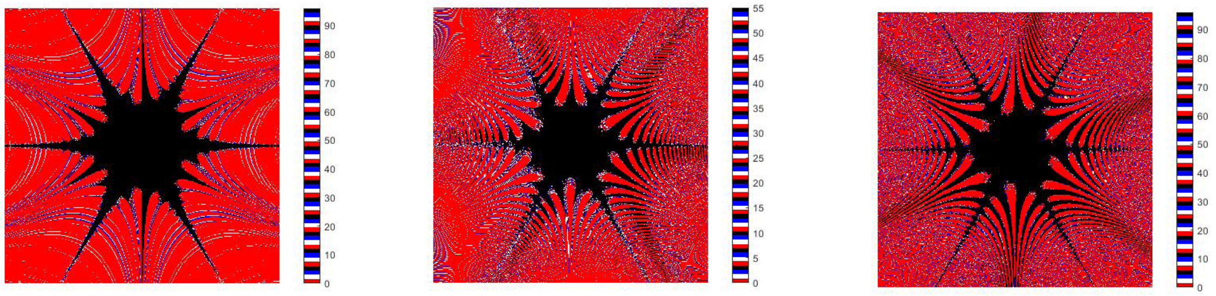
(i) (ii) (iii)

Figure 6: Julia sets for $n = 3$ via four-step iterative method

For values of the parameter as given in Table 7, we get amazing fractal objects, which are visible in Figure 7 [(i) to (iii)]. As the different values of the parameters a, b, c and d as real, pure imaginary and complex, the amount of black colour in the Julia set slightly decreases from the center of the bunches.

Table 7: Effect of parameters a, b, c and d on Julia sets

	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(i)	1	85	13	2.5	0.0004	0.0006	0.0008	0.0009	0.75	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(ii)	i	$11i$	$31i$	$2i$	0.0004	0.0006	0.0008	0.0009	0.75	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(iii)	$4+2i$	$2-i$	$3+2i$	$2+i$	0.0004	0.0006	0.0008	0.0009	0.75	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6



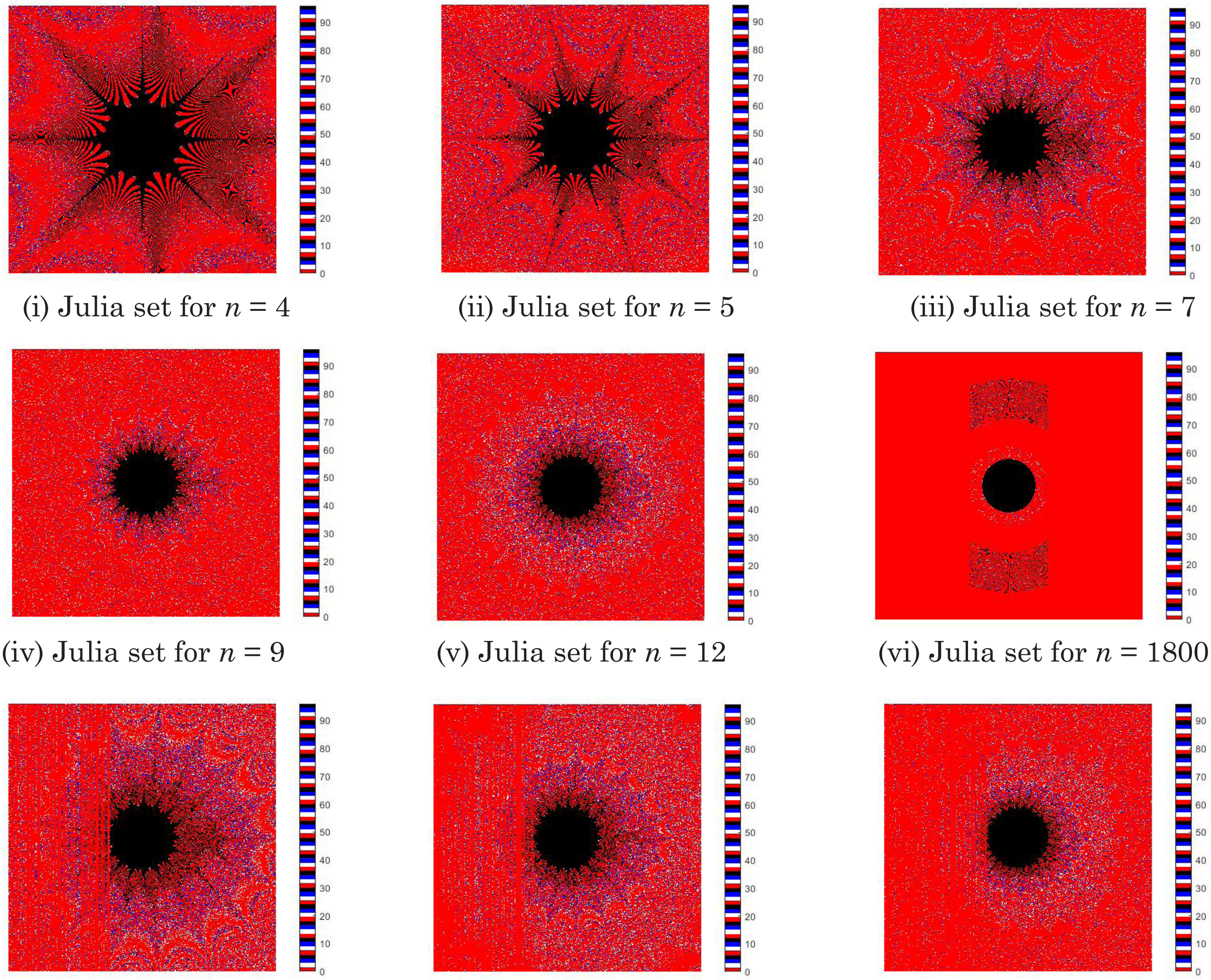
(i) (ii) (iii)

Figure 7: Julia sets for $n = 3$ via four-step iterative method

For values of the parameter as given in Table 8, we get amazing fractal objects, which are visible in Figure 8 ((i) to (ix)). We notice that all Julia sets have bunches of lashes and one of them have symmetry about x-axis. For integer and non-integer values of n, the amount of black colour in the Julia set increases from the center of the bunches, and the red colour decreases.

Table 8: Effect of change in the value of n on Julia sets

	n	a	b	c	d	μ	ν	ξ	η	s	$\lvert\alpha_1\rvert$	$\lvert\alpha_2\rvert$	$\lvert\alpha_3\rvert$	$\lvert\alpha_4\rvert$	$\lvert\omega_1\rvert$	$\lvert\omega_2\rvert$	$\lvert\omega_3\rvert$	$\lvert\omega_4\rvert$
(i)	4	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(ii)	5	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(iii)	7	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(iv)	9	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(v)	12	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(vi)	1800	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(vii)	8.41	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
($viii$)	10.15	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6
(ix)	13.65	$2i$	$31i$	$91i$	$3i$	0.000004	0.000006	0.000002	0.000007	0.9	0.04	0.07	0.08	0.03	0.6	0.8	0.7	0.6



(i) Julia set for $n = 4$ (ii) Julia set for $n = 5$ (iii) Julia set for $n = 7$

(iv) Julia set for $n = 9$ (v) Julia set for $n = 12$ (vi) Julia set for $n = 1800$

(vii) Julia set for $n = 8.31$ (viii) Julia set for $n = 10.15$ (ix) Julia set for $n = 13.65$

Figure 8: Julia sets for integer and non-integer values of n via four-step iterative method

5. Conclusion

Escape criteria is proved by considering a new complex function $p(z) = a\exp[d\sin(z^n)] - bz + c$, where $n, |d| \geq 2$ and $a, b, c, d \in \mathbb{C}$, using Jungck-four step iterative method with s-convexity as well as four step iterative method. These results are implemented in Algorithm 1 and 2 to visualize the Julia sets. We discussed and analyzed the behavior of variants of the Julia sets for different parameter values after obtaining fascinating non-classical variants of the Julia fractals using MATLAB software. We also observed that Julia sets for considered a new complex function had n bunches of lashes and one of them have symmetry about x-axis. The size of lashes gradually decreases from the center of the bunch and the angle between every two bunches is $\frac{K\pi}{n}$, where K represented the positions of attractors from the initial attractor and same argument for Julia set with an extra characteristic that image of Julia sets contains n type of Julia set at center for every n. Also, the size of fractals explored using the Jungck-four step iterative method with s-convexity as well as four step iterative method depends on the parameter b, μ, ν, ξ, η and s. As n increases the area occupied by the fractals decrees. We hope that these findings are useful to study different types of fractals which were mentioned initially. The results of this paper can also be used in cloth industry for designing and printing purposes.

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