Results in Nonlinear Analysis 7 (2023) No. 2, 43–52
https://doi.org/10.31838/rna/2023.07.02.004
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Interplay of Quasi covered ideals and quasi bases in semigroup theory

Fahad Sikander[a], Shahnawaz Ali[b,*], M. Y. Abbasi[b], S. A. Khan[b]

[a]Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Jeddah Branch-K.S.A; [b]Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.

Abstract

This research article introduces and investigates the concepts of quasi covered ideals and quasi bases within the context of semigroup theory a fundamental field of study in algebra. Quasi covered ideals represent a novel subset of semigroups, offering a versatile perspective that extends beyond conventional ideals, enabling a more flexible analysis of semigroup structures. In this paper, we delve into the properties and attributes of quasi covered ideals, providing a comprehensive exploration of their characteristics. Additionally, we establish intricate relationship between covered ideals, the greatest ideal, quasi covered ideals, and quasi bases, shedding light on the interconnections among these fundamental elements within the realm of semigroup theory.

Keywords: Quasi covered ideal; Greatest quasi covered ideal; Maximal ideal; Quasi base; Semigroup.

Mathematics Subject Classification (2010): 18B40; 20M12.

1. Introduction and Preliminaries

Semigroup theory provides a fertile ground for exploring algebraic structures with diverse applications in various fields of mathematics. Fabrici [1, 2], introduced the notion of covered ideal (C-ideal) for dealing ideals with the complement of a set. Due to relationship between complement of a set and ideals, this idea has received much attention in the field of algebraic structures such as semigroups [3, 4, 5], ordered semigroups [6, 7, 8], ordered semihypergroups [9], Γ-semihypergroups [10], ternary semigroups [11], and so on.

Email addresses: f.sikander@seu.edu.sa (Fahad Sikander); shahnawazali57@gmail.com (Shahnawaz Ali)*; mabbasi@jmi.ac.in (M. Y. Abbasi); khansabahat361@gmail.com (S. A. Khan)

Received January 17, 2024; Accepted February 22, 2024; Online March 12, 2024

We are familiar that one-sided ideals are extension of ideals, quasi ideals are extension of left (right) ideals and bi-ideals are extension of quasi ideals. In [12, 13], Steinfeld developed the concept of quasi ideals in semigroups and then in rings. In [14], Good and Hughes have developed the concept of bi-ideals in semigroups and then Lajos introduced bi-ideals in associative rings [15, 16].

Motivated by Fabrici, while studying and analyzing the work related to covered left (right, two-sided) ideals. We got an idea to generalize this concept to quasi covered ideal (QC-ideal) and try to convert results into QC-ideal. In this paper, we introduce QC-ideal, greatest QC-ideal and quasi base in a semigroup. We discuss their properties and interconnection with each other. Also, we introduce the concept of quasi bases in semigroups with some of their properties. Furthermore, we discuss the relationship of covered ideal and greatest ideal with QC-ideal.

Definition 1.1: [1] *Let $\mathcal{C}$ is proper left ideal of $\mathcal{T}$. Then $\mathcal{C}$ is said to be covered left ideal (shortly, CL-ideal) of $\mathcal{T}$, if $\mathcal{C} \subseteq \mathcal{T}(\mathcal{T} - \mathcal{C})$. Accordingly, $\mathcal{C}$ is said to be covered right ideal (CR-ideal), if $\mathcal{C} \subseteq (\mathcal{T} - \mathcal{C})\mathcal{T}$.*

Definition 1.2: [2] *Let $\mathcal{C}$ is proper two-sided ideal of $\mathcal{T}$. Then $\mathcal{C}$ is said to be covered ideal (shortly, C-ideal), if $\mathcal{C} \subseteq \mathcal{T}(\mathcal{T} - \mathcal{C})\mathcal{T}$.*

Remark: [2] The L-class containing α_1 is defined by, $(\alpha_1)_T$ = $\{\alpha_2 \in \mathcal{T} : (\alpha_1)_T = \alpha_1 \cup \mathcal{T}\alpha_1 \cup \alpha_1\mathcal{T} \cup \mathcal{T}\alpha_1\mathcal{T} = \alpha_2 \cup \mathcal{T}\alpha_2 \cup \alpha_2\mathcal{T} \cup \mathcal{T}\alpha_2\mathcal{T} = (\alpha_2)_T\}$. The L-class $\mathcal{C}^{\alpha_1}$ is maximal, if $(\alpha_1)_T$ isn't a proper subset of any principal ideal of $\mathcal{T}$. The L-class $\mathcal{C}^{\alpha_1}$ is maximal if and only if its complement is a maximal ideal of $\mathcal{T}$. Throughout in this paper, we shall denote $\mathcal{T}$ as a semigroup.

2. Quasi Covered Ideals in Semigroups

In this portion, we describe QC-ideals in semigroup with some examples and explore some of their properties.

Definition 2.1: *An ideal $\mathcal{C}$ is said to be quasi covered ideal (shortly, QC-ideal) of $\mathcal{T}$. If*

$$\mathcal{C} \subseteq (\mathcal{T} - \mathcal{C})\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C}) \cup \mathcal{T}(\mathcal{T} - \mathcal{C})\mathcal{T}$$

Example 1: *Let us Consider $\mathcal{T} = \{1,2,3,4,5...\}$. Define the binary operation $'*'$ by $\alpha_1 * \alpha_2 = \alpha_1 + \alpha_2$, $\forall \alpha_1, \alpha_2 \in \mathcal{T}$. Then $\mathcal{T}, *$ is a semigroup and $\mathcal{C}_p = \{p, p+1, p+2, ...\}$, $\forall p \in \mathcal{T}$ is an QC-ideal of $\mathcal{T}$.*

Example 2: *Let $\mathcal{T} = \{0,1,2,3,4,5 - - - n, - - -\}$ be a semigroup define the binary operation $'*'$ by $\alpha * \beta = min\{\alpha, \beta\}$ and let $\mathcal{C} = \{0,1,2,3, - - - \gamma\}$. Then $\mathcal{C}$ is an QC-ideal of $\mathcal{T}$.*

Example 3: *Consider $\mathcal{T} = \{0,1,2,3\}$ is a semigroup with the binary operation $'\circ'$:*

$\circ$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	1	0	3

Then $\mathcal{C} = \{0,1\}$ is an QC-ideal of $\mathcal{T}$.

Theorem 2.1: *Every C-ideal is an QC-ideal of $\mathcal{T}$. Contrary need not be true.*

Proof. Let $\mathcal{C}$ be an C-ideal of $\mathcal{T}$. Then, we have $\mathcal{C} \subseteq \mathcal{T}(\mathcal{T} - \mathcal{C})$ and $\mathcal{C} \subseteq (\mathcal{T} - \mathcal{C})\mathcal{T}$. This implies $\mathcal{C} \subseteq \mathcal{T}(\mathcal{T} - \mathcal{C}) \cup (\mathcal{T} - \mathcal{C})\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C})\mathcal{T}$. Hence, $\mathcal{C}$ is an QC-ideal of $\mathcal{T}$. The example below demonstrates that the converse need not be correct.

Example 4: *Consider $\mathcal{T} = \{0,1,2,3\}$ is a semigroup with the binary operation '$\circ$':Â*

$\circ$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	1	1

Then $(\mathcal{T},\circ)$ is a semigroup and $\mathcal{C} = \{0,1\}$ is an ideal of $\mathcal{T}$. Now $\mathcal{T}-\mathcal{C} = \{2,3\}$, $\mathcal{T}\circ(\mathcal{T}-\mathcal{C}) = \{0,1\}$, $(\mathcal{T}-\mathcal{C})\circ\mathcal{T} = \{0,1\}$ and $\mathcal{T}\circ(\mathcal{T}-\mathcal{C})\circ\mathcal{T} = \{0\}$. Then, $\mathcal{T}\circ(\mathcal{T}-\mathcal{C}) \cup (\mathcal{T}-\mathcal{C})\circ\mathcal{T} \cup \mathcal{T}\circ(\mathcal{T}-\mathcal{C})\circ\mathcal{T} = \{0,1\}$. Therefore, $\mathcal{C}$ is an QC-ideal of $\mathcal{T}$. But $\mathcal{T}(\mathcal{T}-\mathcal{C})\mathcal{T} = \{0\}$. It implies that $\mathcal{C} \nsubseteq \mathcal{T}(\mathcal{T}-\mathcal{C})\mathcal{T}$. Hence, $\mathcal{C}$ is not an C-ideal of $\mathcal{T}$.

Theorem 2.2: *If $\mathcal{C}_1$, $\mathcal{C}_2$ are two ideals of $\mathcal{T}$ such that $\mathcal{C}_1$ is an QC-ideal of $\mathcal{T}$ and $(\mathcal{C}_1 \cap \mathcal{C}_2) \neq \phi$. Then $(\mathcal{C}_1 \cap \mathcal{C}_2)$ is an QC-ideal of $\mathcal{T}$.*

Proof. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two ideals of $\mathcal{T}$. Since $\mathcal{C}_1$ is an QC-ideal of $\mathcal{T}$. Therefore

$$\begin{aligned}(\mathcal{C}_1 \cap \mathcal{C}_2) &\subseteq \mathcal{C}_1 \\ &\subseteq (\mathcal{T}-\mathcal{C}_1)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}_1) \cup \mathcal{T}(\mathcal{T}-\mathcal{C}_1)\mathcal{T} \\ &\subseteq \mathcal{T}(\mathcal{T}-(\mathcal{C}_1 \cap \mathcal{C}_2)) \cup (\mathcal{T}-(\mathcal{C}_1 \cap \mathcal{C}_2))\mathcal{T} \cup \mathcal{T}(\mathcal{T}-(\mathcal{C}_1 \cap \mathcal{C}_2)\mathcal{T}.\end{aligned}$$

Hence, $(\mathcal{C}_1 \cap \mathcal{C}_2)$ is an QC-ideal of $\mathcal{T}$.

Theorem 2.3: *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two QC-ideals of $\mathcal{T}$. Then $(\mathcal{C}_1 \cap \mathcal{C}_2)$ is an QC-ideal of $\mathcal{T}$.*

Proof. The proof is similar with the theorem's 2.2 proof .

Corollary 2.4: *If $\{\mathcal{C}_\lambda : \lambda \in \mathcal{N}\}$ is the family of QC-ideals. Then $\cap_{\lambda\in\mathcal{N}}\mathcal{C}_\lambda$ is an QC-ideal of $\mathcal{T}$.*

Corollary 2.5: *Suppose that $\mathcal{C}_1$ is an QC-ideal of $\mathcal{T}$ and $\mathcal{T}_1$ is sub-semigroup of $\mathcal{T}$. Then $(\mathcal{C}_1 \cap \mathcal{T}_1)$ is an QC-ideal of $\mathcal{T}$.*

Theorem 2.6: *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are CL-ideal and CR-ideal of $\mathcal{T}$. Then their intersection is an QC-ideal of $\mathcal{T}$.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CL-ideal and CR-ideal of $\mathcal{T}$. Then, we have $\mathcal{C}_1 \subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}_1)$, $\mathcal{C}_2 \subseteq (\mathcal{T}-\mathcal{C}_2)\mathcal{T}$. Thus, we have

$$\begin{aligned}(\mathcal{C}_1 \cap \mathcal{C}_2) &\subseteq \mathcal{C}_1 \subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}_1) \\ &\subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}_1) \cup \mathcal{T}(\mathcal{T}-\mathcal{C}_1)\mathcal{T}.\end{aligned}$$

Also, we have

$$\begin{aligned}(\mathcal{C}_1 \cap \mathcal{C}_2) &\subseteq \mathcal{C}_2 \\ &\subseteq (\mathcal{T}-\mathcal{C}_2)\mathcal{T} \\ &\subseteq (\mathcal{T}-\mathcal{C}_2)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}_2)\mathcal{T}.\end{aligned}$$

Therefore

$$\begin{aligned}(\mathcal{C}_1 \cap \mathcal{C}_2) &\subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}_1) \cup \mathcal{T}(\mathcal{T}-\mathcal{C}_1)\mathcal{T} \cup (\mathcal{T}-\mathcal{C}_2)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}_2)\mathcal{T} \\ &\subseteq \mathcal{T}(\mathcal{T}-(\mathcal{C}_1 \cap \mathcal{C}_2)) \cup \mathcal{T}(\mathcal{T}-(\mathcal{C}_1 \cap \mathcal{C}_2))\mathcal{T} \\ &\cup (\mathcal{T}-(\mathcal{C}_1 \cap \mathcal{C}_2))\mathcal{T} \cup \mathcal{T}(\mathcal{T}-(\mathcal{C}_1 \cap \mathcal{C}_2))\mathcal{T}.\end{aligned}$$

Hence, $(\mathcal{C}_1 \cap \mathcal{C}_2)$ is an QC-ideal of $\mathcal{T}$.

Theorem 2.7: *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two different proper ideals of $\mathcal{T}$ s.t. $(\mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{T}$. Then neither $\mathcal{C}_1$ nor $\mathcal{C}_2$ is an QC-ideal of $\mathcal{T}$.*

Proof. Suppose that $\mathcal{C}_1 \neq \mathcal{C}_2$ s.t. $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{T}$, thus $\mathcal{T} - \mathcal{C}_1 \subset \mathcal{C}_2$, $\mathcal{T} - \mathcal{C}_2 \subset \mathcal{C}_1$. If possible one of them, say $\mathcal{C}_1$ is an QC-ideal of $\mathcal{T}$. Then $\mathcal{C}_1 \subseteq (\mathcal{T} - \mathcal{C}_1)\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_1) \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_1)\mathcal{T}$, which implies $\mathcal{C}_1 \subset \mathcal{T}\mathcal{C}_2 \cup \mathcal{C}_2\mathcal{T} \cup \mathcal{T}\mathcal{C}_2\mathcal{T} \subseteq \mathcal{C}_2$. i.e. $\mathcal{C}_1 \subset \mathcal{C}_2$, which is a contradiction. Similarly, if $\mathcal{C}_2$ is an QC-ideal of $\mathcal{T}$, then we can show that $\mathcal{C}_2 \subset \mathcal{C}_1$, which is again contradiction. Hence, neither $\mathcal{C}_1$ nor $\mathcal{C}_2$ is an QC-ideal of $\mathcal{T}$.

As a result of Theorem 2.7, the following corollary derives.

Corollary 2.8: *If there is more than one maximal ideals in a semigroup $\mathcal{T}$. Then, none of them is an QC-ideal of $\mathcal{T}$.*

Theorem 2.9: *Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two QC-ideals of $\mathcal{T}$ and $(\mathcal{C}_1 \cap \mathcal{C}_2) \neq \phi$. Then $(\mathcal{C}_1 \cup \mathcal{C}_2)$ is an QC-ideal of $\mathcal{T}$.*

Proof. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two QC-ideals of $\mathcal{T}$. To prove $\mathcal{C}_1 \cup \mathcal{C}_2$ is an QC-ideal of $\mathcal{T}$. Let $x \in \mathcal{C}_1$, $\mathcal{C}_1 \subseteq (\mathcal{T} - \mathcal{C}_1)\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_1) \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_1)\mathcal{T}$. It implies that there exists $a \in (\mathcal{T} - \mathcal{C}_1)$ s.t. $x \in (\mathcal{T}a \cup a\mathcal{T} \cup \mathcal{T}a\mathcal{T})$. Thus, we have the following two possibilities:

(i) If $a \in \mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2)$, then $x \in \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2)) \cup (\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T} \cup \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T}$.

(ii) If $a \in (\mathcal{T} - \mathcal{C}_1) \cap \mathcal{C}_2$, then $a \in \mathcal{C}_2 \subset (\mathcal{T} - \mathcal{C}_2)\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_2) \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_2)\mathcal{T}$ so, $\exists b \in (\mathcal{T} - \mathcal{C}_2)$ s.t. $a \in (\mathcal{T}b \cup b\mathcal{T} \cup \mathcal{T}b\mathcal{T})$. This implies $a \in \mathcal{T}b$ or $a \in b\mathcal{T}$ or $a \in \mathcal{T}b\mathcal{T}$. Now, the element $b \notin \mathcal{C}_1$, otherwise $a \in (\mathcal{T}b \cup b\mathcal{T} \cup \mathcal{T}b\mathcal{T}) \subseteq \mathcal{T}\mathcal{C}_1 \cup \mathcal{C}_1\mathcal{T} \cup \mathcal{T}\mathcal{C}_1\mathcal{T} \subset \mathcal{C}_1$, since $\mathcal{C}_1$ is an ideal of $\mathcal{T}$. This implies $a \in \mathcal{C}_1$, which is contradiction, so $b \in (\mathcal{T} - \mathcal{C}_1)$. Therefore, $b \in (\mathcal{T} - \mathcal{C}_1)$ and $b \in (\mathcal{T} - \mathcal{C}_2)$, so $b \in \mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2)$. Now, we have three cases:

Case (1): If $a \in \mathcal{T}b$, then $x \in (\mathcal{T}\mathcal{T}b \cup \mathcal{T}b\mathcal{T} \cup \mathcal{T}\mathcal{T}b\mathcal{T}) \subset (\mathcal{T}b \cup \mathcal{T}b\mathcal{T} \cup \mathcal{T}b\mathcal{T}) \subset (\mathcal{T}b \cup \mathcal{T}b\mathcal{T})$.

Case (2): If $a \in b\mathcal{T}$, then $x \in (\mathcal{T}b\mathcal{T} \cup b\mathcal{T}\mathcal{T} \cup \mathcal{T}b\mathcal{T}\mathcal{T}) \subset (\mathcal{T}b\mathcal{T} \cup b\mathcal{T} \cup \mathcal{T}b\mathcal{T}) \subset (\mathcal{T}b\mathcal{T} \cup b\mathcal{T})$.

Case (3): If $a \in \mathcal{T}b\mathcal{T}$, then $x \in (\mathcal{T}\mathcal{T}b\mathcal{T} \cup \mathcal{T}b\mathcal{T}\mathcal{T} \cup \mathcal{T}\mathcal{T}b\mathcal{T}\mathcal{T}) \subset (\mathcal{T}b\mathcal{T} \cup \mathcal{T}b\mathcal{T} \cup \mathcal{T}b\mathcal{T}) \subset \mathcal{T}b\mathcal{T}$.

In all the three cases, we have $x \in (\mathcal{T}b \cup b\mathcal{T} \cup \mathcal{T}b\mathcal{T}) \subseteq \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2)) \cup (\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T} \cup \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T}$. Thus, $\mathcal{C}_1 \subseteq \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2)) \cup (\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T} \cup \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T}$. In the same way we can prove that $\mathcal{C}_2 \subseteq \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2)) \cup (\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T} \cup \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T}$. Therefore $\mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2)) \cup (\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T} \cup \mathcal{T}(\mathcal{T} - (\mathcal{C}_1 \cup \mathcal{C}_2))\mathcal{T}$. Hence, $(\mathcal{C}_1 \cup \mathcal{C}_2)$ is an QC-ideal of $\mathcal{T}$.

Theorem 2.10: *If $\mathcal{T}$ is not simple semigroup with the condition that any two proper ideals of $\mathcal{T}$ having intersection is non-empty. Then, there exists at least one QC-ideal in $\mathcal{T}$.*

Proof. Let $\mathcal{C}$ be a proper ideal of $\mathcal{T}$ and consider an ideal $\mathcal{C}_1 = \mathcal{T}(\mathcal{T} - \mathcal{C}) \cup (\mathcal{T} - \mathcal{C})\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C})\mathcal{T}$ with the condition that $(\mathcal{C} \cap \mathcal{C}_1) \neq \phi$, then $(\mathcal{C} \cap \mathcal{C}_1)$ is an ideal of $\mathcal{T}$. Consider $(\mathcal{C} \cap \mathcal{C}_1) = \mathcal{C}_2$, thus $\mathcal{C}_2 \subset \mathcal{C}_1$ and $\mathcal{C}_2 \subset \mathcal{C}$. Therefore, $\mathcal{C}_2 \subset \mathcal{C}_1 = \mathcal{T}(\mathcal{T} - \mathcal{C}) \cup (\mathcal{T} - \mathcal{C})\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C})\mathcal{T}$ and $\mathcal{C}_2 \subset \mathcal{C}$, then $\mathcal{T} - \mathcal{C}_2 \supset \mathcal{T} - \mathcal{C}$. Thus, $\mathcal{C}_2 \subset \mathcal{T}(\mathcal{T} - \mathcal{C}_2) \cup (\mathcal{T} - \mathcal{C}_2)\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_2)\mathcal{T}$. Hence, $\mathcal{C}_2$ is an QC-ideal of $\mathcal{T}$.

Theorem 2.11: *Let us consider two ideals $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{T}$ s.t. $\mathcal{C}_1 \subseteq \mathcal{C}_2$. If $\mathcal{C}_2$ is an QC-ideal of $\mathcal{T}$, then $\mathcal{C}_1$ is also an QC-ideal of $\mathcal{T}$.*

Proof. Suppose $\mathcal{C}_2$ is an QC-ideal of $\mathcal{T}$ such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Then, we have $(\mathcal{T} - \mathcal{C}_1) \supseteq (\mathcal{T} - \mathcal{C}_2)$. It implies $\mathcal{C}_2 \subseteq \mathcal{T}(\mathcal{T} - \mathcal{C}_1) \cup (\mathcal{T} - \mathcal{C}_1)\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_1)\mathcal{T}$. Thus, we have $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{T}(\mathcal{T} - \mathcal{C}_1) \cup (\mathcal{T} - \mathcal{C}_1)\mathcal{T} \cup \mathcal{T}(\mathcal{T} - \mathcal{C}_1)\mathcal{T}$. Hence, $\mathcal{C}_1$ is an QC-ideal of $\mathcal{T}$.

Theorem 2.12: *Every proper ideal of $\mathcal{T}$ with identity* 1 *is an QC-ideal.*

Proof. Let $\mathcal{C} \subset \mathcal{T}$. Then, $1 \notin \mathcal{C}$. If possible $1 \in \mathcal{C}$. Then, $\mathcal{T} = \mathcal{T}.1 \subseteq \mathcal{T}\mathcal{C} \subseteq \mathcal{C}$. i.e. $\mathcal{T} \quad \mathcal{C}$, it is a contradiction. Hence $1 \in \mathcal{T} - \mathcal{C}$, it follows that $\mathcal{T}(\mathcal{T} - \mathcal{C}) \cup \mathcal{T}(\mathcal{T} - \mathcal{C})\mathcal{T} \cup (\mathcal{T} - \mathcal{C})\mathcal{T} = \mathcal{T}$.

Hence, $\mathcal{T}(\mathcal{T} - \mathcal{C}) \cup \mathcal{T}(\mathcal{T} - \mathcal{C})\mathcal{T} \cup (\mathcal{T} - \mathcal{C})\mathcal{T} \supseteq \mathcal{C}$. It implies that $\mathcal{C}$ is an QC-ideal of $\mathcal{T}$.

3. The Greatest Ideal and QC-ideal

In the following section, we define the greatest ideal in a semigroup and provide the conditions for the greatest ideal to be an QC-ideal.

Definition 3.1: [2] *An ideal $\mathcal{C} \subset \mathcal{T}$ is said to be greatest ideal of $\mathcal{T}$, if $\mathcal{C}$ contains every proper ideals of $\mathcal{T}$. If such an ideal exists, it is indicated by $\mathcal{C}^*$.*

Example 5: *Consider $\mathcal{T} = \{0,1,2,3\}$ is a semigroup with the binary operation '$\circ$':*

$\circ$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	1	1

Then clearly $\mathcal{C}_1 = \{0,1,2\}$ and $\mathcal{C}_2 = \{0,1\}$ are two ideals of $\mathcal{T}$ s.t. $\mathcal{C}_2 \subset \mathcal{C}_1$. Therefore $\mathcal{C}_1$ is the greatest ideal of $\mathcal{T}$.

Theorem 3.1: *Let there be only a maximal ideal $\mathcal{C}$ of $\mathcal{T}$. Then the greatest ideal $\mathcal{C}$ is an QC-ideal of $\mathcal{T}$.*

Proof. It is simple to show because if $\mathcal{C}_1$ is proper ideal of $\mathcal{T}$, then $\mathcal{C}_1 \subseteq \mathcal{C}$. Therefore, $\mathcal{C} = \mathcal{C}^*$ i.e. $\mathcal{C}$ is greatest ideal of $\mathcal{T}$. By Theorem 2.7, $\mathcal{C}$ is an QC-ideal of $\mathcal{T}$.

Theorem 3.2: *If an ideal $\mathcal{C}^*$ is an QC-ideal of $\mathcal{T}$, then $\mathcal{T}^2 = \mathcal{T}^3$.*

Proof. Let $\mathcal{C}^*$ be an QC-ideal of $\mathcal{T}$. Then, $\mathcal{C}^* \subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T}$. Since $\mathcal{C}^*$ is also a maximal ideal of $\mathcal{T}$. Thus $(\mathcal{T}-\mathcal{C}^*) = I^a$ is the exact one maximal L-class in $\mathcal{T}$ by [2]. So, either $\mathcal{T}^2 \subset \mathcal{T}$ or $\mathcal{T}^2 = \mathcal{T}$. If $\mathcal{T}^2 = \mathcal{T}$, then $\mathcal{T}^3 = \mathcal{T}^2$. If $\mathcal{T}^2 \subset \mathcal{T}$ then either $\mathcal{T}^3 \subset \mathcal{T}^2$ or $\mathcal{T}^3 = \mathcal{T}^2$. If possible $\mathcal{T}^3 \subset \mathcal{T}^2$, then $\mathcal{C}^* \subset \mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T} \subset \mathcal{T}^2 \cup \mathcal{T}^2 \cup \mathcal{T}^3$ i.e. $\mathcal{C}^* \subset \mathcal{T}^2$. Consequently, at least two separate L-classes would have been included in $(\mathcal{T}-\mathcal{C}^*)$, these are $\mathcal{T}^2 - \mathcal{T}^3$ and $\mathcal{T}-\mathcal{T}^2$. This is contradictory, as $(\mathcal{T}-\mathcal{C}^*)$ contains a maximal of one L-class, hence we get $\mathcal{T}^2 = \mathcal{T}^3$.

Theorem 3.3: *Let a semigroup $\mathcal{T}$ satisfies one of the following criteria:*

(i) If $\mathcal{T}$ contains $\mathcal{C}^*$, which is an QC-ideal of $\mathcal{T}$.
(ii) If $\mathcal{T} = \mathcal{T}^2$, for any proper ideal $\mathcal{C}$ and for every principal ideal $(\alpha_1)_q \subset \mathcal{C}$, there is a principal proper ideal $(\alpha_2)_q$, whose generator $\alpha_2 \in (\mathcal{T}-\mathcal{C})$ and $(\alpha_1)_q \subsetneqq (\alpha_2)_q$. Then each and every proper ideal of $\mathcal{T}$ is an QC-ideal of $\mathcal{T}$.

Proof. Consider $\mathcal{C}$ is an proper ideal of $\mathcal{T}$. If (i) is true, then $\mathcal{C} \subset \mathcal{C}^*$. Given that $\mathcal{C}^*$ is an QC-ideal, then $\mathcal{C} \subset \mathcal{C}^* \subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T}$. It implies that $\mathcal{C} \subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}) \cup (\mathcal{T}-\mathcal{C})\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C})\mathcal{T}$. Since $\mathcal{C} \subset \mathcal{C}^*$, then $\mathcal{T}-\mathcal{C} \supset \mathcal{T}-\mathcal{C}^*$. Thus $\mathcal{C}$ is an QC-ideal of $\mathcal{T}$. Let (ii) be satisfied. If $\alpha_3 \in \mathcal{C}$, thus $(\alpha_3)_q \subset \mathcal{C}$, then there exists $\alpha_2 \in (\mathcal{T}-\mathcal{C})$ and $(\alpha_3)_q \subset (\alpha_2)_q$, it is obvious that $(\alpha_3)_q \neq (\alpha_2)_q$. As $\mathcal{T} = \mathcal{T}^2$ implies $\mathcal{T} = \mathcal{T}^3$, and $\alpha_2 \in \mathcal{T}$, then $\alpha_2 \in \mathcal{T}^3$. Thus, we have $\alpha_2 \in \mathcal{T}^3 \cup \mathcal{T}^3 \cup \mathcal{T}^3$, then $\alpha_2 \in \mathcal{T}^2 \cup \mathcal{T}^2 \cup \mathcal{T}^3$. Then $\alpha_2 \in \mathcal{T}\mathcal{T} \cup \mathcal{T}\mathcal{T} \cup \mathcal{T}\mathcal{T}\mathcal{T}$ which implies $\alpha_2 \in \mathcal{T}\alpha_4 \cup \alpha_4\mathcal{T} \cup \mathcal{T}\alpha_4\mathcal{T}$, for some $\alpha_4 \in \mathcal{T}$. Thus $\alpha_2 \in \mathcal{T}\alpha_4$ or $\alpha_2 \in \alpha_4\mathcal{T}$ or $\alpha_2 \in \mathcal{T}\alpha_4\mathcal{T}$, for some $\alpha_4 \in \mathcal{T}$. Let $\alpha_2 \in \mathcal{T}\alpha_4$, for some $\alpha_4 \in \mathcal{T}$, we show that $\alpha_2 \in \mathcal{C}$, if $\alpha_2 \in \mathcal{C}$, then $\mathcal{T}\alpha_4 \subset \mathcal{C}$ and $\alpha_2 \in \mathcal{T}\alpha_4 \subset \mathcal{C}$. Hence $\alpha_2 \in \mathcal{C}$, which is a contradiction as $\alpha_2 \in (\mathcal{T}-\mathcal{C})$. Therefore, for arbitrary $\alpha_3 \in \mathcal{C}$ there exists $\alpha_4 \in (\mathcal{T}-\mathcal{C})$ s.t. $\alpha_3 \in \mathcal{T}\alpha_4$. Thus, $\alpha_3 \in \mathcal{T}(\mathcal{T}-\mathcal{C})$. Similarly, $\alpha_3 \in \alpha_4\mathcal{T}$ implies $\alpha_3 \in (\mathcal{T}-\mathcal{C})\mathcal{T}$ and $\alpha_3 \in \mathcal{T}\alpha_4\mathcal{T}$ implies $\alpha_3 \in \mathcal{T}(\mathcal{T}-\mathcal{C})\mathcal{T}$. Thus, we get $\alpha_3 \in \mathcal{T}(\mathcal{T}-\mathcal{C}) \cup (\mathcal{T}-\mathcal{C})\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C})\mathcal{T}$. It implies that $\mathcal{C} \subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}) \cup (\mathcal{T}-\mathcal{C})\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C})\mathcal{T}$. Hence, $\mathcal{C}$ is an $\mathcal{QC}$-ideal of $\mathcal{T}$.

Theorem 3.4: *Assuming that $\mathcal{C}^*$ is the greatest ideal of $\mathcal{T}$ such that $\mathcal{T} = \mathcal{T}^2$, then $\mathcal{C}^*$ is an QC-ideal.*

Proof. Suppose that $\mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T}$ is an ideal of $\mathcal{T}$ and $\mathcal{C}^*$ be the greatest ideal of $\mathcal{T}$, then either $\mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T} = \mathcal{T}$ or $\mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T} \subseteq \mathcal{C}^*$. Therefore, three cases are obtained.

Case (a): If $\mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T} = \mathcal{T}$, then $\mathcal{C}^* \subseteq \mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T}$. It implies $\mathcal{C}^*$ is an QC-ideal.
Case (b): If $\mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T} = \mathcal{C}^*$, then $\mathcal{C}^*$ is an QC-ideal of $\mathcal{T}$.
Case (c): If $\mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T} \subseteq \mathcal{C}^*$, then as given $\mathcal{T}^2 = \mathcal{T}$ implies $\mathcal{T}^3 = \mathcal{T}$, then $\mathcal{T}^3 = \mathcal{T} \cup \mathcal{T}\mathcal{C}^*\mathcal{T} = \mathcal{T}(\mathcal{T}-\mathcal{C}^*) \cup (\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}(\mathcal{T}-\mathcal{C}^*)\mathcal{T} \cup \mathcal{T}\mathcal{C}^*\mathcal{T} \subseteq \mathcal{C}^* \cup \mathcal{C}^* = \mathcal{C}^* \subset \mathcal{T}$. It implies $\mathcal{T}^3 \subset \mathcal{T}$. This is a contradiction. Hence by case (b) and case(c), $\mathcal{C}^*$ is an QC-ideal.

4. Quasi Base and the Greatest QC-ideal

In the section, we define quasi base and greatest QC-ideal of a semigroup with the support of some examples. Also we have proved some results based on quasi base of a semigroup and given example of a semigroup which do not have any quasi base.

Definition 4.1: *If an QC-ideal $\mathcal{C}$ contains every QC-ideal of $\mathcal{T}$. Then $\mathcal{C}$ is called the greatest QC-ideal of $\mathcal{T}$. If it exist, it is denoted by the symbol $\mathcal{C}^g$.*

Remark: Consider a semigroup $\mathcal{T}$ that contains maximal ideals. If the maximal ideals of $\mathcal{T}$ are $\{\mathcal{C}_\lambda, \lambda \in \mathcal{N}\}$. Then $\dot{\mathcal{C}} = \cap_{\lambda \in \mathcal{N}} \mathcal{C}_\lambda \neq \phi$. If $\mathcal{C}^g$ is contained in $\mathcal{T}$. So it is required that $\mathcal{C}^g \subset \dot{\mathcal{C}}$. But, if there is even one $\mathcal{C}_\lambda$ s.t $\mathcal{C}^g \nsubseteq \mathcal{C}_\lambda$. Then by Theorem 2.11, $\mathcal{C}^g$ is not an QC-ideal of $\mathcal{T}$. We can now demonstrate that even if $\mathcal{T}$ contains maximal ideals, this does not imply that $\mathcal{T}$ also contains $\mathcal{C}^g$.

Example 6: *Consider $\mathcal{T} = \{0,1,2,3\}$ is a semigroup with the binary operation '$\circ$':*

$\circ$	0	1	2	3
	0	0	0	0
	0	0	0	0
	0	0	0	0
	0	0	1	1

Here, $\mathcal{T}$ contains two maximal ideals $\mathcal{C}_1 = \{0,1,2\}$ and $\mathcal{C}_2 = \{0,1,3\}$. Although, the greatest QC-ideal is not contained in $\mathcal{T}$.

Definition 4.2 $(\phi \neq \mathcal{C} \subset \mathcal{T})$ *is called Quasi base (Shortly, $\mathcal{Q}$-base) of $\mathcal{T}$. If*

(i) $\mathcal{C} \cup (\mathcal{T}\mathcal{C} \cap \mathcal{C}\mathcal{T}) \cup \mathcal{T}\mathcal{C}\mathcal{T} = \mathcal{T}$
(ii) There does not exist any proper subset $\mathcal{D} \subset \mathcal{C}$ s.t. $(\mathcal{D})_q = \mathcal{T} = (\mathcal{C})_q$. It is denoted by $(\mathcal{C})_q$

Example 7: *Consider $\mathcal{T} = \{0,1,2,3\}$ is a semigroup with the binary operation '$\circ$':*

$\circ$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	1	0	3

Let $\mathcal{C}=\{2,3\}$ be a subset of $\mathcal{T}$. Then there does not exist any proper subset $\mathcal{D}\subset\mathcal{C}$ s.t. $(\mathcal{D})_q=\mathcal{T}$. Hence, $\mathcal{C}$ is a quasi base of $\mathcal{T}$.

Remark: QB-semigroup is a semigroup of $\mathcal{T}$ that contains at least one quasi base. Quasi base element is an element α_1 in a semigroup $\mathcal{T}$, such that $\alpha_1\cup(\mathcal{T}\alpha_1\cap\alpha_1\mathcal{T})\cup\mathcal{T}\alpha_1\mathcal{T}=\mathcal{T}$. Additionally, we define the principal quasi ideal, which is generated by an element α_1 and it is denoted by $(\alpha_1)_q$. i.e. $(\alpha_1)_q=\{\alpha_1\}\cup(\mathcal{T}\alpha_1\cap\alpha_1\mathcal{T})\cup\mathcal{T}\alpha_1\mathcal{T}$ and we define $\mathcal{Q}$-class containing α_1 by $\mathcal{Q}^{\alpha_1}=\{\beta\in\mathcal{T}:(\alpha_1)_q=\alpha_1\cup(\mathcal{T}\alpha_1\cap\alpha_1\mathcal{T})\cup\mathcal{T}\alpha_1\mathcal{T}=\beta\cup(\mathcal{T}\beta\cap\beta\mathcal{T})\cup\mathcal{T}\beta\mathcal{T}=(\beta)_q\}$. An $\mathcal{Q}$-class $\mathcal{Q}^{\alpha_1}$ is maximal quasi, if there does not exist any principal quasi ideal of $\mathcal{T}$ which properly contains $(\alpha_1)_q$.

Corollary 4.1: *A semigroup can be without any quasi base.*

Example 8: *Consider $\mathcal{T}=\{0,1,2,3\}$ is a semigroup with the binary operation '$\circ$':*

$\circ$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	3
3	0	1	2	3

Let $\mathcal{C}=\{2,3\}$ be a subset of $\mathcal{T}$. Then, $\mathcal{T}\circ\mathcal{C}=\{0,1,2,3\},\mathcal{C}\circ\mathcal{T}=\{0,1,2,3\},\mathcal{T}\circ\mathcal{C}\circ\mathcal{T}=\{0,1,2,3\}$ Now, $(\mathcal{C})_q=\mathcal{C}\cup(\mathcal{T}\circ\mathcal{C}\cap\mathcal{C}\circ\mathcal{T})\cup\mathcal{T}\circ\mathcal{C}\circ\mathcal{T},(\mathcal{C})_q=\{0,1,2,3\}=\mathcal{T}$. We observe that a proper subset exists i.e. $\mathcal{D}\subset\mathcal{C}$ s.t. $(\mathcal{D})_q=\mathcal{T}$. Hence, $\mathcal{C}$ is not a quasi base of $\mathcal{T}$.

Example 9: *Let $\mathcal{N}$ be a collection of all natural numbers with the binary operation defined by a_1a_2 =min$\{a_1,a_2\}$, $\forall a_1,a_2\in\mathcal{N}$. Then we define $(a_1)_q=\{1,2,3,4,5,6...a_1\}$ and $\mathcal{Q}^{a_1}=a_1$, for every $a_1\in\mathcal{N}$, therefore, $(1)_q\subset(2)_q\subset(3)_q\subset(4)_q\subset...(a)_q\subset....$ Hence $\mathcal{N}$ has no quasi base.*

Lemma 4.2: *Consider a quasi base $\mathcal{Q}$ of $\mathcal{T}$, and $\alpha_1,\alpha_2\in\mathcal{Q}$. If $\alpha_1\in(\mathcal{T}\alpha_2\cap\alpha_2\mathcal{T})\cup\mathcal{T}\alpha_2\mathcal{T}$, then $\alpha_1=\alpha_2$.*

Proof. Let $\alpha_1\in(\mathcal{T}\alpha_2\cap\alpha_2\mathcal{T})\cup\mathcal{T}\alpha_2\mathcal{T}$ and if possible $\alpha_1\neq\alpha_2$. Consider $\mathcal{Q}_1=\mathcal{Q}-\alpha_1$, then $\alpha_2\in\mathcal{Q}_1$ and given $\alpha_1\in(\mathcal{T}\alpha_2\cap\alpha_2\mathcal{T})\cup\mathcal{T}\alpha_2\mathcal{T}$ implies $(\alpha_1)_q\subset(\mathcal{T}\alpha_2\cap\alpha_2\mathcal{T})\cup\mathcal{T}\alpha_2\mathcal{T}\subset(\mathcal{Q}_1)_q$, it follows that $\mathcal{T}=\mathcal{Q}\subset(\mathcal{Q}_1)_q$. But this is contradiction because $\mathcal{Q}$ is quasi base. Hence $\alpha_1=\alpha_2$.

Remark: Now, we define a relation which is called quasi ordering relation in $\mathcal{T}$, namely $\alpha_1\leq\alpha_2$ means $\alpha_1\cup(\mathcal{T}\alpha_1\cap\alpha_1\mathcal{T})\cup\mathcal{T}\alpha_1\mathcal{T}\subset\alpha_2\cup(\mathcal{T}\alpha_2\cap\alpha_2\mathcal{T})\cup\mathcal{T}\alpha_2\mathcal{T}$, we write $(\alpha_1)_q\subset(\alpha_2)_q$.

Lemma 4.3: *Let $\mathcal{C}$ be a quasi base of a semigroup $\mathcal{T}$. If $\alpha_1,\alpha_2\in\mathcal{C}$, $\alpha_1\neq\alpha_2$, then neither $\alpha_1\leq\alpha_2$, nor $\alpha_2\leq\alpha_1$.*

Proof. Consider that $\alpha_1\leq\alpha_2$, then $(\alpha_1)_q\subset(\alpha_2)_q$. It implies $\alpha_1\in(\mathcal{T}\alpha_2\cap\alpha_2\mathcal{T})\cup\mathcal{T}\alpha_2\mathcal{T}$. Lemma 4.2 implies that $\alpha_1=\alpha_2$, which is contradiction. Similarly if $\alpha_2\leq\alpha_1$, then we have a contradiction. Hence neither $\alpha_1\leq\alpha_2$ nor $\alpha_2\leq\alpha_1$.

Theorem 4.4: *($\phi\neq\mathcal{Q}\subset\mathcal{T}$) is quasi base of $\mathcal{T}$ if and only if $\mathcal{Q}$ satisfies the following:*

(i) For $\alpha\in\mathcal{T}$, there exists $\alpha_1\in\mathcal{Q}$ s.t $\alpha\leq\alpha_1$.
(ii) If $\alpha_1,\alpha_2\in\mathcal{Q}$ s.t. $\alpha_1\neq\alpha_2$, then neither $\alpha_1\leq\alpha_2$, nor $\alpha_2\leq\alpha_1$.

Proof. Let us consider (i) and (ii) holds for $\mathcal{Q}$, let $\alpha\in\mathcal{T}$, then it implies $\alpha\leq\alpha_1\in\mathcal{Q}$, i.e. $\alpha\in(\alpha_1)_q\subset(\mathcal{Q})_q$. Thus, it follows $\mathcal{T}\subset(\mathcal{Q})_q$ that would be $\mathcal{T}=(\mathcal{Q})_q$. That is yet left to prove $\mathcal{Q}$ is the smallest subset with the condition $\mathcal{T}=(\mathcal{Q})_q$. Let $\mathcal{Q}_1\subset\mathcal{Q}$ and $\mathcal{Q}_1\neq\mathcal{Q}$ s.t. $\mathcal{T}=(\mathcal{Q}_1)_q$, if $\alpha_1\in\mathcal{Q}-\mathcal{Q}_1$, there exists $\alpha_2\in\mathcal{Q}_1$ s.t. $\alpha_1\in(\mathcal{T}\alpha_2\cap\alpha_2 S)\cup\mathcal{T}\alpha_2\mathcal{T}$. Then we have, $(\alpha_1)_q\subset(\alpha_2)_q$, However, this is contradicts with *(ii)*. Hence $\mathcal{Q}$

is quasi base of $\mathcal{T}$. Conversely, let us suppose that $\mathcal{Q}$ is quasi base of $\mathcal{T}$. Then $\mathcal{T}=(\mathcal{Q})_q$. Thus if $\alpha\in\mathcal{T}$, then $\alpha\in(\mathcal{Q})_q$, there exists $\alpha_1\in\mathcal{Q}$ such that $\alpha\in(\alpha_1)_q$. It implies $\alpha\le\alpha_1$, therefore (i) is satisfied, and by Lemma 4.3 validity for (ii) is satisfied.

Theorem 4.5: *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two quasi bases of a semigroup $\mathcal{T}$. Then both quasi bases have the same cardinality.*

Proof. Let a mapping $\psi:\mathcal{C}_1\to\mathcal{C}_2$ is defined as if $\alpha\in\mathcal{C}_1$, then $\psi(\alpha)=\beta$, $\beta\in\mathcal{C}_2$ if and only if $\beta\in\mathcal{C}^{\alpha}$. We show that this mapping is defined for every $\alpha\in\mathcal{C}_1$. As $\mathcal{C}_2$ is a quasi base, there exists $\beta\in\mathcal{C}_2$ s.t. $\alpha\le\beta$, because $\mathcal{C}_1$ is a quasi base of $\mathcal{T}$. Also for the element $\beta\in\mathcal{C}_2$, there exists $\gamma\in\mathcal{C}_1$ such that $\beta\le\gamma$. We get $\alpha\le\beta\le\gamma$. It implies $\alpha\le\gamma$ and therefore $\alpha=\gamma$, this implies $(\alpha)_q=(\beta)_q$, so $\beta\in\mathcal{C}^{\alpha}$. We show that ψ is one-one and onto. Let $\alpha_1,\alpha_2\in\mathcal{C}_1$, such that $\psi(\alpha_1)=\psi(\alpha_2)$, then $(\alpha_1)_q=(\alpha_2)_q$ the condition (ii) of Theorem 4.4 implies that $\alpha_1=\alpha_2$. Now for onto, if $\beta\in\mathcal{C}_2$ then there exists $\alpha_1\in\mathcal{C}_1$ s.t $\beta\le\alpha_1$. For the same reason, an element $\alpha_1\in\mathcal{C}_1$ there exists some $\beta_1\in\mathcal{C}_2$ s.t. $\alpha_1\le\beta_1$, thus $\beta\le\alpha_1\le\beta_1,\beta\in\mathcal{C}_2$, therefore by condition (ii) of Theorem 4.4, $\beta=(\beta)_1$, so $(\beta)_q=(\beta_1)_q$ and $(\alpha)_q=(\beta)_q$ i.e. $\psi(\alpha_1)=\beta_1$ for $\alpha_1\in\mathcal{C}_1$. Therefore, ψ is onto. Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same cardinality.

Lemma 4.6: *Let $\mathcal{Q}$ be quasi base of $\mathcal{T}$ and any element α_1 of $\mathcal{Q}$. If $(\alpha_1)_q=(\alpha_2)_q$ for some $\alpha_2\in\mathcal{T}$ and $\alpha_2\ne\alpha_1$. Then any element α_2 belongs to quasi base of $\mathcal{T}$ distinct from $\mathcal{Q}$.*

Proof. Let $\mathcal{Q}_1=(\mathcal{Q}-\beta)\cup\{\alpha_2\}$. Without any uncertainty, $\mathcal{Q}\ne\mathcal{Q}_1$. To prove that $\mathcal{Q}_1$ is also quasi base of $\mathcal{T}$. It is sufficient to show that $\mathcal{Q}_1$ satisfies condition (i) of Theorem 4.4. Let γ is arbitrary element of $\mathcal{T}$. Then, because $\mathcal{Q}$ is quasi base of $\mathcal{T}$, if $\exists\ \alpha\in\mathcal{Q}$ such that $\gamma\le\alpha$. Now there are two possibilities: (i) $\alpha_1\ne\beta$ (ii) $\alpha_1=\beta$. If $\alpha_1\ne\beta$, then $\alpha_1\in\mathcal{Q}_1$. If $\alpha_1=\beta$, then $\alpha_1\notin\mathcal{Q}_1$, but $(\beta)_q=(\alpha_2)_q$, so if $\gamma\le\alpha$ then, $\{\gamma\}\cup(\mathcal{T}\gamma\cap\gamma\mathcal{T})\cup\mathcal{T}\gamma\mathcal{T}\subseteq\{\alpha_1\}\cup(\mathcal{T}\alpha_1\cap\alpha_1\mathcal{T})\cup\mathcal{T}\alpha_1\mathcal{T}=\{\alpha_2\}\cup(\mathcal{T}\alpha_2\cap\alpha_2\mathcal{T})\cup\mathcal{T}\alpha_2\mathcal{T}$, it follows that $\gamma\le\alpha_2$ and $\alpha_2\in\mathcal{Q}_1$. It means that $\mathcal{Q}_1$ satisfies condition (i) of Theorem 4.4. Now, let $\beta_1,\beta_2\in\mathcal{Q}_1$ be two distinct arbitrary elements. If both elements are distinct form α_2, then $\beta_1,\beta_2\in\mathcal{Q}$ and $\mathcal{Q}$ is a quasi base of $\mathcal{T}$. Thus neither $\beta_1\le\beta_2$, nor $\beta_2\le\beta_1$. But, for $\beta_1=\alpha_2$. If $\beta_1\le\beta_2$ thus $\beta\le\beta_2$, where $\beta\in\mathcal{Q}$, $\beta_2\in\mathcal{Q}$. But $\mathcal{Q}$ is quasi base of $\mathcal{T}$, Consequently, this is not feasible similarly, we can prove that the relationship $\beta_2\le\beta_1$ can not be satisfied. It implies that $\mathcal{Q}_1$ fulfils the condition (ii) of Theorem 4.4 therefore, $\mathcal{Q}_1$ is quasi base of $\mathcal{T}$ different from $\mathcal{Q}$. Hence α_2 is an element of quasi base $\mathcal{Q}_1$ of $\mathcal{T}$ which is different from $\mathcal{Q}$.

Theorem 4.7:*Let $\mathcal{D}$ be the union of all quasi bases of $\mathcal{T}$. If $\mathcal{C}=(\mathcal{T}-\mathcal{D})\ne\phi$. Then $\mathcal{C}$ is always left or right ideal but need not be an QC-ideal of $\mathcal{T}$.*

Proof. We must prove this if $\beta\in\mathcal{T},\alpha_1\in\mathcal{C}=\mathcal{T}-\mathcal{D}$. It implies $\beta\alpha_1\in\mathcal{C}$. For if we assume $\beta\alpha_1\notin\mathcal{C}$. Thus $\alpha_2=\beta\alpha_1\in\mathcal{D}$ and then $\alpha_2\in\mathcal{D}_1$ (at least one quasi base) of $\mathcal{T}$ and that is here $\alpha_2\in\mathcal{T}\alpha_1$, hence $\mathcal{T}\alpha_2\subseteq\mathcal{T}\alpha_1\Rightarrow\alpha_2\cup\mathcal{T}\alpha_2\subseteq\alpha_1\cup\mathcal{T}\alpha_1$, now we will show that $(\alpha_2)_q\ne(\alpha_1)_q$. If $(\alpha_2)_q=(\alpha_1)_q$, therefore $\alpha_2\in\mathcal{T}$, by Lemma 4.6 $\alpha_1\in\mathcal{T}$. But this is contradiction with the assumption $\alpha_1\in\mathcal{T}-\mathcal{D}$. It implies that $(\alpha_2)_q\le(\alpha_1)_q,(\alpha_2)_q\ne(\alpha_1)_q$ and since $\mathcal{D}_1$ is quasi base, then for an element α_1, $\exists\ \alpha_3\in\mathcal{D}_1$ s.t. $\alpha_1\le\alpha_3$, we know $\alpha_2\le\alpha_1\le\alpha_3$, therefore $\alpha_2\le\alpha_3$ but this is contradiction to Lemma 4.3, because $\alpha_2,\alpha_3\in\mathcal{D}_1$. Therefore $\beta\alpha_1\in\mathcal{T}-\mathcal{D}$, implies $\beta\alpha_1\in\mathcal{C}$. Hence $\mathcal{C}$ is left ideal of $\mathcal{T}$. With the help of a counter example, we can demonstrate that it need not be an QC-ideal.

Example 10: *Consider $\mathcal{T}=\{0,1,2,3\}$ is a semigroup with the binary operation '$\circ$':*

$\circ$	0	1	2	3
	0	1	0	0
	0	1	0	0
	0	1	2	2
	0	1	3	3

Let $\mathcal{C}_1 = \{1,2\}$, $\mathcal{C}_2 = \{1,3\}$, $\mathcal{D} = \mathcal{C}_1 \cup \mathcal{C}_2 = \{1,2,3\}$ and $\mathcal{C} \quad \mathcal{T} - \mathcal{D} = \{0\}$. s.t $\mathcal{T} \circ \mathcal{C} = \{0\} \subset \mathcal{C}, \mathcal{C} \circ \mathcal{T} = \{0,1\} \not\subseteq \mathcal{C}$. So, $\mathcal{C}$ is not an ideal of $\mathcal{T}$. Therefore $\mathcal{C}$ need not be an QC-ideal of $\mathcal{T}$.

Lemma 4.8: *If $\mathcal{T} \neq \mathcal{T}^2$ then, $(\alpha)_q = \alpha$, for any $\alpha \in \mathcal{T} - \mathcal{T}^2$.*

Proof. Let $\alpha \in \mathcal{T} - \mathcal{T}^2$, then it implies $(\alpha)_q = \alpha$. For if, we assume that $(\alpha)_q \neq \alpha$ then $\exists\ \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{T}$. s.t $\alpha_1\alpha > \alpha$, or $\alpha_2\alpha > \alpha$, or $\beta_1\alpha\beta_2 > \alpha$. Since $\alpha_1\alpha, \alpha_2\alpha$, and $\beta_1\alpha\beta_2 \in \mathcal{T}^2$, therefore we have $\alpha \in \mathcal{T}^2$ this contradict to $\alpha \in \mathcal{T} - \mathcal{T}^2$. Hence, for any $\alpha \in \mathcal{T} - \mathcal{T}^2$, if $\mathcal{T} \neq \mathcal{T}^2$, then $(\alpha)_q = \alpha$.

5. Open Problems

In this section, we highlight some open problems and questions that arise from our research on quasi bases and greatest QC-ideals in semigroup theory. These open problems can serve as directions for future research in this area.

Problem 1: Investigate necessary and sufficient conditions under which a semigroup possesses a unique quasi base. Are there any algebraic or structural properties that can determine when a semigroup has exactly one quasi base?

Problem 2: Investigate the properties of greatest QC-ideals in specific classes of semigroups, such as regular semigroups or inverse semigroups. How do these properties vary in different semigroup structures, and can they be used to simplify the analysis of such semigroups?

6. Conclusion

This research paper has explored the concepts of quasi covered ideals and quasi bases within semigroup theory, shedding light on their significance in understanding the structure and properties of semigroups. The study introduces the notion of the greatest quasi covered ideals, providing valuable insights into semigroup analysis. Quasi bases, distinct subsets of semigroups, have also been examined for their unique properties in relation to quasi covered ideals. These findings underscore the importance of quasi covered ideals and quasi bases as essential tools for dissecting semigroup structures.

In summary, this research enhances our understanding of semigroup theory, offering valuable insights into quasi covered ideals and quasi bases, which serve as foundational elements for the study of semigroups and related mathematical structures. These concepts continue to be instrumental in advancing our knowledge of semigroups, paving the way for further exploration in this field.

Conflict of interests

The Authors declare that there is no conflict of interests.

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