Results in Nonlinear Analysis 7 (2024) No. 2, 107–114
https://doi.org/10.31838/rna/2024.07.02.009
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556
Peer Reviewed Scientific Journal

Product of operators and ∂-Spectrum preservers

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Abstract

Consider an infinite-dimentional Banach space denoted as X, and designate $\mathcal{B}(X)$ as the algebra of all bounded linear operators on X. Moreover, let $\sigma(A)$ denote the spectrum of $A \in \mathcal{B}(X)$, and $\partial(\sigma(A))$ indicate the boundary of $\sigma(A)$. A map $\Delta : \mathcal{B}(X) \to 2^{\mathbb{C}}$ is termed a ∂-spectrum if $\partial(\sigma(A)) \subseteq \Delta(A) \subseteq \sigma(A)$ for all $A \in \mathcal{B}(X)$. In this paper, we characterize all surjective maps ϕ_1 and ϕ_2 on $\mathcal{B}(X)$ satisfying $\Delta(\phi_1(A)\phi_2(B)) = \Delta(AB)$ for all $A, B \in \mathcal{B}(X)$.

Key words and phrases: Nonlinear preserver problem, Product of operators, ∂-spectrum.

Mathematics Subject Classification (2010): 47B49, 47A10, 47A11

1. Introduction and Statement of the Main Result

In this paper, X represent an infinite-dimentional Banach space, and X^* denote its dual space. As per usual, $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on X. The identity operator on X is denoted by $\mathbf{1}$.

For any operator A in $\mathcal{B}(X)$, we use $\sigma(A)$, $\partial(\sigma(A))$, $\sigma_{ap}(A)$, $\sigma_r(A)$, $\sigma_l(A)$ and $\sigma_{sur}(A)$ to represent the spectrum, the boundary of spectrum of A, the approximate point spectrum, the right spectrum, the left spectrum, and the surjectivity spectrum, respectively. Let $\Delta(.)$ represent any one of $\sigma(.)$, $\sigma_{ap}(.)$, $\sigma_r(.)$, $\sigma_l(.)$, $\sigma_{sur}(.)$, then the map Δ is a spectral function on $\mathcal{B}(X)$ that adheres to the conditions

$$\partial(\sigma(A)) \subseteq \Delta(A) \subseteq \sigma(A),$$

for all $A \in \mathcal{B}(X)$.

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Received February 21, 2024; Accepted March 16, 2024, Online April 2, 2024

The spectral function $\Delta(.)$ is considered a ∂-spectrum when it satisfies the condition

$$\partial(\sigma(A)) \subseteq \Delta(A) \subseteq \sigma(A), \quad A \in \mathcal{B}(X). \tag{1.1}$$

The ∂-spectrum $\Delta(.)$ is characterized by the property that $\Delta(A) \neq \emptyset$ for all $A \in \mathcal{B}(X)$, and $\Delta(A)$ is countable if and only if $\sigma(A)$ is countable.

It is noteworthy that alternative spectra, such as the Kato spectrum $\sigma_K(.)$, the Saphar spectrum $\sigma_{rr}(.)$, and the generalized spectrum $\sigma_g(.)$, all satisfy property (1.1). Further details can be found in references [1, 2].

Extensive focus has been directed toward exploring problems related to preserving local spectra, both in linear and nonlinear contexts. The pivotal contribution to this field was initially made by Bourhim and Ransford in [3], who characterized all additive maps on $\mathcal{B}(X)$ preserving the local spectrum of operators for each vector in X.

In recent years, there has been a noticeable surge in interest regarding the more general problem of characterizing maps, whether linear or nonlinear, that preserve various spectral sets and quantities; see for instance [4–14].

Cui and Hou, in [15], characterized surjective linear map φ operating from a semi-simple Banach algebra $\mathcal{A}$ to another algebra $\mathcal{B}$. This characterization states that $\Delta(\varphi(A)) \subset \Delta(A)$ for all $A \in \mathcal{A}$, where Δ denotes a ∂-spectrum.

Additionally, in [16], Miura and Honma delved into multiplicatively peripheral-spectrum preserving surjections between standard operator algebras on complex Banach spaces. Benbouziane et al., as detailed in [17], furnished characterizations for surjective maps preserving the ∂-spectrum of the product or triple product of operators. In [18], Bourhim and Lee provided a comprehensive analysis concerning the structure of surjective maps ϕ_1 and ϕ_2 on $B(X)$ that satisfy the condition of having the same local spectrum of $\phi_1(T)\phi_2(S)$ and TS for all T and S in $B(X)$.

The focal point of this paper is to explore nonlinear maps that preserve any part of the spectrum, including the ∂-spectrum, of the product of operators. Employing an alternative approach, our proofs draw inspiration from the main results presented in the referenced papers [16, 18].

The main result of this paper is the theorem below.

Theorem 1.1: *Let ϕ_2 and ϕ_2 be two surjective maps on $\mathcal{B}(X)$. If ϕ_1 and ϕ_2 satisfying the condition*

$$\Delta(\phi_1(A)\phi_2(B)) = \Delta(AB), \quad (A, B \in \mathcal{B}(X)), \tag{1.2}$$

then one of the statements below is true:

1. *There exists an operator $M \in \mathcal{B}(X)$ such that for every $A \in \mathcal{B}(X)$,*

$$\phi_1(A) = MA(\phi_2(\mathit{1})M)^{-1} \quad \text{and} \quad \phi_2(A) = \phi_2(\mathit{1})MAM^{-1}.$$

2. *There exists an operator $N \in \mathcal{B}(X^*, X)$ such that for every $A \in \mathcal{B}(X)$,*

$$\phi_1(A) = NA^*(\phi_2(\mathit{1})N)^{-1} \quad \text{and} \quad \phi_2(T) = \phi_2(\mathit{1})NA^*N^{-1}.$$

*Where **1** is the identity operator on X.*

2. Preliminaries

For any x in X and f in X^*, take $x \otimes f$ to represent a rank-one operator defined as $(x \otimes f)y = f(y)x$ for every $y \in X$. Importantly, it should be emphasized that any finite-rank operator in $\mathcal{B}(X)$ can be represented as a finite sum of rank-one operators. Take note that $(x \otimes f)^* = f \otimes \tilde{x}$, where $\tilde{x}$ signifies the canonical representation of x in the bidual space X^{**}. Additionally, the spectrum $\sigma(x \otimes f) = \{f(x), 0\}$. The notations $\mathcal{F}_1(X)$ and $\mathcal{F}(X)$ represent, respectively, the set of all rank one operators and the ideal of all finite rank operators in $\mathcal{B}(X)$.

In this section, we introduce certain results essential for establishing our main result. We commence with the following notation, wich will be employed throughout this paper.

$$\Delta^*(A) = \begin{cases} \Delta(A)\setminus\{0\} & \text{if } \Delta(A) \neq 0 \\ \{0\} & \text{if } \Delta(A) = 0 \end{cases}, \tag{2.1}$$

where $\Delta(.)$ is ∂-spectrum and $A \in \mathcal{B}(X)$. Hence,

$$\Delta^*(x \otimes f) = \{f(x)\}, \quad x \in X, \quad f \in X^*. \tag{2.2}$$

The subsequent lemma provides the conditions both necessary and sufficient for two operators to be identical in term of the ∂-spectrum.

Lemma 2.1: *Consider* $A, B \in \mathcal{B}(X)$. *The following assertions are equivalent.*

1. $A = B$.
2. $\Delta(AP) = \Delta(BP)$ for all $P \in \mathcal{F}_1(X)$.
3. $\Delta^*(AP) = \Delta^*(BP)$ for all $P \in \mathcal{F}_1(X)$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are evident.

To establish $(c) \Rightarrow (a)$, suppose that $\Delta^*(AP) = \Delta^*(BP)$ for every $P = x \otimes f \in \mathcal{F}_1(X)$, where $x \in X$ and $f \in X^*$. Then,

$$\begin{aligned} \{f(Bx)\} &= \Delta^*(Bx \otimes f) \\ &= \Delta^*(Ax \otimes f) \\ &= \{f(Ax)\}. \end{aligned}$$

Consequently, $f(Ax) = f(Bx)$, leading to $Ax = Bx$. Since this holds for any arbitrarily chosen x, it is evident that $A = B$. □

The subsequent result characterizes all rank one operators in relation to the ∂-spectrum.

Lemma 2.2: *Consider a nonzero operator* P *in* $\mathcal{B}(X)$, *then the following statements are equivalent.*

1. $P \in \mathcal{F}_1(X)$.
2. $\Delta^*(PA)$ is a singleton for all $A \in \mathcal{B}(X)$.

Proof. The implication $(a) \Rightarrow (b)$ stems from Eqs. (2.1) and (2.2).

Now, to establish $(b) \Rightarrow (a)$, consider that $\Delta^*(PA)$ consists of only one element for every $A \in \mathcal{B}(X)$, then $\Delta^*(PA) = \sigma_\pi(PA)$, where $\sigma_\pi(A) := \{\lambda \in \sigma(A) / \quad |\lambda| = r(A)\}$ represents the peripheral spectrum of A. As a result, based on [16, Lemma 2.1], it follows that P has rank one. □

Lemma 2.3: *For any rank-one operator* $P \in \mathcal{F}_1(X)$ *and any operators* $A, B \in \mathcal{B}(X)$, *the following holds:*

$$\Delta^*((A+B)P) = \Delta^*(AP) + \Delta^*(BP).$$

Proof. Consider $P = x \otimes f \in \mathcal{F}_1(X)$ as a rank-one operator, where $x \in X$ and $f \in X^*$.

Utilizing Eqs. (2.1) and (2.2), we derive that

$$\begin{aligned} \Delta^*((A+B)P) &= \Delta^*((A+B)x \otimes f) \\ &= \{f((A+B)x)\} \\ &= \{f(Ax) + f(Bx)\} \\ &= \{f(Ax)\} + \{f(Bx)\} \\ &= \Delta^*(Ax \otimes f) + \Delta^*(Bx \otimes f) \\ &= \Delta^*(AP) + \Delta^*(BP). \end{aligned}$$

□

The following Lemma, proved in [13, Theorem 3.3], describes bijective linear maps on $\mathcal{F}(X)$ that preserve the rank one operators in both directions.

Lemma 2.4: *(See [13, Theorem 3.3]) Consider a complex Banach space X with $dim X \geq 1$. Assuming $\phi : \mathcal{F}(X) \to \mathcal{F}(X)$ is an additive surjective map that preserves rank-one operators in both directions, then one of the statements below holds:*

1. *There is a bijective linear transformations $M : X \to X$ and $M' : X^* \to X^*$ such that*

$$\phi(x \otimes f) = Mx \otimes M'f,$$

for all $x \in X$ and $f \in X^*$.

2. *There is a bijective linear transformations $N : X^* \to X$ and $N' : X \to X^*$ such that*

$$\phi(x \otimes f) = Nf \otimes N'x,$$

for all $x \in X$ and $f \in X^*$.

3. Proof of the Main Result

Within this section, we move forward to prove our main theorem, namely Theorem 1.1. The proof is extensive, so we've organized it into six steps.

Step 1. ϕ_1 *is injective and* $\phi_1(0) = 0$.

Firstly, suppose that $\phi_1(A) = \phi_1(B)$ for some A and B in $\mathcal{B}(X)$, then

$$\begin{aligned}\Delta^*(AP) &= \Delta^*(\phi_1(A)\phi_2(P)) \\ &= \Delta^*(\phi_1(B)\phi_2(P)) \\ &= \Delta^*(BP)\end{aligned}$$

for all $P \in \mathcal{F}_1(X)$. Lemma 2.1 entails that $A = B$. Thus, ϕ_1 is injective.

For the second part, consider $x \in X$ and $f \in X^*$ such that $f(x) = 0$. Given the fact that ϕ_2 is surjective, there exists $A \in \mathcal{B}(X)$ such that $\phi_2(A) = x \otimes f$. Then,

$$\begin{aligned}\{f(\phi_1(0)x)\} &= \Delta^*(\phi_1(0)x \otimes f) \\ &= \Delta^*(\phi_1(0)\phi_2(A)) \\ &= \Delta^*(0A) \\ &= \{0\}.\end{aligned}$$

We get that $\phi_1(0) = \lambda 1$ for some $\lambda \in \mathbb{C}$. Now, consider $y \in X$ and $g \in X^*$ such that $g(y) = 1$. Similary to the previous method, there is an operator $B \in \mathcal{B}(X)$ such that $\phi_2(B) = y \otimes g$. Therefore,

$$\begin{aligned}\{\lambda\} &= \Delta^*(\phi_1(0)y \otimes g) \\ &= \Delta^*(\phi_1(0)\phi_2(B)) \\ &= \Delta^*(0B) \\ &= \{0\}.\end{aligned}$$

Consequently, $\lambda = 0$, leading to $\phi_1(0) = 0$.

Step 2. ϕ_1 *and* ϕ_2 *preserve rank-one operators in both directions.*

Consider a rank-one operator $P \in \mathcal{F}_1(X)$ and notice that $\phi_1(P) \neq 0$. Let $A \in \mathcal{B}(X)$ be an operator, then there exists $B \in \mathcal{B}(X)$ such that $\phi_2(B) = A$. Thus Eqs. (1.2), (2.1) and Lemma 2.2 tell us that

$$\begin{aligned}\Delta^*(\phi_1(P)A) &= \Delta^*(\phi_1(P)\phi_2(B))\\ &= \Delta^*(PB)\end{aligned}$$

contains one element for all $B \in \mathcal{B}(X)$. Using Lemma 2.2 again, we deduce that $\phi_1(P)$ has a rank one.

Conversely, assume that $\phi_1(P) \in \mathcal{F}_1(X)$ for some operator $P \in \mathcal{B}(X)$ and note that $P \neq 0$. Eqs. (1.2), (2.1) and Lemma 2.2 implies that $\Delta^*(PA) = \Delta^*(\phi_1(P)\phi_2(A))$ is a singleton for every $A \in \mathcal{B}(X)$. Again, from Lemma 2.2, we conclude that P has rank one.

Using similar discussion, we deduce an equivalent outcome for ϕ_2.

Step 3. ϕ_1 *is linear.*

To begin, let's demonstrate the homogeneity of ϕ_1. Consider $P \in \mathcal{F}_1(X)$ as a rank one operator, then we obtain

$$\begin{aligned}\Delta^*(\lambda\phi_1(A)\phi_2(P)) &= \lambda\Delta^*(\phi_1(A)\phi_2(P))\\ &= \lambda\Delta^*(AP)\\ &= \Delta^*((\lambda A)P)\\ &= \Delta^*(\phi_1(\lambda A)\phi_2(P)),\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $A \in \mathcal{B}(X)$. The surjectivity of ϕ_2, Lemma 2.1, and Step 2 entails that $\phi_1(\lambda A) = \lambda\phi_1(A)$.

To finalize the proof, let's demonstrate the additivity of ϕ_1. Consider a rank-one operator $P \in \mathcal{F}_1(X)$. Since $\phi_1(P)$ has a rank one too, then for every $A, B \in \mathcal{B}(X)$ Lemma 2.3 ensures that

$$\begin{aligned}\Delta^*(\phi_1(A+B)\phi_2(P)) &= \Delta^*((A+B)P)\\ &= \Delta^*(AP) + \Delta^*(BP)\\ &= \Delta^*(\phi_1(A)\phi_2(P)) + \Delta^*(\phi_1(B)\phi_2(P))\\ &= \Delta^*((\phi_1(A)+\phi_1(B))\phi_2(P)).\end{aligned}$$

Lemma 2.1 implies that $\phi_1(A+B) = \phi_1(A) + \phi_1(B)$ and thus ϕ_1 is additive.

Step 4. *Either there exists a bijective linear transformations* $M : X \to X$ *and* $M' : X^* \to X^*$ *such that*

$$f(x) = (M'f)(\phi_2(1)Mx), \quad x \in X, \quad f \in X^*,$$

or there exists a bijective linear transformations $N : X^* \to X$ and $N' : X \to X^*$ such that

$$f(x) = (N'x)(\phi_2(1)Nf), \quad x \in X, \quad f \in X^*.$$

Given the fact that ϕ_1 is a bijective linear map on $\mathcal{F}(X)$ that preserves rank one operators bidirectionally, then Lemma 2.4 ensures that either there is a pair of bijective linear transformations, $M : X \to X$ and $M' : X^* \to X^*$, such that

$$\phi_1(x \otimes f) = Mx \otimes M'f \quad \textit{for all} \quad x \in X \quad \textit{and} \quad f \in X^*, \tag{3.1}$$

or there exists another pair of bijective linear transformations, $N : X^* \to X$ and $N' : X \to X^*$, such that

$$\phi_1(x \otimes f) = Nf \otimes N'x \quad \textit{for all} \quad x \in X \quad \textit{and} \quad f \in X^*. \tag{3.2}$$

Case 1: Suppose that ϕ_1 take the first form (3.1). Eqs. (2.1) and (2.2) implies that

$$\begin{aligned}\{f(x)\} &= \Delta^*(x \otimes f)\\ &= \Delta^*(\phi_1(x \otimes f)\phi_2(1))\\ &= \Delta^*((Mx \otimes M' f)\phi_2(1))\\ &= \{(M' f)(\phi_2(1)Mx)\},\end{aligned}$$

for every $x \in X$ and $f \in X^*$. Then,

$$f(x) = (M' f)(\phi_2(1)Mx), \quad x \in X, \quad f \in X^*. \tag{3.3}$$

Case 2: Assume that ϕ_1 take the second form (3.4). Using the same approach of the first case, we find that

$$f(x) = (N' x)(\phi_2(1)Nf), \quad \textit{for all} \quad x \in X \quad \textit{and} \quad f \in X^*. \tag{3.4}$$

Step 5. *M and N are continuous and the invertibility of $\phi_2(1)$ is affirmed.*

Firstly, let's demonstrate the injectivity of $\phi_2(1)$. Suppose by the way of contradiction that $\phi_2(1)$ is not injective, and let $y \in X$ such that $\phi_2(1)y = 0$.

Case 1: Suppose that the case (3.3) occurs, and take $x \in X$ and $f \in X^*$ such that $Mx = y$ and $f(x) \neq 0$. The previous step entails that

$$\begin{aligned}\{0\} \neq \{f(x)\} &= \{(M' f)(\phi_2(1)Mx)\}\\ &= \{(M' f)(\phi_2(1)y)\}\\ &= \{0\}.\end{aligned}$$

This is a contradiction. Therefore $\phi_2(1)$ is injective.

Case 2: If the case (3.4) occurs, then $N : X^* \to X$ and $N' : X \to X^*$ are invertible. Through a comparable discussion, we determine that $\phi_2(1)$ is injective.

Secondly, let's demonstrate the continuity of both M and N.

Case 1: If the case (3.3) occurs, consider a sequence $(x_n)_n$ in X converging to $x \in X$, and let $y \in X$ such that $\lim_{n\to+\infty} Mx_n = y$. For every $f \in X^*$, we obtain

$$\begin{aligned}(M' f)(\phi_2(1)y) &= \lim_{n\to+\infty}(M' f)(\phi_2(1)Mx_n)\\ &= \lim_{n\to+\infty} f(x_n)\\ &= f(x)\\ &= (M' f)(\phi_2(1)Mx).\end{aligned}$$

Given that $\phi_2(1)$ is injective and taking into account the arbitrariness of the linear functional $f \in X^*$, the closed graph theorem entails the continuity of M. Consequently, we have $(\phi_2(1)M)^* M' f(x) = (M' f)(\phi_2(1)Mx) = f(x)$ for every $x \in X$ and $f \in X^*$. Therefore, $1_{X^*} = (\phi_2(1)M)^* M'$. Given the invertibility of both M and M', it follows that the inverse of $\phi_2(1)$ exists and $M'^* = (\phi_2(1)M)^{-1}$.

Case 2: If the case (3.4) occurs, then let π the canonical embedding of X in X^{**}. Since N and N' are invertible and using similar discussion, we show that N is continuous and $1_{X^*} = N'^* \pi \phi_2(1)N$. Hence, $\phi_2(1)$ is invertible and X is reflexive, and $N' = (\phi_2(1)N)^{-1}$.

Step 6. ϕ_1 *and* ϕ_2 *takes the desired forms.*

Case 1: If the case (3.3) occurs, then based on the preceding step, we find that for every $x \in X$ and $f \in X^*$

$$\begin{aligned}\phi_1(x \otimes f) &= Mx \otimes M' f \\ &= M(x \otimes f)M'^*.\end{aligned}$$

As a result, Eqs. (1.2) and (2.1) implies that

$$\begin{aligned}\Delta^*(M(x \otimes f)M'^*\phi_2(A)) &= \Delta^*(\phi_1(x \otimes f)\phi_2(A)) \\ &= \Delta^*((x \otimes f)A) \\ &= \Delta^*(M(x \otimes f)M'^*(M'^*)^{-1}AM^{-1}),\end{aligned}$$

for every $A \in \mathcal{B}(X)$. Therefore, Lemma 2.1 enthails that $\phi_2(A) = (M'^*)^{-1}AM^{-1} = \phi_2(1)MAM^{-1}$ for all $A \in \mathcal{B}(X)$.

Now, observe that for any $A \in \mathcal{B}(X)$ and $P \in \mathcal{F}_1(X)$ we have

$$\begin{aligned}\Delta^*(MAM'^*\phi_2(P)) &= \Delta^*(MAM'^*(M'^*)^{-1}PM^{-1}) \\ &= \Delta^*(AP) \\ &= \Delta^*(\phi_1(A)\phi_2(P)).\end{aligned}$$

By employing Lemma 2.1 once more, we conclude that $\phi_1(A) = MAM'^* = MA(\phi_2(1)M)^{-1}$ for all $A \in \mathcal{B}(X)$.

Case 2: If the case (3.4) occurs, a similar analysis reveals that

$$\begin{aligned}\phi_1(x \otimes f) &= Nf \otimes N' x \\ &= N(f \otimes x^*)N' \\ &= N(x \otimes f)^* N',\end{aligned}$$

for every $x \in X$ and $f \in X^*$. Thus, Eqs. (1.2) and (2.1) entails that

$$\begin{aligned}\Delta^*(N(x \otimes f)^* N'\phi_2(A)) &= \Delta^*(\phi_1(x \otimes f)\phi_2(A)) \\ &= \Delta^*((x \otimes f)A) \\ &= \Delta^*(N(x \otimes f)^* N' N'^{-1} A^* N^{-1}),\end{aligned}$$

for all $A \in \mathcal{B}(X)$. As a result, Lemma 2.1 indicates that $\phi_2(A) = N'^{-1}A^*N^{-1} = \phi_2(1)NA^*N^{-1}$ for all $A \in \mathcal{B}(X)$.

Now, consider any $A \in \mathcal{B}(X)$ and $P \in \mathcal{F}_1(X)$. It holds that

$$\begin{aligned}\Delta^*(NA^*N'\phi_2(P)) &= \Delta^*(NA^*N'N'^{-1}P^*N^{-1}) \\ &= \Delta^*(A^*P^*) \\ &= \Delta^*(AP) \\ &= \Delta^*(\phi_1(A)\phi_2(P)).\end{aligned}$$

Using Lemma 2.1, we infer that $\phi_1(A) = NA^*N' = NA^*(\phi_2(1)N)^{-1}$ for every $A \in \mathcal{B}(X)$.

This concludes the proof. ■

4. Acknowledgment

The authors convey their heartfelt gratitude to the reviewers for providing exceptionally valuable feedback, significantly enhancing the quality of the paper.

Furthermore, the authors do not have any financial or non-finantial interests to declare.

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