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Two–energy group neutron diffusion model in spherical reactors

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Abstract

This paper investigates the neutron diffusion model with two energy groups in spherical reactors. In particular, the integer-order two energy groups neutron diffusion model in spherical reactors is resolved using the Laplace transform method by regarding the spherical radius r as a time domain. Next, we transform the neutron diffusion model into fractional-order versions using the Caputo differentiator, resulting in what is referred to as the fractional-order two-energy-group neutron diffusion model. To address this fractional-order system, we introduce a novel approach to reduce a system of 2α-order to a duplicated system of α-order, where $0 < \alpha \leq 1$. This converted system is then solved using one of the recent modifications of the fractional Euler method called the Modified Fractional Euler Method (MFEM). Several numerical simulations are depicted to verify our findings.

Keywords: Two energy groups neutron diffusion model; spherical reactors; Modified Fractional Euler Method; Fractional Calculus.

Mathematics Subject Classification: 34A08, 34C20

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1. Introduction

Control rods, coolants, reflectors, and other components make up the intricate system competition of the nuclear reactor. Such reactors' design and analysis for various operating techniques is a complicated endeavor combining multiple nuclear engineering fields [1–3]. The cylindrically symmetric fractional Helmholtz equation was analytically addressed in an isotropic medium [4]. The solution of the neutron diffusion equation is significant to identify the behavior of the neutrons in the nuclear reactors [5]. With the use of a combination of Laplace transform and residual power series techniques, the system of coupled fractional neutron diffusion equations was efficiently solved with delayed neutrons, and by taking the non-Gaussian case with different fractional-order values, the anomalous diffusion was dealt with [6]. With the help of the homotopy perturbation method (HPM), the complex neutron diffusion equations was addressed in a spherical nuclear reactor [7, 8]. To reconstruct the neutron flux of a reactor from the nuclear parameters, initial and boundary conditions, the one-dimensional, mono-energetic diffusion kinetic equation was used in Cartesian geometry with one delayed neutron precursor concentration [9].

To explore the nuclear reactors' behaviors, it is crucial to find a solution the time-dependent two energy groups neutron diffusion model. The two energy groups point kinetics equations were analytically solved with delayed neutrons derived from the neutron diffusion equations in the presence of the time-dependent external neutron source, see reference[10]. The distribution of the neutron population in a nuclear reactor can be described by using transport equations. One possible solution of the fractional neutron transport equation is given by the fractional neutron diffusion equation [11]. A nonlocal scheme for a neutron diffusion equation with a memory might be constructed in terms of moments of the displacement kernel with a modified geometric buckling. Such a scheme can lead to a class of partial differential equations (PDEs) which belongs to the Swift-Hohenberg equations and the Fisher-Kolmogorov family [12]. In literature, there are two schemes that can be applied for homogeneous case of those equations: computational and analytical methods [13]. The space-time neutron diffusion equations with multigroup of delayed neutrons are a couple of the stiff nonlinear differential equations [14]. In [15], the interaction between two hemispheres was theoretically discussed with opposite flat faces. With the aim of creating a criticality verification benchmark test set, a couple of numerical solutions were summarized in [16] for analytic eigenvalue and eigenfunction equations.

The neutron diffusion equation is among the most significant PDEs which can be used to express the neutron behavior in nuclear reactors and numerous physical applications [17]. In the past few decades, this topic has drawn a lot of interest from the reactor physics community [18, 19]. Nuclear reactors depend on achieving criticality, a precise balance between neutron production and neutron loss. This equilibrium is expressed mathematically by the steady-state neutron transport equation where the case is independent of time. However, because of its intricacy, fear simplification is used to make practical analysis easier. An appropriate simplification is provided by Fick's law, which creates a connection between the neutron current and the neutron flux. Engineers can solve steady state, time-independent, and neutron diffusion equations to ensure criticality, which ensures a controlled and self-sustaining chain reaction inside the nuclear reactor. The concept of cross-section is essential to nuclear reactor physics in order to calculate the likelihood that specific reactions will occur. A cross-section is the size of a target nucleus that is appropriate for a particular reaction. A measure of the probability that a nuclear contact will take place is provided by the interaction of a particle, like a neutron, with a nucleus; for further information, consult references [20–23].

The opening part of the contribution of this work introduces a sufficient procedure based on the Laplace Transform Method (LTM) to provide a general solution for the integer-order two energy groups neutron diffusion model in spherical geometry. This would be achieved by considering the spherical radius r as a time domain. The LTM is an effective and a suitable technique to solve such a problem without discretization or perturbation. The second part of this work focuses on developing a scheme to reduce a fractional-order system of order 2α into another duplicated fractional-order system of order α, where $0 < \alpha \leq 1$. This would be carried out by establishing a new result which introduces some auxiliary variables, and hence can serve us in performing this reduction. After converting the 2α-order

system into an α-order one, we use one of the recent modifications of the fractional Euler method called the Modified Fractional Euler Method (MFEM) to solve the resultant system. In a few words, by reducing a higher fractional-order differential equations to a duplicated system of fractional-order equations, we simplify the problem and make it amenable to numerical methods and computational techniques, and this what we have carried out for the fractional-order two energy groups neutron diffusion model as it is of order 2α.

Mathematicians who study fractional calculus expand on the ideas of differentiation and integration to non-integer orders. It permits differentiation and integration of any real or complex order, including fractional-orders, as opposed to limiting these operations to whole integers alone [24]. Fractional calculus has several uses in the field of physics, particularly in the research of complex systems, anomalous diffusion, and fractal phenomena. A foundation for describing physical events that show memory or long-range dependency is provided by fractional calculus. In order to simulate systems having memory effects, such as viscoelastic materials, electrical circuits, and biological systems, fractional differential equations (FDEs) must take fractional derivatives into account. The diffusion equation, a second-order PDE, is used in classical physics to model diffusion. However, in some systems, diffusion behaves atypically and cannot be adequately predicted by the conventional diffusion theory [25–27]. These anomalous diffusion processes may be modeled and examined using fractional calculus, where the order of the derivative is a fractional-order value. In this regard, with the aim of addressing numerous fractional-order models that appear broadly in biological, physics, and applied mathematics, a lot of numerical methods have been recently proposed and implemented. In [28], a new modification for the fractional Euler method (FEM) called by modified fractional Euler method (MFEM) proved its efficiency in solving these models, see references [29]. From this point of view, we intend to apply this method to solve the duplicated α-order system generated from the reduction scheme.

2. Preliminaries

In the following content, we review certain primary definitions and theorems connected with fractional calculus. This would lay the foundation to the fractionalization of the two energy groups neutron diffusion model in the spherical reactors later on.

Definition 1: [32, 33] The fractional Riemann-Liouville integral of a function $f(t)$ of order $\mu > 0$ is typically expressed by

$$J^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t f(\tau)(t-\tau)^{\mu-1} d\tau, \quad t > 0, \mu > 0. \tag{1}$$

Remark 1: [32, 33] *A number of Riemann-Liouville integral properties are provided below for the sake of completeness:*

1. $J^0 f(t) = f(t)$.
2. $J^{\mu}(t-a)^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\mu+\gamma+1)}(t-a)^{\mu+\gamma}, \quad \gamma \geq -, a \in \mathbb{R}$.
3. $J^{\mu} J^{\beta} f(t) = J^{\beta} J^{\mu} f(t), \quad \mu, \beta \geq 0$.
4. $J^{\mu} J^{\beta} f(t) = J^{\mu+\beta} f(t), \quad \mu, \beta \geq 0$.

Definition 2: [32, 33] *The Caputo fractional derivative of order $\alpha > 0$ is defined as*

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{-\alpha+n-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m f(t)}{dt^m}, & \alpha = m, \end{cases} \tag{2}$$

where m is the smallest integer number greater than α and $t > 0$.

Remark 2: [32, 33] *A number of Caputo derivative properties are provided below for the sake of completeness:*

1. $D^{\alpha}c = 0$, *where* c *is constant.*
2. *For* $a \in \mathbb{R}$, *we have*

$$D^{\alpha}(t-a)^{\rho} = \begin{cases} \dfrac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}(t-a)^{\rho-\alpha}, & \rho > \alpha - 1, \\ 0, & \textit{otherwise.} \end{cases} \tag{3}$$

3. D^{α} *is a linear operator, i.e.,*

$$D^{\alpha}(\mu f(t) + \omega k(t)) = \mu D^{\alpha}(f(t)) + \omega D^{\alpha}(k(t)),$$

where μ *and* ω *are constants.*

4. *If* $m-1 < \alpha \le m$, $m \in \mathbb{N}$, *then we have*

$$J^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{i=1}^{n} f^{i}(0^{+})\frac{t^{i}}{i!}, \ t > 0. \tag{4}$$

Definition 3: [32, 33] *Let* $\alpha \in \mathbb{R}^{+}$ *and* $m = \lceil \alpha \rceil$. *The Caputo fractional-order derivative operator* D^{α} *can be defined in terms of the Riemann-Liouville fractional integral operator as follows:*

$$D^{\alpha}f = J^{m-\alpha}D^{m}f. \tag{5}$$

Theorem 1: [28] *Suppose that* $D^{k\alpha}f(x) \in C(0,b]$ *for* $k = 0,1,2,\cdots,n+1$, *where* $0 < \alpha \le 1$. *Then we can expand the function* f *about the node* x_0 *as follows:*

$$f(x) = \sum_{i=0}^{n} \frac{(x-x_0)^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha}f(x_0) + \frac{(x-x_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} D^{(n+1)\alpha}f(\xi), \tag{6}$$

$\forall x \in (0,b]$ *with* $0 < \xi < x$.

Definition 4: *Suppose that the function* f *is defined on* $[0,\infty)$. *Then the Laplace transform* $\mathcal{L}\{f\}$ *is another function* $F(s)$, *which can be outlined as*

$$F(s) = \mathcal{L}\{f\} := \int_{0}^{\infty} e^{-st} f(t)dt. \tag{7}$$

Remark 3: *Some properties of Laplace transform are listed below for completeness:*

1. $\mathcal{L}\{ty\} = -\dfrac{d}{ds}\mathcal{L}\{y\}$.
2. $\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f\} = sF(s) - f(0)$.
3. $\mathcal{L}\{f''(t)\} = s^{2}F(s) - sf(0) - f'(0)$.

In the subsequent material, we purpose to briefly review the MFEM and illustrate how it can be used in solving the following initial value problem:

$$D^{\alpha}z(t) = \Theta(t, z(t)), \tag{8}$$

with the initial condition:

$$z(0) = z_0, \tag{9}$$

where $0 < \alpha \leq 1$. For this purpose, we assume that $0 = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \ldots < t_n = t_0 + nh = b$ whereby the mesh points are $t_i = t_0 + ih$, $i = 1,2,\ldots,n$, with the step size $h = \dfrac{b-a}{n}$. Consequently, with the use of the first three terms of Theorem 1, and performing some substitutions, we can obtain

$$\begin{aligned} &\varphi_0 = z_0, \\ &\varphi_{i+1} = \varphi_i + \frac{h^\alpha}{\Gamma(\alpha+1)}\Theta\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \varphi_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}\Theta(t_i, \varphi_i)\right), \end{aligned} \tag{10}$$

where φ_i denotes the numerical solution of problem (8), for $i = 1,2,\cdots,n-1$. However, to get a full overview about the MFEM algorithm and its analysis, the reader may refer to the reference [28].

3. Two Energy Groups of Neutron Diffusion Model

This part aims to first address the integer-order two energy groups neutron diffusion model in spherical reactors via LTM, and then this model will be dealt with the use of MFEM in its fractional-order case.

3.1. Integer-order model

In what follows, with the use of LTM and by letting that the spherical radius r be a time domain, the integer-order two energy groups neutron diffusion model is resolved in spherical reactors. For this purpose, it is assumed that the integer-order two energy groups neutron diffusion system has a single solution in the interval of integration, and has the following form [1, 2]:

$$\begin{aligned} &\nabla^2\phi_1(r) + C_{11}\phi_1(r) + C_{12}\phi_2(r) = 0, \\ &\nabla^2\phi_2(r) + C_{21}\phi_1(r) + C_{22}\phi_2(r) = 0, \end{aligned} \tag{11}$$

where C_{ij} is a constant connection between fluxes in different energy groups of neutrons and C_{ii} is known as a group buckling. In particular, these constants can be respectively defined by

$$\begin{aligned} C_{ij} &= \frac{\sum\sum_{sij} + x_i \upsilon_j \sum_{fi}}{D_i}, \\ C_{ii} &= \frac{x_i \upsilon_i \sum_{fi} - (\sum_{\gamma i} + \sum\sum_{sij})}{D_i}, \end{aligned} \tag{12}$$

where D_i is a group diffusion coefficient that can be defined by

$$D_i = \frac{1}{3(\sum_{fi} + \sum_{sii} + \sum\sum_{sij} + \sum_{\gamma i})}. \tag{13}$$

The constants reported in (12) and (13) are identified in terms of different macroscopic cross-sections, the fraction of fission neutrons that are emitted with energies in the i^{th} group (x_i), and the number of neutrons produced per fission for each group (υ_i). In fact, system (11) characterizes the behavior of the neutrons in nuclear reactors, where each flux ϕ_i exhibits the neutron flux with a specific speed. Each flux is maximum at the center of the reactor, and its derivative vanishes. Hence, the initial conditions might be expressed as

$$\phi_i(0^+) = h_i, \phi_i'(0^+) = k_i, i = 1,2, \tag{14}$$

where the fluxes $\phi_i(r)$ are functions of independent variable r, and $h_i, k_i \in \mathbb{R}$, for $i = 1,2$. Throughout it is supposed that $\phi_i(r)$ are analytic functions for $r > 0$ and $i = 1,2$.

System (11) might be simplified by taking into account the fact of nuclear reactor theory which states that the material buckling must be equal to the geometrical buckling B^2 for all energy group in the criticality case, i.e.,

$$\begin{aligned} \nabla^2\phi_1(r) + B^2\phi_1(r) &= 0, \\ \nabla^2\phi_2(r) + B^2\phi_2(r) &= 0. \end{aligned} \tag{15}$$

Substituting of the values of B^2 into above system yields

$$\begin{aligned} -B^2\phi_1(r) + C_{11}\phi_1(r) + C_{12}\phi_2(r) &= 0, \\ -B^2\phi_2(r) + C_{21}\phi_1(r) + C_{22}\phi_2(r) &= 0. \end{aligned} \tag{16}$$

The above system demonstrates that the ratio of each two fluxes is constant. In this regard, a proper conventional method such as Cramer's rule, can identified the value of B^2 along with finding each flux separately for any number of energy group for any reactor geometry.

The energy spectrum of neutrons in a two energy groups neutron diffusion model is split into fast and thermal energy groups. A set of diffusion equations roughly approximates the neutron behavior inside each energy group, which represents a particular range of neutron energies. In the following content, the integer-order two energy groups neutron diffusion model is resolved in spherical reactors. This would be achieved with the use of LTM and by letting the spherical radius r be a time domain. To do so, we should review the following fact:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{17}$$

Accordingly, system (11) can be rewritten as

$$\begin{aligned} r\phi_1''(r) + 2\phi_1'(r) + rC_{11}\phi_1(r) + rC_{12}\phi_2(r) &= 0, \\ r\phi_2''(r) + 2\phi_2'(r) + rC_{21}\phi_1(r) + rC_{22}\phi_2(r) &= 0, \end{aligned} \tag{18}$$

with the following initial conditions:

$$\begin{aligned} \phi_i(0) &= a_i,\ i = 1,2, \\ \phi_i'(0) &= b_i,\ i = 1,2. \end{aligned} \tag{19}$$

By taking the Laplace transform to the both sides of (19), we get

$$\begin{aligned} \mathcal{L}(r,\phi_1''(r)) + 2\mathcal{L}(\phi_1'(r))) + C_{11}\mathcal{L}(r\phi_1(r)) + C_{12}\mathcal{L}(r\phi_2(r)) &= 0, \\ \mathcal{L}(r,\phi_2''(r)) + 2\mathcal{L}(\phi_2'(r))) + C_{21}\mathcal{L}(r\phi_1(r)) + C_{22}\mathcal{L}(r\phi_2(r)) &= 0. \end{aligned} \tag{20}$$

By using Remark 3, we can have

$$\begin{aligned} &\frac{-d}{ds}\left(s^2\mathcal{L}\{\phi_1(r)\} - s\phi_1(0) - \phi_1'(0)\right) + 2\left(s\mathcal{L}\{\phi_1(r)\} - \phi_1(0)\right) + C_{11}\left(\frac{-d}{ds}\mathcal{L}\{\phi_1(r)\}\right) + C_{12}\left(\frac{-d}{ds}\mathcal{L}\{\phi_2(r)\}\right) = 0, \\ &\frac{-d}{ds}\left(s^2\mathcal{L}\{\phi_2(r)\} - s\phi_2(0) - \phi_2'(0)\right) + 2\left(s\mathcal{L}\{\phi_2(r)\} - \phi_2(0)\right) + C_{21}\left(\frac{-d}{ds}\mathcal{L}\{\phi_1(r)\}\right) + C_{22}\left(\frac{-d}{ds}\mathcal{L}\{\phi_2(r)\}\right) = 0. \end{aligned} \tag{21}$$

With the help of assuming $\mathcal{L}\{\phi_i(r)\}) = T_i(s)$, for all $i = 1,2$, we obtain

$$\begin{aligned} \frac{-d}{ds}(s^2T_1(s) - a_1s - b_1) + 2(sT_1(s) - a_1) - C_{11}T_1'(s) - C_{12}T_2'(s) &= 0, \\ \frac{-d}{ds}(s^2T_2(s) - a_2s - b_2) + 2(sT_2(s) - a_2) - C_{21}T_1'(s) - C_{22}T_2'(s) &= 0. \end{aligned} \tag{22}$$

Consequently, simplifying system (22) yields

$$\begin{aligned} -(s^2 + C_{11})T_1'(s) - C_{12}T_2'(s) &= a_1, \\ -C_{21}T_1'(s) - (s^2 + C_{22})T_2'(s) &= a_2. \end{aligned} \tag{23}$$

The above system can be written in its corresponding matrix form as

$$\begin{bmatrix} -(s^2 + C_{11}) & -C_{12} \\ -C_{21} & -(s^2 + C_{22}) \end{bmatrix} \begin{bmatrix} T_1'(s) \\ T_2'(s) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tag{24}$$

or

$$A(s)T'(s) = K, \tag{25}$$

where

$$A(s) = \begin{bmatrix} -(s^2 + C_{11}) & -C_{12} \\ -C_{21} & -(s^2 + C_{22}) \end{bmatrix}, T'(s) = \begin{bmatrix} T_1'(s) \\ T_2'(s) \end{bmatrix}, K = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{26}$$

As a consequence, by assuming that $A(s)$ is invertible, we can have

$$T'(s) = A^{-1}(s)K \tag{27}$$

Thus, system (27) can be then solved numerically by using a prepared MATLAB code with noting that $\phi_i(r) = \mathcal{L}^{-1}\{T_i(s)\}$, for all $i = 1,2$. This would give the solution $\phi_i(r)$ of system (18).

3.2. Fractional-order model

In this subsection, we intend first to fractionalize system (18) by applying the Caputo differentiator operator to its equations. This would generate the fractional-order two energy groups neutron diffusion model in the spherical reactors, which would be in the form

$$\begin{aligned} D^{2\alpha}\phi_1(r) + \frac{2}{r}D^{\alpha}\phi_1(r) + C_{11}\phi_1(r) + C_{12}\phi_2(r) &= 0, \\ D^{2\alpha}\phi_2(r) + \frac{2}{r}D^{\alpha}\phi_2(r) + C_{21}\phi_1(r) + C_{22}\phi_2(r) &= 0, \end{aligned} \tag{28}$$

with the following initial conditions:

$$\begin{aligned} &\phi_i(0) = a_i, i = 1,2, \\ &\frac{\partial^{\alpha}}{\partial r^{\alpha}}\phi_i(0) = b_i, i = 1,2. \end{aligned} \tag{29}$$

It is clear that system (28) is of order 2α. For this reason, we need to develop a manner that can deal with such a system with such a fractional-order. To do so, we introduce the following lemma that aims to reduce the system of 2α-order into another duplicated system of α-order, where $0 < \alpha \leq 1$.

Lemma 1: *Any FDE of order $n\alpha$, $n \in \mathbb{Z}^+$ and $\alpha \in (0,1]$, with functions possessing values in $\mathbb{R}$ can be converted into a system of FDEs of order α with values in $\mathbb{R}^{nd}$.*

Proof 1. *To prove this result, we should first take the scalar case that takes place whenever $d = 1$, and then we will consider the remaining case that is hold when $d > 1$. For this reason, we should note that the general form of the FDE of order $n\alpha$ in its scaler case can be given by*

$$D^{n\alpha}y(t) = G(t, y(t), D^{\alpha}y(t), D^{2\alpha}y(t), \cdots, D^{(n-1)\alpha}y(t)), \tag{30}$$

where G is a continuous function defined on the subset $I\times\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$, so that it takes values in $\mathbb{R}$ for a given intend I. Now, define

$$\Psi(t,v_0,v_1,\cdots,v_{n-1})=(v_1,v_2,\cdots,G(t,v_0,v_1,\cdots,v_{n-1})) \tag{31}$$

as a continuous function on $I\times\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$ as G, but it takes the values in $\mathbb{R}^n$. In this regard, we consider the following equation:

$$D^{\alpha}Y(t)=\Psi(t,Y(t)),\ \text{for } t\in I. \tag{32}$$

Now, we want to show that $x:I\to\mathbb{R}$ is a solution of equation (31) if and only if the function

$$\begin{aligned} X:\ & I\to\mathbb{R}^n,\\ & t\to(x(t),D^{\alpha}x(t),D^{2\alpha}x(t),\cdots,D^{(n-1)\alpha}x(t)) \end{aligned} \tag{33}$$

is a solution of equation (32). To this end, we assume that X is a solution to equation (31) such that X is defined above. Then we have

$$D^{\alpha}X(t)=\begin{pmatrix} D^{\alpha}x(t)\\ D^{2\alpha}x(t)\\ \vdots\\ D^{(n-1)\alpha}x(t)\\ D^{n\alpha}x(t)\end{pmatrix}=\begin{pmatrix} D^{\alpha}x(t)\\ D^{2\alpha}x(t)\\ \vdots\\ D^{(n-1)\alpha}x(t)\\ G(t,x(t),D^{\alpha}x(t),D^{2\alpha}x(t),\cdots,D^{(n-1)\alpha}x(t))\end{pmatrix}=\Psi(t,X(t)). \tag{34}$$

Herein, the converse of the above discussion is similar. Now, for the case of $d>1$, one can reread the above proof again, and substitute each occurrence of $\mathbb{R}$ by $\mathbb{R}^d$ to get the result.

In light of Lemma 1, we look forward to convert system (28), which is of order 2α, into its corresponding fractional system of order α, where $0<\alpha\le 1$. To do so, we can rewrite system (28) again as

$$\begin{aligned} D^{2\alpha}\phi_1(r)&=-(\frac{2}{r}D^{\alpha}\phi_1(r)+C_{11}\phi_1(r)+C_{12}\phi_2(r)),\\ D^{2\alpha}\phi_2(r)&=-(\frac{2}{r}D^{\alpha}\phi_2(r)+C_{21}\phi_1(r)+C_{22}\phi_2(r)). \end{aligned} \tag{35}$$

Now, we take the assumptions below into account:

$$\begin{aligned} f_1\left(r,\phi_1(r),D^{\alpha}\phi_1(r),\phi_2(r),D^{\alpha}\phi_2(r)\right)&=-(\frac{2}{r}D^{\alpha}\phi_1(r)+C_{11}\phi_1(r)+C_{12}\phi_2(r)),\\ f_2\left(r,\phi_1(r),D^{\alpha}\phi_1(r),\phi_2(r),D^{\alpha}\phi_2(r)\right)&=-(\frac{2}{r}D^{\alpha}\phi_2(r)+C_{21}\phi_1(r)+C_{22}\phi_2(r)). \end{aligned} \tag{36}$$

Then system (35) becomes

$$\begin{aligned} D^{2\alpha}\phi_1(r)&=f_1\left(r,\phi_1(r),D^{\alpha}\phi_1(r),\phi_2(r),D^{\alpha}\phi_2(r)\right),\\ D^{2\alpha}\phi_2(r)&=f_2\left(r,\phi_1(r),D^{\alpha}\phi_1(r),\phi_2(r),D^{\alpha}\phi_2(r)\right). \end{aligned} \tag{37}$$

Let $u_i(r)=D^{\alpha}\phi_i(r)$, for all $i=1,2$. Then we have

$$\begin{aligned} D^{\alpha}\phi_1(r)&=u_1(r)=g_1\left(r,\phi_1(r),u_1(r),\phi_2(r),u_2(r)\right),\\ D^{\alpha}u_1(r)&=D^{2\alpha}\phi_1(r)=f_1\left(r,\phi_1(r),u_1(r),\phi_2(r),u_2(r)\right),\\ D^{\alpha}\phi_2(r)&=u_2(r)=g_2\left(r,\phi_1(r),u_1(r),\phi_2(r),u_2(r)\right),\\ D^{\alpha}u_2(r)&=D^{2\alpha}\phi_2(r)=f_2\left(r,\phi_1(r),u_1(r),\phi_2(r),u_2(r)\right), \end{aligned} \tag{38}$$

with the following initial conditions:

$$\phi_i(0) = a_i, u_i(0) = b_i, \text{forall } i = 1,2. \tag{39}$$

In what follows, we intend here to consider the spherical radius r as a time domain, as well. Thus, in order to solve the converted system (38) with the help of MFEM [34], we divide the interval $I = [0,T]$ as $0 = r_0 < r_1 = r_0 + h < r_2 = r_0 + 2h < \cdots < r_n = r_0 + nh = T$ such that $r_i = r_0 + ih$ and $h = \frac{T}{n}$, for $i = 1,2$. For simplicity, we denote respectively $g_i(r,\phi_1(r),u_1(r),\phi_2(r),u_2(r))$ and $f_i(r,\phi_1(r),u_1(r),\phi_2(r),u_2(r))$ by $g_i(F)$ and $f_i(F)$, where $F = (r,\phi_1(r),u_1(r),\phi_2(r),u_2(r))$, for all $i = 1,2$. Now, based on the main formula of the MFEM (10), we can obtain the following states:

$$\begin{aligned}
\phi_1(r_{i+1}) = \quad & \phi_1(r_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} g_1(r_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \phi_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_1(F), u_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_1(F), \\
& \phi_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_1(F), u_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_1(F)), \\
u_1(r_{i+1}) = \quad & u_1(r_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1(r_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \phi_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_1(F), u_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_1(F), \\
& \phi_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_1(F), u_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_1(F)), \\
\phi_2(r_{i+1}) = \quad & \phi_2(r_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} g_2(r_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \phi_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_2(F), u_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_2(F), \\
& \phi_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_2(F), u_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_2(F)), \\
u_2(r_{i+1}) = \quad & u_2(r_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_2(r_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \phi_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_2(F), u_1(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_2(F) \\
& \phi_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_2(F), u_2(r_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_2(F)),
\end{aligned} \tag{40}$$

for $i = 1,2$. System (40) represents an approximate solution of system (38) and hence $(\phi_1(t),\phi_2(t),)$ is then the defined solution of system (28).

4. Numerical Experiments

In numerical simulations and mathematical models used to explore physical processes, boundary conditions are a key component. For appropriate findings to be obtained when examining reactor systems, boundary conditions must be accurately represented. Since it has an impact on a number of reactor performance and safety factors, the behavior of the flux near the reactor's surface is of significant interest. It is frequently believed that the flux, which represents the passage of neutrons through the reactor, disappears at the surface. The zero flux boundary condition is frequently used to mimic reactor activity. The zero flux boundary condition aims to mimic the behavior of the real system by assuming zero flux at the surface. In actuality, the flow may continue at the reactor's surface despite declining in magnitude due to phenomena including dispersion and leakage. An extrapolated boundary condition (EBC) can be used to overcome this restriction. The EBC assumes that the flux diminishes at a short distance from the surface in order to account for the behavior of the flux beyond the reactor's surface. The EBC gives a more accurate picture of the reactor system by extrapolating the flux behavior. Physical factors and the unique features of the reactor under study can be used to define the distance over which the flux is supposed to disappear in the EBC. The EBC gives a more

accurate simulation of the real-world problem while the zero flux boundary condition offers simplicity and computing economy.

The purpose of this section is to look at how boundary conditions, namely the zero flux boundary condition and the extrapolated boundary condition, affect the numerical outcomes of reactor simulations. We compare the results produced by various boundary conditions in an effort to determine how these decisions affect the precision and dependability of the simulation results. In what follows, we aim to depict the numerical solutions of the integer- and fractional-order two energy groups neutron diffusion model in spherical reactors by using the LTM and MFEM, respectively. For this purpose, we list the parameters'values in Table 1 and Table 2, which are taken from the reference [35].

Table 1: Data of the two energy groups neutron diffusion model

Fast Energy group		
$\sum_{f_1} = 0.0010484 cm^{-1}$	$\sum_{\gamma_1} = 0.0010046 cm^{-1}$	$\sum_{S11} = 0.62568 cm^{-1}$
$\sum_{S12} = 0.029227 cm^{-1}$	$\upsilon_1 = 2.5$	$\chi_1 = 1.0$
Thermal Energy group		
$\sum_{f_2} = 0.05063 cm^{-1}$	$\sum_{\gamma_2} = 0.025788 cm^{-1}$	$\sum_{S22} = 2.443838 cm^{-1}$
$\sum_{S21} = 0.00000 cm^{-1}$	$\upsilon_2 = 2.5$	$\chi_2 = 0.0$

Table 2: The values of the coefficients C_{ij} calculated based on (12)

C_{ij}	$i=1$	$j=2$
$i=1$	–0.0564834	0.220978
$j=2$	0.249474	–0.577793

The integer-order two energy groups of neutron reactor diffusion system related to the spherical reactor can be redescribed as follows:

$$\begin{aligned} &r\phi_1''(r) + 2\phi_1'(r) + rC_{11}\phi_1(r) + rC_{12}\phi_2(r) = 0, \\ &r\phi_2''(r) + 2\phi_2'(r) + rC_{21}\phi_1(r) + rC_{22}\phi_2(r) = 0, \end{aligned} \tag{41}$$

with initial conditions

$$\phi_1(0) = 2.766976, \phi_2(0) = 1, \phi_1'(0) = 1, \phi_2'(0) = 0. \tag{42}$$

On the other hand, the fractional-order two energy groups of neutron reactor diffusion system related to the spherical reactor can be redescribed in the following manner:

$$\begin{aligned} &rD^{2\alpha}\phi_1(r) + 2D^{\alpha}\phi_1(r) + rC_{11}\phi_1(r) + rC_{12}\phi_2(r) = 0, \\ &rD^{2\alpha}\phi_2(r) + 2D^{\alpha}\phi_2(r) + rC_{21}\phi_1(r) + rC_{22}\phi_2(r) = 0, \end{aligned} \tag{43}$$

with initial conditions

$$\phi_1(0) = 2.766976, \phi_2(0) = 1, D^{\alpha}\phi_1(0) = 0, D^{\alpha}\phi_2(0) = 0. \tag{44}$$

Now, in order to verify the validity of the scheme of fractionalization performed via Lemma 1, we make a numerical comparison between the LTM's solution of system (18) and the MFEM's solution

of system (28) in Figure 1. In light of such figure, it can obviously notice that those two solution are completely coincided, and hence the two energy groups of neutron reactor diffusion model is well simulated when $\alpha = 1$ with the use of MATLAB.

In Figure 1, we make comparisons between MFEM and the exact solution when $\alpha = 1$ for two-energy groups of neutrons reactors diffusions system. In the context of computational simulations, it has been observed that as the mesh is refined, the solution obtained using the (MEFM) converges to the exact solution at every point within the solution domain. This convergence demonstrates the accuracy and reliability of the MEFM as the mesh density increases.

In Figure 2 and Figure 3, we depict respectively two further numerical comparisons for $\phi_1(r)$ and $\phi_2(r)$, which are performed between several MFEM's solutions according to different fractional-order values, i.e. $\alpha = 0.95, 0.975, 1$.

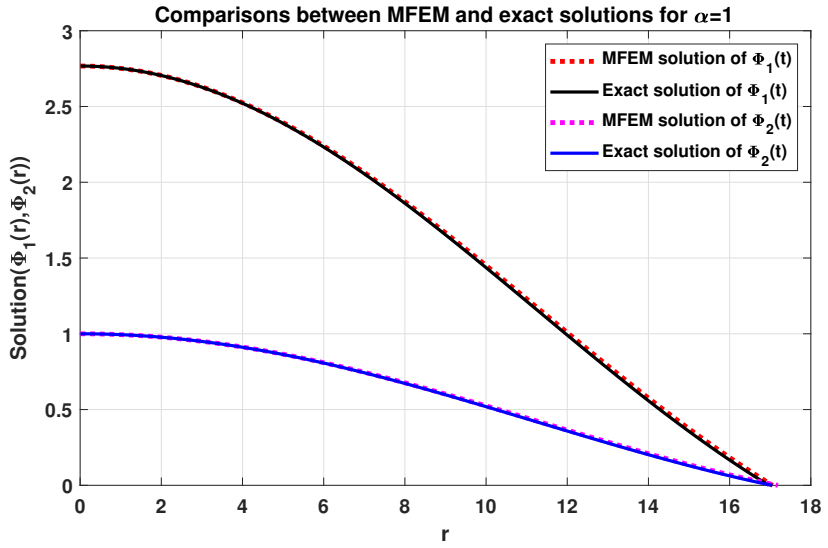

Figure 1: The LTM's solution and the MFEM's solution for the two energy groups of neutrons reactors diffusion system, where the radius r in centimeters.

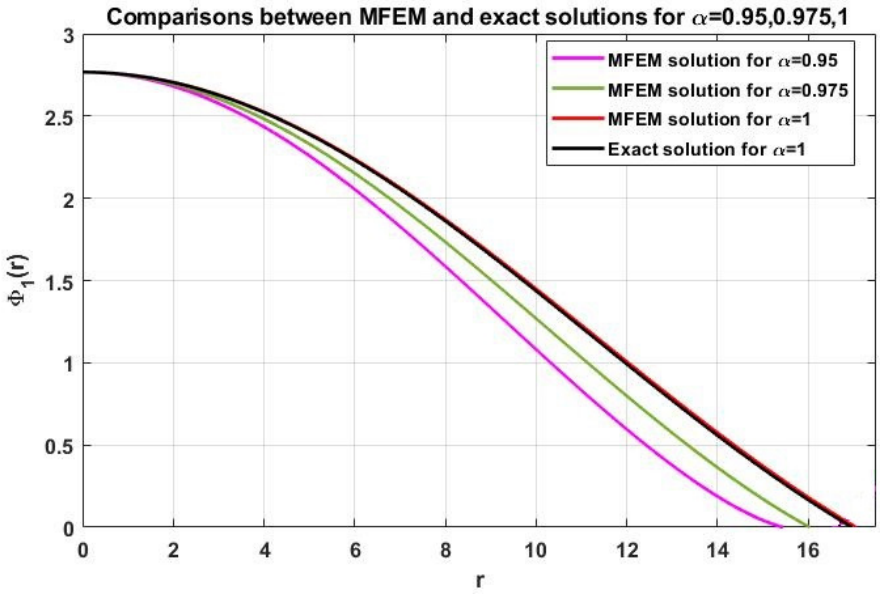

Figure 2: MFEM's numerical solutions for $\phi_1(r)$ according to $\alpha = 0.95, 0.975, 1$, where the radius r in centimeters.

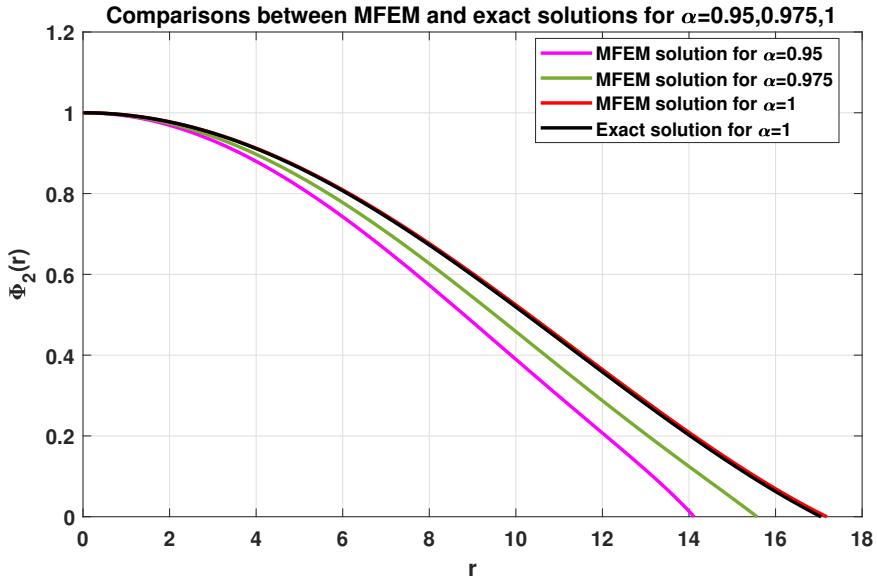

Figure 3: MFEM's numerical solutions for $\phi_2(r)$ according to $\alpha = 0.95, 0.975, 1$, where the radius r in centimeters.

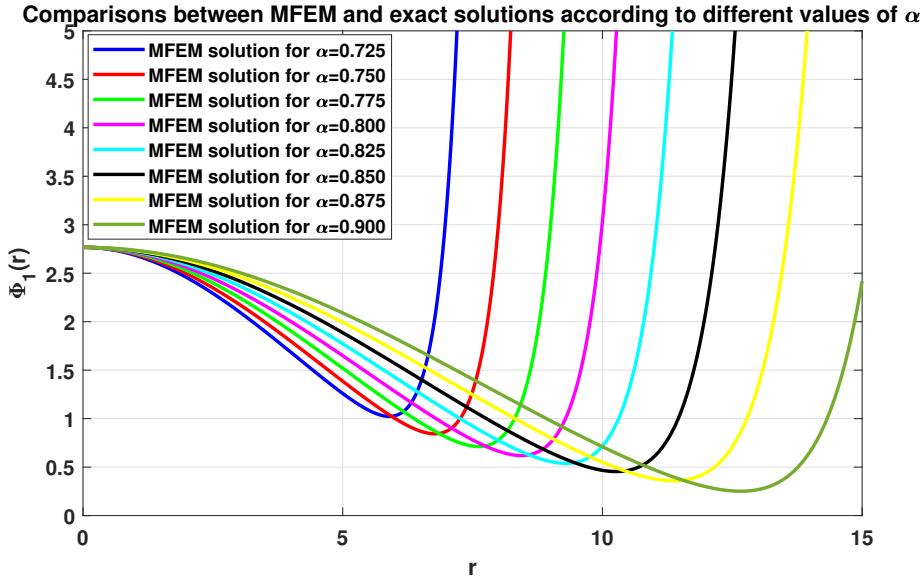

Figure 4: MFEM's solutions for different for $\phi_1(r)$ according to different fractional-order values, where the radius r in centimeters.

In Figure 4, we present a comparison for $\phi_1(r)$ between MFEM and exact solutions when $\alpha = 0.95, 0.975, 1$. In this figure 4, the flux distribution in the system is visually depicted. The highest flux value is observed at the sphere's axis $r = 0$, and it gradually diminishes toward the surface. Notably, it has been observed that the solution obtained using the Modified Equation Fractional Method (MEFM) converges to the exact solution at all points within the solution domain."

For further insights on the MFEM's numerical solutions $(\phi_1(r), \phi_2(r))$ of system (28), we plot respectively in Figure 4 and Figure 5, several MFEM's solutions of such system in accordance with different fractional-order values, i.e. $\alpha = 0.95, 0.975, 1$.

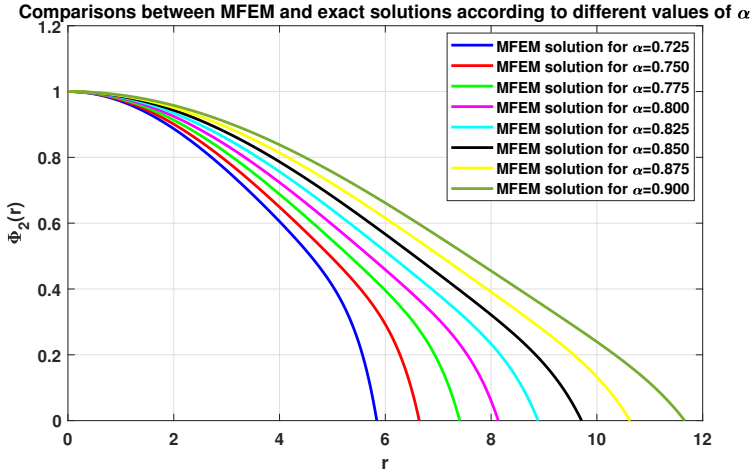

Figure 5: MFEM's solutions for different for $\phi_2(r)$ according to different fractional-order values, where the radius r in centimeters.

5. Conclusion

In this paper, we successfully developed two schemes to solve the integer- and fractional-order two energy group neutron diffusion model in spherical reactors. We solved the integer-order model using the Laplace transform method, and the fractional-order model with the use of the Modified Fractional Euler Method (MFEM). In specifically, we presented a novel method to reduce a 2α-order system to a duplicated α-order system, where $0 < \alpha \leq 1$, in order to address the fractional-order system. We can obviously observe that the solution of the integer-order two energy group neutron diffusion model, which was generated by the Laplace transform method, is completely coincided with the solution of the fractional-order version of the same model when we use the MFEM at $\alpha = 1$. This assertion was verified by performing several numerical comparisons.

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