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Geometric properties of submanifolds of a Riemannian manifold in tangent bundles

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Abstract

The authors consider a quarter-symmetric semi-metric (QSSM) connection in the tangent bundle and study the connection on submanifold of co-dimension 2 and hypersurface concerning the QSSM connection in the tangent bundle. Totally geodesic (TG), totally umbilical (TU), Gauss, Weingarten and Codazzi equations concerning the QSSM connection on submanifold of co-dimension 2 and hypersurface in the tangent bundle are obtained. Finally, we deduce Riemannian curvature tensor, Gauss and Codazzi equations on a submanifold of co-dimension 2 and hypersurface of Riemannian manifold concerning the quarter symmetric semi-metric connection in the tangent bundle.

Key words and phrases. Tangent bundle, Mathematical operators, Induced metric, Connection, Gauss equation, Weingarten equation, Codazzi equation, Curvature tensor, Hypersurface, Submanifold.

Mathematics Subject Classification (2010): 53C03, 53B25, 58A30

1. Introduction

The study of semi-symmetric metric connection on a differentiable manifold M was initiated and developed by Friedmann and Schouten [1] in 1924. It is well known that a linear connection is called a semi-symmetric connection if its torsion tensor T is of the form $T(\mathcal{X}_0, \mathcal{Y}_0) = \omega(\mathcal{Y}_0)\mathcal{X}_0 - \omega(\mathcal{X}_0)\mathcal{Y}_0$,

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where the 1-form ω is definedby $\omega(\mathcal{X}_0) = g(\mathcal{X}_0, U)$ and U is a vector field. A metric connection with non-zero torsion on a Riemannian manifold was introduced by Hayden in 1932 and known as Hayden connection. Later on, Golab [2] introduced the quarter-symmetric metric connection in M with the linear connection ∇ in 1975. A linear connection ∇ is said to be a quarter-symmetric connection if its torsion tensor T satisfies $T(\mathcal{X}_0, \mathcal{Y}_0) = \eta(\mathcal{Y}_0)\phi X_0 - \eta(\mathcal{X}_0)\phi\mathcal{Y}_0$ where $\mathcal{X}_0, \mathcal{Y}_0$ are arbitrary vector fields, η is a 1-form and ϕ is a (1,1) tensor field. A QSSM connection $\bar{\nabla}$ defined by $\nabla_{\mathcal{X}_0}\mathcal{Y}_0 = \nabla_{\mathcal{X}_0}\mathcal{Y}_0 - \eta(\mathcal{X}_0)\mathcal{Y}_0 + g(\phi\mathcal{X}_0, \mathcal{Y}_0)\xi$, where $\mathcal{X}_0, \mathcal{Y}_0$ are arbitrary vector fields, ∇ denotes the Levi-Civita connection concerning Riemannian metric g and ξ the vector field defined by $g(\xi, \mathcal{X}_0) = \eta(\mathcal{X}_0)$. The connections such as symmetric, semi-symmetric, quarter-symmetric non-metric connection have been recently discussed by ([3–15]).

On the other hand, in the foundation of the differentiable geometry of tangent bundles, it is classical to study some geometrical structures and connections deploy natural operations transforming structures and connections on base manifold to its tangent bundle. Tani introduced the notion of prolongations of surfaces to tangent bundle and developed the theory of the surface prolonged to the tangent bundle concerning the metric tensor [16]. Lifts of a semi-symmetric non-metric connection (SSNMC) from statistical manifolds to the tangent bundle studied by Khan et al. [17]. Khan studied the lifts from P-Sasakian and an LP-Sasakian manifold to its tangent bundle associated with a QSM connection in [18] and [19] respectively.

Submanifold theory is an important topic in differential geometry. Gauss Codazzi and Weingarten equations are fundamentals of submanifold theory. We investigate the relation between the connection of the ambient manifold and that of the submanifold in the tangent bundle. Also, We have deduced Weingarten, Gauss and Codazzi equations for submanifold of codimension 2 and hypersurface of a Riemannian manifold with a QSSM connection in the tangent bundle.

The paper is organized as follows. In Section 2, a brief account of tangent bundle, vertical and complete lifts. Section 3 deals with the study of submanifold of codimension 2 and hypersurface concerning QSSM connection in the tangent bundle. Totally geodesic and totally umbilical submanifold of codimension 2 and hypersurface concerning such connection in the tangent bundle are investigated in Section 4. Moreover, We establish Weingarten equations concerning QSSM connection in the tangent bundle in Section 5. Finally, we calculate the Riemannian curvature tensor, Gauss and Codazzi equations for a QSSM connection on a submanifold of codimension 2 and hypersurface in the tangent bundle.

2. Preliminaries

2.1. Vertical and complete lifts

Let TM_n be tangent bundle of n-dimensional differentiable manifold over M_n with the bundle projection $\pi_M n : TM_n \to M_n$. The vertical and complete (V & C) lifts of a function f, a vector field X_0, 1-form ω, (1,1) tensor field F and an affine connection ∇ are $f^V, \mathcal{X}_0^V, \omega^V, F^V, \nabla^C$ and $f^C, X_0^C, \omega^C, F^C, \nabla^V$ correspondingly ([20–22]).

The characteristics of V & C lifts with mathematical operators are presented as ([23], [24])

$$(f_0\mathcal{X}_0)^V = f_0^V\mathcal{X}_0^V, (f_0\mathcal{X}_0)^C = f_0^C\mathcal{X}_0^V + f_0^V\mathcal{X}_0^C, \tag{2.1}$$

$$\mathcal{X}_0^V f_0^V = 0, \mathcal{X}_0^V f_0^C = \mathcal{X}_0^C f_0^V = (\mathcal{X}_0 f_0)^V, \mathcal{X}_0^C f_0^C = (\mathcal{X}_0 f_0)^C, \tag{2.2}$$

$$\omega_0^V(f_0^V) = 0, \omega_0^V(\mathcal{X}_0^C) = \omega_0^C(\mathcal{X}_0^V) = \omega_0(\mathcal{X}_0)^V, \omega_0^C(\mathcal{X}_0^C) = \omega_0(\mathcal{X}_0)^C, \tag{2.3}$$

$$f_0^V\mathcal{X}_0^C = (f_0\mathcal{X}_0)^V, f_0^C\mathcal{X}_0^C = (f_0\mathcal{X}_0)^C, \tag{2.4}$$

$$\mathcal{X}_0 \mathcal{Y}_0^V = \mathcal{X}_0^C \ \mathcal{Y}_0^V = \mathcal{X}_0^V \ \mathcal{Y}_0^C \quad \mathcal{X}_0 \ \mathcal{Y}_0{}^C = \mathcal{X}_0^C \ \mathcal{Y}_0^C \tag{2.5}$$

$$\nabla^C_{\mathcal{X}_0^C} \mathcal{Y}_0^C = (\nabla_{\mathcal{X}_0} \mathcal{Y}_0)^C, \quad \nabla^C_{\mathcal{X}_0^C} \mathcal{Y}_0^V = (\nabla_{\mathcal{X}_0} \mathcal{Y}_0)^V \tag{2.6}$$

2.2. Vertical and complete lifts of $\Im^r_s(M_{n-1}, M_{n+1})$ to TM_{n+1}

If $\bar{f}_0$ is a function on M_{n-1}, the vertical lift $\bar{f}_0^{\bar{V}}$ of $\bar{f}_0$ to TM_{n+1} is given by $\bar{f}_0^{\bar{V}} = \bar{f}_0 \circ \pi M_{n-1}$. Let U be neighborhood of p in M_{n+1}. Then the function $\hat{f}$ fits with $\bar{f}_0$ in $U \cup M_{n+1}$ containing p. The complete lift $\hat{f}^{\hat{C}}$ of $\hat{f}$ is given as $\hat{f}^{\hat{C}} = \mathcal{Y}_0^i \partial_i \hat{f}$ in $\pi^{-1}_{M_{n+1}}(U)$. If $\bar{X}_0$ is an element of $\Im^r_s(M_{n-1}, M_{n+1})$, the vertical lift $\overline{\mathcal{X}_0}^{\bar{V}}$ to TM_{n+1} is defined by $\overline{\mathcal{X}_0}^{\bar{V}} \hat{f}^C = (\overline{\mathcal{X}_0} \hat{f})^{\bar{V}}$ and complete lift $\overline{\mathcal{X}_0}^{\bar{C}}$ to TM_{n+1} is defined as $\overline{\mathcal{X}_0}^{\bar{C}} \hat{f}^C = (\overline{\mathcal{X}_0} \hat{f})^{\bar{C}}$, for each $\hat{f} \in \Im^0_0(M_{n+1})$ along M_{n-1}. Similarly, If $\bar{\omega}_0$ is an element of $\Im^0_1(M_{n-1}, M_{n+1})$. The vertical lift $\bar{\omega}_0^{\bar{V}}$ and complete lift $\bar{\omega}_0^{\bar{C}}$ to TM_{n+1} are defined by $\bar{\omega}_0^{\bar{V}}(\overline{\mathcal{X}_0}^{\bar{C}}) = (\bar{\omega}_0(\overline{\mathcal{X}_0}))^{\bar{V}}$ and $\bar{\omega}_0^{\bar{C}}(\overline{\mathcal{X}_0}^{\bar{C}}) = (\bar{\omega}_0(\overline{\mathcal{X}_0}))^{\bar{C}}$ for each $\overline{\mathcal{X}_0} \in \Im^0_1(M_{n+1})$ respectively ([19], [25], [26], [27]).

2.3. Submanifold of codimension 2

Let M_{n+1} (dim=n + 1) be a differentiable manifold and M_{n-1} (dim=n − 1) submanifold submerged in M_{n+1} by mapping $\tau : M_{n-1} \to M_{n+1}$. The differentaibility $d\tau$ of the submerged τ is shownby B ([28–29]). Assume that the Riemannian manifold M_{n+1} has a metric tensor of $\tilde{g}$. In such case, the submanifold M_{n-1} likewise has a metric tensor g, making it a Riemannian manifold such that

$$g(\phi \mathcal{X}_0, \mathcal{Y}_0) = \tilde{g}(B\phi \mathcal{X}_0, \tilde{B} \mathcal{Y}_0), \tag{3.1}$$

for all $\mathcal{X}_0$, $\mathcal{Y}_0$ in M_{n-1}.

If M_{n-1} and M_{n+1} are orientable, then mutually orthogonal unit normals N_1 and N_2 defined along M_{n-1} such that

$$\tilde{g}(B\phi \mathcal{X}_0, N_1) = \tilde{g}(B\phi \mathcal{X}_0, N_2) = \tilde{g}(N_1, N_2)$$

$$\tilde{g}(N_1, N_1) = \tilde{g}(N_2, N_2) = 0 \tag{3.2}$$

forall $\mathcal{X}_0$ in M_{n-1}.

A QSSM connection $\tilde{\nabla}$on manifold M_{n+1} provided by ([18], [30], [31])

$$\tilde{\nabla}_{\widetilde{\mathcal{X}_0}} \widetilde{\mathcal{Y}_0} = \dot{\tilde{\nabla}}_{\widetilde{\mathcal{X}_0}} \widetilde{\mathcal{Y}_0} - \tilde{\eta}(\widetilde{\mathcal{X}_0}) \tilde{\phi} \widetilde{\mathcal{Y}_0} + \tilde{g}(\tilde{\phi} \widetilde{\mathcal{X}_0}, \widetilde{\mathcal{Y}_0}) \tilde{P}, \tag{3.3}$$

where $\tilde{\nabla}$be Levi-Civita connection concerning to the Riemannian metric $\tilde{g}$, $\tilde{\eta}$ is a 1-form, $\tilde{\phi}$ is a tensor of type (1,1) such that $\tilde{g}(\tilde{\phi}\tilde{\mathcal{X}}0, \tilde{\mathcal{Y}}0) = \tilde{g}(\tilde{\mathcal{X}}_0, \tilde{\phi}\tilde{\mathcal{Y}}_0)$ and the vector field $\tilde{P}$ given by $\tilde{g}(\tilde{P}, \tilde{\mathcal{X}}_0) = \tilde{\eta}(\tilde{\mathcal{X}}_0)$.

Let us put

$$\tilde{P} = BP + \lambda N_1 + \mu N_2, \tag{3.4}$$

P is a vector field in the tangent space and λ and μ functions of M_{n-1}.

Let $\dot{\nabla}$ Riemannian connection induced on M_{n-1} form $\dot{\nabla}$ on the enveloping manifold wrt normals N_1 and N_2, then we infer

$$\dot{\tilde{\nabla}}_{B\mathcal{X}_0} B\mathcal{Y}_0 = B(\dot{\nabla}_{\mathcal{X}_0} \mathcal{Y}_0) + h(\mathcal{X}_0, \mathcal{Y}_0) N_1 + k(\mathcal{X}_0, \mathcal{Y}_0) N_2, \tag{3.5}$$

where for all $\mathcal{X}_0$, $\mathcal{Y}_0$ in M_{n-1}, h and k denote II fundamental tensors of M_{n-1}. In the same way, if the connection ∇ be induced on M_{n-1} from the QSSM connection $\tilde{\nabla}$ on M_{n-1}, we infer

$$\tilde{\nabla}_{B\mathcal{X}_0} B\mathcal{Y}_0 = B(\nabla_{\mathcal{X}_0}\mathcal{Y}_0) + m(\mathcal{X}_0,\mathcal{Y}_0)N_1 + n(\mathcal{X}_0,\mathcal{Y}_0)N_2, \tag{3.6}$$

m and n are (0,2) tensorfields of M_{n-1}([32], [33], [34]).

Let TM_{n-1} and TM_{n+1} be the tangent bundles of Riemannian manifolds M_{n-1} and M_{n-1} respectively. Let $\tilde{g}^C$ be the complete lift of a Riemannian metric $\tilde{g}$ in TM_{n-1} and g^C induced metric from $\tilde{g}^C$ such that

$$g^C((\phi\mathcal{X}_0)^C, \mathcal{Y}_0^C) = \tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C, \tilde{B}\mathcal{Y}_0^C), \tag{3.7}$$

for all $\mathcal{X}_0^C$, $\mathcal{Y}_0^C$ in TM_{n-1}.

Operating complete liftby mathematical operators on both sides of the equation (3.2), we get

$$\begin{aligned}
\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C, N_1^{\bar{C}}) &= \tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C, N_1^{\bar{V}}) = 0,\\
\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C, N_2^{\bar{C}}) &= \tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C, N_2^{\bar{V}}) = 0,\\
\tilde{g}^C(N_1^{\bar{C}}, N_1^{\bar{C}}) &= \tilde{g}^C(N_1^{\bar{V}}, N_1^{\bar{V}}) = 0,\\
\tilde{g}^C(N_2^{\bar{C}}, N_2^{\bar{C}}) &= \tilde{g}^C(N_2^{\bar{V}}, N_2^{\bar{V}}) = 0,\\
\tilde{g}^C(N_1^{\bar{C}}, N_2^{\bar{C}}) &= \tilde{g}^C(N_1^{\bar{V}}, N_2^{\bar{C}}) = 0,\\
\tilde{g}^C(N_1^{\bar{V}}, N_1^{\bar{C}}) &= \tilde{g}^C(N_2^{\bar{V}}, N_2^{\bar{C}}) = 1,
\end{aligned} \tag{3.8}$$

where $N_1^{\bar{V}}, N_1^{\bar{C}}, N_2^{\bar{V}}$ and $N_2^{\bar{C}}$ are V & C lifts of N_1 and N_2, accordingly along with submanifold TM_{n-1}.

Operating complete liftby mathematical operators on both sides of the equations (3.3) and (3.4), we get

$$\begin{aligned}
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C = \tilde{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C - (\tilde{\eta}(B\mathcal{X}_0)(B\phi\mathcal{Y}_0))^C + (\tilde{g}(B\phi\mathcal{X}_0, B\mathcal{Y}_0)\tilde{P})^C\\
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C = \tilde{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C - (\tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V) - (\tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C)\\
&+(\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^C, \tilde{B}\mathcal{Y}_0^C)\tilde{P}^V + (\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^V, \tilde{B}\mathcal{Y}_0^C)\tilde{P}^C
\end{aligned} \tag{3.9}$$

for all $\mathcal{X}_0^C$, $\mathcal{Y}_0^C$ in TM_{n-1}, where $\tilde{\tilde{\nabla}}^C$ denotes complete lift of $\tilde{\tilde{\nabla}}$ wrt $\tilde{g}^C$ determined by $\tilde{g}^C(\tilde{P}^C, \tilde{\mathcal{X}}_0^C) = (\tilde{\eta}(\tilde{\mathcal{X}}))^C$ where $\tilde{\eta}^C$, $\tilde{\phi}^C$, $\tilde{P}^C$ are complete lifts of form η, (1,1) tensorfield ϕ and vector field $\tilde{P}$.

$$\begin{aligned}
\tilde{P}^C &= \tilde{B}P^C + \lambda N_1^{\bar{C}} + \mu N_2^{\bar{C}},\\
\tilde{P}^V &= \tilde{B}P^V + \lambda N_1^{\bar{V}} + \mu N_2^{\bar{V}},
\end{aligned} \tag{3.10}$$

where P is a vector field and λ and μ are functions of M_{n-1}. Now, we are going the prove the following theorem:

Theorem 3.1 *The connection $\dot{\tilde{\nabla}}^C$ induced on the submanifold T (M_{n-1}) from $\dot{\tilde{\nabla}}^C$ of a Riemannian manifold with a QSSM connection is also a QSSM connection.*

Proof: Let $\dot{\tilde{\nabla}}^C$ be the induced connection from $\dot{\tilde{\nabla}}^C$ on the submanifold T (M_{n-1}) from the connection $\dot{\tilde{\nabla}}^C$ on the enveloping manifold concerning the unit normals N_1 and N_2 whose complete and vertical lifts are $N_1^{\bar{C}}$, $N_1^{\bar{V}}$, $N_2^{\bar{C}}$ and $N_2^{\bar{V}}$ respectively.

Operating complete lift with mathematical operators on both sides of equation (3.5), we obtain

$$\begin{aligned}
\dot{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C = B(\dot{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + h^C(\mathcal{X}_0^C, \mathcal{Y}_0^C)N_1^{\bar{V}} + h^V(\mathcal{X}_0^C, \mathcal{Y}_0^C)N_1^{\bar{C}}\\
+k^C(\mathcal{X}_0^C, \mathcal{Y}_0^C)N_2^{\bar{V}} + k^V(\mathcal{X}_0^C, \mathcal{Y}_0^C)N_2^{\bar{C}},
\end{aligned} \tag{3.11}$$

where h^V, h^C, k^V and k^C are V & C lifts of II fundamental tensors h and k respectively of M_{n-1}.

In the same way, if ∇^C be connection induced on $T(M_{n-1})$ from the QSSM connection $\tilde{\nabla}$ C on $T(M_{n-1})$, we have

$$\begin{aligned}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C &= B(\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + m^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_1^{\bar{V}} + m^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_1^{\bar{C}} \\ &+ n^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_2^{\bar{V}} + n^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_2^{\bar{C}},\end{aligned} \tag{3.12}$$

where m^V, m^C, n^V and n^C are V & C lifts of II fundamental tensors m and n respectively of M_{n-1}.

In the view of equations (3.9), (3.10), (3.11) and (3.12), we have

$$\begin{aligned}B(\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) &+ m^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_1^{\bar{V}} + m^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_1^{\bar{C}} \\ &+ n^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_2^{\bar{V}} + n^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_2^{\bar{C}} \\ &= B(\dot{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + h^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_1^{\bar{V}} + h^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_1^{\bar{C}} \\ &+ k^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_2^{\bar{V}} + k^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N_2^{\bar{C}} \\ &- (\tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V) - (\tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C) \\ &+ (\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C)(\tilde{B}P^V + \lambda N_1^{\bar{V}} + \mu N_2^{\bar{V}}) \\ &+ (\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C)(\tilde{B}P^C + \lambda N_1^{\bar{C}} + \mu N_2^{\bar{C}}).\end{aligned} \tag{3.13}$$

Comparison of tangential and normal vector fields, we get

$$\begin{aligned}\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C &= \dot{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C - \tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V - \tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C \\ &+ \tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C)P^V + \tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C)P^C,\end{aligned}$$

where λ and μ are choosen such that

$$\begin{aligned}m^C(\mathcal{X}_0^C,\mathcal{Y}_0^C) &= h^C(\mathcal{X}_0^C,\mathcal{Y}_0^C) + \lambda\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C), \\ m^V(\mathcal{X}_0^C,\mathcal{Y}_0^C) &= h^V(\mathcal{X}_0^C,\mathcal{Y}_0^C) + \lambda\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C), \\ n^C(\mathcal{X}_0^C,\mathcal{Y}_0^C) &= k^C(\mathcal{X}_0^C,\mathcal{Y}_0^C) + \mu\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C), \\ n^V(\mathcal{X}_0^C,\mathcal{Y}_0^C) &= k^V(\mathcal{X}_0^C,\mathcal{Y}_0^C) + \mu\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C).\end{aligned} \tag{3.14}$$

Thus,

$$\begin{aligned}\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C - \nabla^C_{\mathcal{Y}_0^C}\mathcal{X}_0^C - [\mathcal{X}_0^C,\mathcal{Y}_0^C] &= -\tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V \\ &- \tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C \\ &+ \tilde{\eta}^C(\tilde{B}\mathcal{Y}_0^C)(\tilde{B}\phi\mathcal{X}_0)^V \\ &+ \tilde{\eta}^V(\tilde{B}\mathcal{Y}_0^C)(\tilde{B}\phi\mathcal{X}_0)^C.\end{aligned} \tag{3.15}$$

Hence, ∇^C induced on TM_{n-1} is the QSSM connection. Hence the proof is completed.

Let M_{n+1} (dim=$(n + 1)$) be a differentiable manifold and M_n be hypersurface in M_{n+1} by mapping $\tau : M_{n+1} \rightarrow M_n$ and by B the mapping induced by τ from $T(M_n)$ to $T(M_{n+1})$, where $T(M_n)$ and $T(M_{n+1})$ denote tangent bundles of manifold M_n and M_{n+1} respectively.

As an immediate consequence of the above theorem, we have the following corollary:

Corollary 3.1 *The connection induced on the hypersurface TM_n from of a Riemannian manifold with a QSSM connection concerning the unit normals $N^{\bar{C}}$ and $N^{\bar{V}}$ is also a QSSM connection.*

Proof: Let $\dot{\nabla}^C$ be the induced connection from $\dot{\tilde{\nabla}}^C$ on the hypersurface TM_n concerning the unit normal N whose complete and vertical lifts are $N^{\bar{C}}$ $N^{\bar{V}}$. Then we have,
and

$$\dot{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C = B(\dot{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + h^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{V}} + h^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{C}}, \tag{3.16}$$

where for all $\mathcal{X}_0^C$, $\mathcal{Y}_0^C$ on TM_n and h is the II fundamental tensor of the hypersurface M_n whose C & V lifts are h^C and h^V respectively on $T(M_n)$.

Let ∇^C be connection induced on hypersurface from $\tilde{\nabla}^C$ concerning the unit normal N whose C & V lifts are $N^{\bar{C}}$ and $N^{\bar{V}}$.

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C = B(\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + m^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{V}} + m^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{C}} \tag{3.17}$$

where m^C and m^V are complete and vertical lifts of (0,2) tensor field m on M_n.

From equation (3.9), we have

$$\begin{aligned} B(\tilde{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) &= \dot{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Y}_0^C \\ &-(\hat{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V) - (\hat{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C)n \\ &+(\hat{g}^C(\tilde{B}\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C)\tilde{P}^V + (\hat{g}^C(\tilde{B}\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C)\tilde{P}^C. \end{aligned} \tag{3.18}$$

In view of equations (3.16) and (3.17) in the above equation, we get

$$\begin{aligned} &B(\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + m^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{V}} + m^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{C}} \\ &= B(\dot{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + h^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{V}} + h^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{C}} \\ &-(\tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V) - (\tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C) \\ &+(\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C)\tilde{P}^V + (\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C)\tilde{P}^C. \end{aligned} \tag{3.19}$$

Making use of equation (3.10) in equation (3.19), we get

$$\begin{aligned} &B(\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + m^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{V}} + m^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{C}} \\ &= B(\dot{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C) + h^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{V}} + h^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)N^{\bar{C}} \\ &-(\tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V) - (\tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C) \\ &+(\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C)(\tilde{B}P^V + \lambda N^{\bar{V}}) \\ &+(\tilde{g}^C(\tilde{B}\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C)(\tilde{B}P^C + \lambda N^{\bar{C}}). \end{aligned} \tag{3.20}$$

Comparison of tangential and normal vector fields, we get

$$\begin{aligned} \nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C &= \dot{\nabla}^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C - \tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V - \tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C \\ &+\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C)P^V + \tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C)P^C \end{aligned}$$

$$\begin{aligned} m^C(\mathcal{X}_0^C,\mathcal{Y}_0^C) &= h^C(\mathcal{X}_0^C,\mathcal{Y}_0^C) + \lambda\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^C,\tilde{B}\mathcal{Y}_0^C)P^V \\ m^V(\mathcal{X}_0^C,\mathcal{Y}_0^C) &= h^V(\mathcal{X}_0^C,\mathcal{Y}_0^C) + \lambda\tilde{g}^C(\tilde{B}(\phi\mathcal{X}_0)^V,\tilde{B}\mathcal{Y}_0^C)P^C. \end{aligned}$$

Thus,

$$\nabla^C_{\mathcal{X}_0^C}\mathcal{Y}_0^C - \nabla^C_{\mathcal{Y}_0^C}\mathcal{X}_0^C - [\mathcal{X}_0^C,\mathcal{Y}_0^C] = -\tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^V \\ -\tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\tilde{B}\phi\mathcal{Y}_0)^C \\ +\tilde{\eta}^C(\tilde{B}\mathcal{Y}_0^C)(\tilde{B}\phi\mathcal{X}_0)^V \\ +\tilde{\eta}^V(\tilde{B}\mathcal{Y}_0^C)(\tilde{B}\phi\mathcal{X}_0)^C. \tag{3.21}$$

Hence, ∇^C induced on M_n is QSSM connection. Thus, the proof is completed.

4. Applications

Let $e_1, e_2, \ldots.., e_{n-1}$ be $(n-1)$-orthonormal vector fields on the submanifold M_{n-1}. Then the function

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}[h(e_i,e_i)+k(e_i,e_i)]$$

is mean curvature of M_{n-1} wrt $\dot{\nabla}$ and

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}[m(e_i,e_i)+n(e_i,e_i)]$$

is mean curvature of M_{n-1} wrt ∇.

Definition 4.1. If h and k are zero, M_{n-1} is said to be TG wrt the Riemannian connection $\dot{\nabla}$.

Definition 4.2. M_{n-1} is said to be TU wrt $\dot{\nabla}$ if h and k are propotional to g.

Now, we call TM_{n-1} is TG and TU wrt the QSSM connection ∇^C for m^C, m^V, n^C and n^V are zero individually and are proportional to g^C respectively.

Theorem 4.1. *In order that the mean curvature of TM_{n-1} wrt the connection $\dot{\nabla}^C$ may coincide with that of TM_{n-1} wrt the connection ∇^C it is necessary and sufficient that $\tilde{P}^C$ and $\tilde{P}^V$ are in the tangent space of TM_{n+1}.*

Proof: In the view of equations (3.14), we have

$$m^C(e_i^C,e_i^C)+n^C(e_i^C,e_i^C)=h^C(e_i^C,e_i^C)+k^C(e_i^C,e_i^C)+(\lambda+\mu)\tilde{g}^C(\tilde{B}(\phi e_i)^C,\tilde{B}e_i^C)$$
$$m^V(e_i^C,e_i^C)+n^V(e_i^C,e_i^C)=h^V(e_i^C,e_i^C)+k^V(e_i^C,e_i^C)+(\lambda+\mu)\tilde{g}^C(\tilde{B}(\phi e_i)^V,\tilde{B}e_i^C).$$

Summing up for $i = 1, 2, \ldots, (n-1)$ and dividingby $2(n-1)$, we get

$$m^C(e_i^C,e_i^C)+n^C(e_i^C,e_i^C)=h^C(e_i^C,e_i^C)+k^C(e_i^C,e_i^C)$$
$$m^V(e_i^C,e_i^C)+n^V(e_i^C,e_i^C)=h^V(e_i^C,e_i^C)+k^V(e_i^C,e_i^C)$$

iff $\lambda = \mu = 0$.

In the view of equation (3.10), it follows that $\tilde{P}C = BP^C$ and $\tilde{P}V = BP^V$. Thus the vector fields $\tilde{P}^C$ and $\tilde{P}^V$ are in the tangent space of TM_{n-1}. Hence, the proof is completed.

Theorem 4.2. *The submanifold TM_{n-1} is TU wrt the Riemannian connection $\dot{\tilde{\nabla}}^C$ iff it is TU wrt the QSSM connection ∇^C.*

Proof: From equation (3.14), the proof is simply obtained.

As an immediate consequence of theorem 4.1 and theorem 4.2, we have the following corollaries on hypersurface:

Corollary 4.1. *In order that the mean curvature of TM_{n-1} wrt the connection $\dot{\nabla}^C$ may coincide with that of TM_{n-1} wrt the connection ∇^C it is necessary and sufficient that $\tilde{P}C$ and $\tilde{P}V$ are in the tangent space of TM_{n+1}.*

Proof: The proof is trivial.

Corollary 4.2. *The hypersurface TM_n is TU wrt the Riemannian connection $\dot{\nabla}^C$ iff it is TU wrt the QSSM ∇^C.*

Proof: The proof is trivial.

5. Weingarten equations for the QSSM connection in the tangent bundle

In this section, Weingarten Equations concerning the QSSM connection $\tilde{\nabla}^C$ on the submanifold TM_{n-1} in TM_{n+1} are investigated.

The Weingarten equations for $\dot{\nabla}^C$ are presented by

$$\begin{aligned}
&(a)\dot{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_1^{\bar{V}} = -\tilde{B}H^V\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_2^{\bar{V}},\\
&(b)\dot{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_1^{\bar{C}} = -\tilde{B}H^C\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_2^{\bar{C}},\\
&(c)\dot{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_2^{\bar{V}} = -\tilde{B}K^V\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_1^{\bar{V}},\\
&(d)\dot{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_2^{\bar{C}} = -\tilde{B}K^C\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_1^{\bar{C}},
\end{aligned} \tag{5.1}$$

where H^C, H^V, K^C and K^V are complete and vertical lifts of tensor fields H and K of type (1,1) such that

$$\begin{aligned}
&(a)\tilde{g}^C(H^C\mathcal{X}_0^C,\mathcal{Y}_0^C) = h^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)\\
&(b)\tilde{g}^C(K^C\mathcal{X}_0^C,\mathcal{Y}_0^C) = k^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)\\
&(c)\tilde{g}^V(H^V\mathcal{X}_0^C,\mathcal{Y}_0^C) = h^V(\mathcal{X}_0^C,\mathcal{Y}_0^C)\\
&(d)\tilde{g}^V(K^V\mathcal{X}_0^C,\mathcal{Y}_0^C) = k^V(\mathcal{X}_0^C,\mathcal{Y}_0^C).
\end{aligned}$$

Making use of (3.3) and (5.1a), we get

$$\begin{aligned}
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_1^{\bar{C}} = -\tilde{B}H^C\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_2^{\bar{C}} + \tilde{B}(\eta^C(\mathcal{X}_0^C)\xi_1^V + \eta^V(\mathcal{X}_0^C)\xi_1^C)\\
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_1^{\bar{C}} = -\tilde{B}M_1\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_2^{\bar{C}},
\end{aligned} \tag{5.2}$$

where $M_1\mathcal{X}_0^C = H^C\mathcal{X}_0^C - \eta^C(\mathcal{X}_0^C)\xi_1^V - \eta^V(\mathcal{X}_0^C)\xi_1^C$

$$\begin{aligned}
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_2^{\bar{C}} = -\tilde{B}K^C\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_1^{\bar{C}} + \tilde{B}(\eta^C(\mathcal{X}_0^C)\xi_2^V + \eta^V(\mathcal{X}_0^C)\xi_2^C)\\
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_2^{\bar{C}} = -\tilde{B}M\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_1^{\bar{C}},
\end{aligned} \tag{5.3}$$

where $M_2\mathcal{X}_0^C = K^C\mathcal{X}_0^C - \eta^C(\mathcal{X}_0^C)\xi_2^V - \eta^V(\mathcal{X}_0^C)\xi_2^C$.

Similarly,

$$\begin{aligned}
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_1^{\bar{V}} = -\tilde{B}H^V\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_2^{\bar{V}} + \tilde{B}(\eta^C(\mathcal{X}_0^C)\xi_1^V\\
&\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_1^{\bar{V}} = -\tilde{B}M_1\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_2^{\bar{V}},
\end{aligned} \tag{5.4}$$

where $M^V\mathcal{X}_0^C = H^V\mathcal{X}_0^C - \eta^C(\mathcal{X}_0^C)\xi_1^V$

$$\begin{aligned}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_2^{\bar{V}} &= -\tilde{B}K^V\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_1^{\bar{V}} + \tilde{B}(\eta^C(\mathcal{X}_0^C)\xi_1^V \\ \tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N_2^{\bar{V}} &= -\tilde{B}M_2\mathcal{X}_0^C + l(\mathcal{X}_0^C)N_1^{\bar{V}}.\end{aligned} \tag{5.5}$$

where $M_2^V\mathcal{X}_0^C = H^V\mathcal{X}_0^C - \eta^C(\mathcal{X}_0^C)\xi_1{}^V$.

The equations (5.2), (5.3), (5.4) and (5.5) are Weingarten equations concerning the QSSM connection in the tangent bundle. We have the following theorem:

Theorem 5.1. *The connection $\dot{\nabla}^C$ induced on the submanifold $T(M_{n-1})$ from $\dot{\tilde{\nabla}}^C$ of a Riemannian manifold with a QSSM connection. The Weingarten equations concerning the QSSM connection are given by (5.2), (5.3), (5.4) and (5.5).*

As an immediate consequence of the above theorem, we have the following corollary on hypersurface:

Corollary 5.1 *The connection induced on TM_n from the Riemannian manifold concerning the QSSM connection $\dot{\tilde{\nabla}}^C$. The Weingarten equations concerning the QSSM connection $\tilde{\nabla}^C$ on TM_n in TM_{n+1} are given by (5.11) and (5.13).*

Proof: The Weingarten equations are given in the following form

$$\begin{aligned}\dot{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{V}} &= -\tilde{B}H^V\mathcal{X}_0^C \\ \dot{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} &= -\tilde{B}H^C\mathcal{X}_0^C\end{aligned} \tag{5.6}$$

where for all $\mathcal{X}_0$, $\mathcal{Y}_0$ on M_n and H is a (1,1)-tensor field of M_n defined by

$$\tilde{g}^C(H^C\mathcal{X}_0^C, \mathcal{Y}_0^C) = h^C(\mathcal{X}_0^C, \mathcal{Y}_0^C) \tag{5.7}$$

$$\tilde{g}^V(H^V\mathcal{X}_0^C, \mathcal{Y}_0^C) = h^V(\mathcal{X}_0^C, \mathcal{Y}_0^C) \tag{5.8}$$

In the view of equation (3.3), we have

$$\begin{aligned}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} &= \dot{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} - \tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\phi N)^{\bar{V}} - \tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\phi N)^{\bar{C}} \\ &= \dot{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} - \tilde{\eta}^C(\tilde{B}\mathcal{X}_0^C)(\phi N)^{\bar{V}} - \tilde{\eta}^V(\tilde{B}\mathcal{X}_0^C)(\phi N)^{\bar{C}}.\end{aligned} \tag{5.9}$$

Put $\phi N = -B\xi$, where ξ is a vector field on M_n.

$$\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} = \dot{\tilde{\nabla}}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} + \tilde{B}((\eta^C\mathcal{X}_0^C)B\xi^V + (\eta^V\mathcal{X}_0^C)B\xi^C) \tag{5.10}$$

Making use of equation (5.7) in equation (5.10), we get

$$\begin{aligned}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} &= -\tilde{B}H^C\mathcal{X}_0^C + \tilde{B}(\eta^C(\mathcal{X}_0^C)\xi^V + \eta^V(\mathcal{X}_0^C)\xi^C) \\ \tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C} N^{\bar{C}} &= -\tilde{B}M^C\mathcal{X}_0^C,\end{aligned} \tag{5.11}$$

where

$$M\mathcal{X}_0^C = H^C\mathcal{X}_0^C - \eta^C(\mathcal{X}_0^C)\xi^V - \eta^V(\mathcal{X}_0^C)\xi^C$$

for arbitrary vector field X_0 on M_n.

Similarly,

$$\tilde{\nabla}^{C}_{\tilde{B}\mathcal{X}_0^C} N^{\bar{V}} = \dot{\tilde{\nabla}}^{C}_{\tilde{B}\mathcal{X}_0^C} N^{\bar{V}} + \tilde{B}(\eta^C \mathcal{X}_0^C)\xi^V \tag{5.12}$$

Making use of equation (5.7) in equation (5.12), we get

$$\begin{aligned}\tilde{\nabla}^{C}_{\tilde{B}\mathcal{X}_0^C} N^{\bar{V}} &= -\tilde{B}H^V \mathcal{X}_0^C + \tilde{B}(\eta^C(\mathcal{X}_0^C)\xi^V \\ &= -\tilde{B}M^V \mathcal{X}_0^C,\end{aligned} \tag{5.13}$$

where

$$M^V \mathcal{X}_0^C = H^V \mathcal{X}_0^C - \eta^C(\mathcal{X}_0^C)\xi^V.$$

The equations (5.11) and (5.13) are Weingarten equations concerning the QSSM connection on TM_n in TM_{n+1}. Hence, the proof of corollary is completed.

6. Riemannian curvature tensor and Gauss and Codazzi equations for the QSSM connection in the tangent bundle

This section deals with the study of Riemannian curvature and equations of Gauss and Codazzi concerning the QSSM connection on TM_{n-1} in TM_{n+1}.

Let $\tilde{K}^C$ and K^C be the curvature tensors of TM_n and TM_{n+1} concerning $\dot{\tilde{\nabla}}^C$ and $\dot{\nabla}^C$ respectively. Thus

$$\begin{aligned}\tilde{K}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C &= \dot{\tilde{\nabla}}^{C}_{\tilde{B}\mathcal{X}_0^C} \dot{\tilde{\nabla}}^{C}_{\tilde{B}\mathcal{Y}_0^C} \tilde{B}\mathcal{Z}_0^C - \dot{\tilde{\nabla}}^{C}_{\tilde{B}\mathcal{Y}_0^C} \dot{\tilde{\nabla}}^{C}_{\tilde{B}\mathcal{X}_0^C} \tilde{B}\mathcal{Z}_0^C \\ &-\dot{\tilde{\nabla}}^{C}_{[\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C]} \tilde{B}\mathcal{Z}_0^C\end{aligned} \tag{6.1}$$

and

$$K^C(\mathcal{X}_0^C, \mathcal{Y}_0^C)\mathcal{Z}_0^C = \nabla^C_{\mathcal{X}_0^C}\nabla^C_{\mathcal{Y}_0^C}\mathcal{Z}_0^C - \nabla^C_{\mathcal{Y}_0^C}\nabla^C_{\mathcal{X}_0^C}\mathcal{Z}_0^C - \nabla^C_{[\mathcal{X}_0,\mathcal{Y}_0]^C}\mathcal{Z}_0^C \tag{6.2}$$

Then the equation of Gauss is given by

$$\begin{aligned}\tilde{K}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C, \tilde{B}\mathcal{Z}_0^C, \tilde{B}U^C) = K^C&(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C, \tilde{B}\mathcal{Z}_0^C, \tilde{B}U^C) \\ &+h^V(\mathcal{X}_0^C, U^C)h^C(\mathcal{Y}_0^C, \mathcal{Z}_0^C) \\ &+h^C(\mathcal{X}_0^C, U^C)h^V(\mathcal{Y}_0^C, \mathcal{Z}_0^C) \\ &-h^V(\mathcal{Y}_0^C, U^C)h^C(\mathcal{X}_0^C, \mathcal{Z}_0^C) \\ &-h^C(\mathcal{Y}_0^C, U^C)h^V(\mathcal{X}_0^C, \mathcal{Z}_0^C).\end{aligned}$$

where $\tilde{K}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C, \tilde{B}\mathcal{Z}_0^C, \tilde{B}U_0^C) = \tilde{g}^C(\tilde{K}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C, \tilde{B}\mathcal{Z}_0^C, \tilde{B}U_0^C)$ and the similar expression for $K^C(\mathcal{X}_0^C, \mathcal{Y}_0^C, \mathcal{Z}_0^C, U_0^C)$ for M_{n+1}.

The equation of Codazzi is given by

$$\begin{aligned}&\tilde{K}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)N^{\bar{V}} = \tilde{B}(\dot{\nabla}^C_{\mathcal{X}_0^C} H^V \mathcal{Y}_0^C - \dot{\nabla}^C_{\mathcal{X}_0^C} H^V \mathcal{X}_0^C) \\ &\tilde{K}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)N^{\bar{C}} = \tilde{B}(\dot{\nabla}^C_{\mathcal{X}_0^C} H^C \mathcal{Y}_0^C - \dot{\nabla}^C_{\mathcal{X}_0^C} H^C \mathcal{X}_0^C) \\ &\tilde{K}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}\mathcal{X}_0^C = 0.\end{aligned}$$

Let $\tilde{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C$ be the Riemannian curvature tensor field of the enveloping manifold TM_{n+1} concerning the QSSM connection $\tilde{\nabla}^C$. Then

$$\begin{aligned}\tilde{R}^C(\tilde{B}\mathcal{X}_0^C,\tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C &= \tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{Y}_0^C}\tilde{B}\mathcal{Z}_0^C\\ &-\tilde{\nabla}^C_{\tilde{B}\mathcal{Y}_0^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Z}_0^C-\tilde{\nabla}^C_{[\tilde{B}\mathcal{X}_0^C,\tilde{B}\mathcal{Y}_0^C]}\tilde{B}\mathcal{Z}_0^C\end{aligned}$$

In the view of the equations (3.12), (5.2), (5.3), (5.4), (5.5) and (3.15), we get

$$\begin{aligned}\tilde{R}^C(\tilde{B}\mathcal{X}_0^C,\tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C &= B\{R^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)\mathcal{Z}_0^C\\
&+m^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^V-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N_1^{\bar{C}}\\
&+m^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^C+\tilde{\eta}(\mathcal{Y}_0))^C(\phi\mathcal{X}_0)^V\\
&-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^C-\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^C,\mathcal{Z}_0^C)\}N_1^{\bar{V}}\\
&+m^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^V-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N_1^{\bar{V}}\\
&+B\{m^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)H^C\mathcal{Y}_0^C+m^C(\mathcal{X}_0^C,\mathcal{Z}_0^C)H^V\mathcal{Y}_0^C\\
&-m^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)H^C\mathcal{X}_0^C-m^C(\mathcal{Y}_0^C,\mathcal{Z}_0^C)H^V\mathcal{X}_0^C\\
&+n^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)H^C\mathcal{Y}_0^C+n^C(\mathcal{X}_0^C,\mathcal{Z}_0^C)H^V\mathcal{Y}_0^C\\
&-n^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)H^C\mathcal{X}_0^C-n^C(\mathcal{Y}_0^C,\mathcal{Z}_0^C)H^V\mathcal{X}_0^C\}\\
&-B\{m^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{Y}_0^C)+m^C(\mathcal{X}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{Y}_0^C)\\
&-m^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{X}_0^C)-m^C(\mathcal{Y}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{X}_0^C)\}\xi_1^V\\
&-B\{m^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{Y}_0^C)-m^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{X}_0^C)\}\xi_1^C\\
&-B\{n^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{Y}_0^C)+n^C(\mathcal{X}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{Y}_0^C)\\
&-n^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{X}_0^C)-n^C(\mathcal{Y}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{X}_0^C)\}\xi_2^V\\
&-B\{n^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{Y}_0^C)-n^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{X}_0^C)\}\xi_2^C\\
&+n^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^V-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N_2^{\bar{C}}\\
&+n^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^C+(\tilde{\eta}(\mathcal{Y}_0))^C(\phi\mathcal{X}_0)^V\\
&-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^C-(\tilde{\eta}(\mathcal{X}_0))^C(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N_2^{\bar{V}}\\
&+n^C\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^V-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N_2^{\bar{V}}\\
&+\{(\nabla^C_{\mathcal{X}_0^C}m^V)(\mathcal{Y}_0^C,\mathcal{Z}_0^C)-(\nabla^C_{\mathcal{Y}_0^C}m^V)(\mathcal{X}_0^C,\mathcal{Z}_0^C)\}N_1^{\bar{C}}\\
&+\{(\nabla^C_{\mathcal{X}_0^C}m^C)(\mathcal{Y}_0^C,\mathcal{Z}_0^C)-(\nabla^C_{\mathcal{Y}_0^C}m^C)(\mathcal{X}_0^C,\mathcal{Z}_0^C)\}N_1^{\bar{V}}\\
&+\{(\nabla^C_{\mathcal{X}_0^C}n^V)(\mathcal{Y}_0^C,\mathcal{Z}_0^C)-(\nabla^C_{\mathcal{Y}_0^C}n^V)(\mathcal{X}_0^C,\mathcal{Z}_0^C)\}N_2^{\bar{C}}\\
&+\{(\nabla^C_{\mathcal{X}_0^C}n^C)(\mathcal{Y}_0^C,\mathcal{Z}_0^C)-(\nabla^C_{\mathcal{Y}_0^C}n^C)(\mathcal{X}_0^C,\mathcal{Z}_0^C)\}N_2^{\bar{V}}\\
&+l(\mathcal{X}_0^C)\{m^C(\mathcal{Y}_0^C,\mathcal{Z}_0^C)N_2^{\bar{V}}+m^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)N_2^{\bar{C}}\\
&-n^C(\mathcal{Y}_0^C,\mathcal{Z}_0^C)N_1^{\bar{V}}-n^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)N_1^{\bar{C}}\}\\
&-l(\mathcal{Y}_0^C)\{m^C(\mathcal{X}_0^C,\mathcal{Z}_0^C)N_2^{\bar{V}}+m^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)N_2^{\bar{C}}\\
&-n^C(\mathcal{X}_0^C,\mathcal{Z}_0^C)N_1^{\bar{V}}-n^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)N_1^{\bar{C}}\},\end{aligned} \tag{6.3}$$

where $R^C(\mathcal{X}_0^C, \mathcal{X}_0^C)\mathcal{Z}_0^C$ being the Riemannian curvature tensor of the submanifold TM_{n-1} with QSSM connection ∇^C. We have the following theorem:

Theorem 6.1. *Let $R^C(\mathcal{X}_0^C, \mathcal{X}_0^C)\mathcal{Z}_0^C$ be the Riemannian curvature tensor of submanifold TM_{n-1} with QSSM connection ∇^C, then the Riemannian curvature tensor $\tilde{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C\ \tilde{\nabla}^C$ of the enveloping manifold TM_{n+1} concerning the QSSM $\tilde{\nabla}^C$ is given by equation (6.3).*

Substituting

$$\tilde{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C, \tilde{B}\mathcal{Z}_0^C, \tilde{B}U_0^C) = \tilde{g}^C(\tilde{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C, \tilde{B}U_0^C)$$

and

$$R^C(\mathcal{X}_0^C, \mathcal{Y}_0^C, \mathcal{Z}_0^C, U_0^C) = g^C(R^C(\mathcal{X}_0^C, \mathcal{Y}_0^C)\mathcal{Z}_0^C, U_0^C).$$

Then from (6.6) we can easily show that

$$\begin{aligned}\hat{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C, \tilde{B}\mathcal{Z}_0^C, \tilde{B}U^C) = {} & R^C(\mathcal{X}_0^C, \mathcal{Y}_0^C, \mathcal{Z}_0^C, U^C) \\ & + m^V(\mathcal{X}_0^C, \mathcal{Z}_0^C) g(H^C\mathcal{Y}_0^C, U^C) \\ & + m^C(\mathcal{X}_0^C, \mathcal{Z}_0^C) g(H^V\mathcal{Y}_0^C, U^C) \\ & - m^V(\mathcal{Y}_0^C, \mathcal{Z}_0^C) g(H^C\mathcal{X}_0^C, U^C) \\ & - m^C(\mathcal{Y}_0^C, \mathcal{Z}_0^C) g(H^V\mathcal{X}_0^C, U^C)\end{aligned} \tag{6.4}$$

$$\begin{aligned}\tilde{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C, \tilde{B}\mathcal{Z}_0^C, N^{\bar{C}}) = {} & m^V(\mathcal{X}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{Y}_0^C) + m^C(\mathcal{X}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^V\mathcal{Y}_0^C \\ & - m^V(\mathcal{Y}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{X}_0^C) - m^C(\mathcal{Y}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{X}_0^C) \\ & + m^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^C + (\tilde{\eta}(\mathcal{Y}_0))^C(\phi\mathcal{X}_0)^V \\ & - (\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^C - (\tilde{\eta}(\mathcal{X}_0))^C(\phi\mathcal{Y}_0)^V, \mathcal{Z}_0^C)\} \\ & + m^C\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^V - (\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^V, \mathcal{Z}_0^C)\}\end{aligned} \tag{6.5}$$

The equations (6.4) and (6.5) are known as Gauss and Codazzi equations concerning the QSSM connection in the tangent bundle. We have the following theorem:

Theorem 6.2. *Let $\tilde{K}$ C and K^C be the curvature tensors of TM_{n+1} and TM_{n-1} concerning $\dot{\tilde{\nabla}}^C$ and $\dot{\nabla}^C$ respectively. The Gauss and Codazzi concerning the QSSM connection are given by equations (6.4) and (6.5).*

The curvature tensor concerning the QSSM connection $\tilde{\nabla}^C$ of TM_n is

$$\begin{aligned}\tilde{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C = {} & \tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{Y}_0^C}\tilde{B}\mathcal{Z}_0^C \\ & - \tilde{\nabla}^C_{\tilde{B}\mathcal{Y}_0^C}\tilde{\nabla}^C_{\tilde{B}\mathcal{X}_0^C}\tilde{B}\mathcal{Z}_0^C - \tilde{\nabla}^C_{[\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C]}\tilde{B}\mathcal{Z}_0^C.\end{aligned}$$

By virtue of (3.17), (5.11) and (3.21), we get

$$\begin{aligned}\tilde{R}^C(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C = {} & B\{R^C(\mathcal{X}_0^C, \mathcal{Y}_0^C)\mathcal{Z}_0^C + m^V(\mathcal{X}_0^C, \mathcal{Z}_0^C)H^C\mathcal{Y}_0^C \\ & + m^C(\mathcal{X}_0^C, \mathcal{Z}_0^C)H^V\mathcal{Y}_0^C - m^V(\mathcal{Y}_0^C, \mathcal{Z}_0^C)H^C\mathcal{X}_0^C \\ & - m^C(\mathcal{Y}_0^C, \mathcal{Z}_0^C)H^V\mathcal{X}_0^C\} - B\{m^V(\mathcal{X}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{Y}_0^C) \\ & + m^C(\mathcal{X}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^V\mathcal{Y}_0^C - m^V(\mathcal{Y}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^C(\mathcal{X}_0^C) \\ & - m^C(\mathcal{Y}_0^C, \mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{X}_0^C)\}\mathcal{X}_0 i^V\end{aligned} \tag{6.6}$$

$$-B\{m^V(\mathcal{X}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{Y}_0^C)-m^V(\mathcal{Y}_0^C,\mathcal{Z}_0^C)\tilde{\eta}^V(\mathcal{X}_0^C)\}\xi^C$$
$$+m^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^V-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N^{\bar{C}}$$
$$+m^V\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^C+(\tilde{\eta}(\mathcal{Y}_0))^C(\phi\mathcal{X}_0)^V$$
$$-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^C-(\tilde{\eta}(\mathcal{X}_0))^C(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N^{\bar{V}}$$
$$+m^C\{(\tilde{\eta}(\mathcal{Y}_0))^V(\phi\mathcal{X}_0)^V-(\tilde{\eta}(\mathcal{X}_0))^V(\phi\mathcal{Y}_0)^V,\mathcal{Z}_0^C)\}N^{\bar{V}}$$

where $R^C(\mathcal{X}_0^C,\mathcal{Y}_0^C)\mathcal{Z}_0^C=\nabla^C_{\mathcal{X}_0^C}\nabla^C_{\mathcal{Y}_0^C}\mathcal{Z}_0^C-\nabla^C_{\mathcal{Y}_0^C}\nabla^C_{\mathcal{X}_0^C}\mathcal{Z}_0^C-\nabla^C_{[\mathcal{X}_0^C,\mathcal{Y}_0^C]}\mathcal{Z}_0^C$ is curvature tensor of the QSSM connection.

As an immediate consequence of the theorem (5.1) and theorem (5.2), we have the following corollaries:

Corollary 6.1. *Let $R^C(\mathcal{X}_0^C, \mathcal{Y}_0^C)\mathcal{Z}_0^C$ be the Riemannian curvature tensor of hypersurface TM_n with QSSM connection ∇^C, then the Riemannian curvature tensor $\tilde{R}^C$ $(\tilde{B}\mathcal{X}_0^C, \tilde{B}\mathcal{Y}_0^C)\tilde{B}\mathcal{Z}_0^C$ $\tilde{\nabla}^C$ of the enveloping manifold TM_n concerning the QSSM connection $\tilde{\nabla}^C$ is given by equation (6.6).*

Corollary 6.2. *Let $\tilde{K}^C$ and K^C be the curvature tensors of TM_n and TM_{n+1} concerning $\dot{\tilde{\nabla}}^C$ and $\dot{\nabla}^C$ respectively. The Gauss and Codazzi equations con cerning the QSSM connection are similar equations obtained from Theorem 6.2.*

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