Results in Nonlinear Analysis 7 (2024) No. 2, 115–126
https://doi.org/10.31838/rna/2024.07.02.010
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Blow-up phenomena for pseudo-parabolic Kirchhoff equations with logarithmic nonlinearity

Abdelatif Toualbia[a], Nabila Barrouk[b]

[a]Department of Mathematics and Informatics, Faculty of Exact Sciences And Natural and Life Sciences, LAMIS Laboratory, Echahid Cheikh Larbi Tebessi University, Tebessa 12000, Algeria; [b]Department of Mathematics, Faculty of Sciences And Technology, Mohamed Cherif Messaadia University, P.O.Box 1553, Souk Ahras 41000, Algeria.

Abstract

We consider pseudo-parabolic equations with $r(x)$-Kirchhoff coefficient and logarithmic nonlinear term subject to Dirichlet boundary condions

$$\upsilon_t - \mu\Delta\upsilon_t - M\left(\|\nabla\upsilon\|_{r(x)}^{r(x)}\right)\Delta_{r(x)}\upsilon = |\upsilon|^{s(x)-2}\,\upsilon\ln|\upsilon|.$$

Using a method based on differential inequalities, we prove that the solutions become unbounded at a finite time T, and, we ascertain an upper limit for this time in the case of negative initial energy. Additionally, we determine a lower limit for the time at which blow-up occurs.

Key words and phrases: Pseudo-parabolic equation Blow-up variable exponent source.

Mathematics Subject Classification (2010): 35K70, 35B44

1. Introduction

This paper is concerned with the following pseudo-parabolic equation involving an unknown function $\upsilon = \upsilon(x,t)$

$$\upsilon_t - \mu\Delta\upsilon_t - M\left(\|\nabla\upsilon\|_{r(x)}^{r(x)}\right)\Delta_{r(x)}\upsilon = |\upsilon|^{s(x)-2}\,\upsilon\ \text{In}\ |\upsilon|,\ \text{in}\ \Omega\times(0,\infty), \tag{1}$$

with homogeneous Dirichlet boundary condition

Email addresses: abdelatif@univ-tebessa.dz (Abdelatif Toualbia)*; n.barrouk@univ-soukahras.dz (Nabila Barrouk)

Received December 7, *2023; Accepted March 19, 2024; Online April 3, 2024*

$$v(x,t) = 0, \text{ on } \partial\Omega \times (0,\infty), \tag{2}$$

supplemented with the initial conditions

$$v(x,0) = v_0(x), \; x \in \Omega, \tag{3}$$

where Ω denotes an set of $\mathbb{R}^n, n > 1$, with smooth boundary $\partial\Omega$, Δ is the Laplace operator, $vt = \frac{\partial v}{\partial t}$, $\Delta_{r(x)}$ is the so-called $r(x)$-Laplace operator which is given by

$$\Delta_{r(x)} = div\left(\left|\nabla v\right|^{r(x)-2} \nabla v\right),$$

and for $a,b > 0$ we define

$$\begin{cases} M(s) = a + bs, \\ \left\|\nabla v\right\|_{r(x)} = \left(\int_\Omega \frac{1}{r(x)}\left|\nabla v\right|^{r(x)} dx\right)^{\frac{1}{r(x)}}. \end{cases}$$

The logharitmic nonlinearity $|v|^{s(x)-2} v \ln|v|$ plays the role of a source, and the dissipative term Δv_t is a linear strong damping term. The exponents $r(.)$ and $s(.)$ are continuous functions on $\overline{\Omega}$ and satisfy

$$2 < r_- \le r(x) \le r_+ < s_- \le s(x) \le s_+ < \infty, \tag{4}$$

and

$$2|\Omega|\frac{r_+^2}{r_-} < s_- \tag{5}$$

where

$$r_- = \inf r(x), \qquad r_+ = \sup r(x)$$
$$s_- = \inf s(x), \qquad s_+ = \sup s(x)$$

and the Zhikov-Fan conditions:

$$\begin{cases} |r(x) - r(y)| = \frac{-A}{In|x-y|} \\ \qquad \text{and} \qquad\qquad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta, \\ |s(x) - s(y)| = \frac{-B}{In|x-y|} \end{cases} \tag{6}$$

where A, $B > 0$ and $0 < \delta < 1$.

In recent years, logarithmic nonlinearity appears frequently in partial differential equations which describes important physical phenomena (see [10, 11, 32]) and the references therein).

One of the main features of the system (1)-(3) is that the coefficient of $\Delta_{r(x)}v$ depends on the integration of the gradient of the unknown, such equations are usually refrred to as $r(x)$-Kirchhoff equations or non-local equations. When the function $r(x) = r = 2$, we commonly refer to them as Kirchhoff equations. The coefficient of diffusion, denoted as $M(.)$ can depict a potential alteration in the overall condition of population density, fluid, or gas resulting from the corresponding movement within the examined medium. The investigation of these equations in math began with Kirchhoff's research [35].

The problem (1)-(3) is not only $r(x)$-Kirchhoff equations one but also includes logarithmic nonlinearity, which is widely used in various fields such as nuclear physics, geophysics, and optics [5, 7, 18]. These equations naturally arise in inflation cosmology, physics of semiconductors, and quantum mechanics, among other areas [2, 3, 6, 15, 24, 25]. Obviously, if $M(s) = \mu = 1$, $r(x) = 2$, $s(x) = s$, then the equation (1) reduces to the following pseudo-parabolic equation

$$v_t - \Delta v_t - \Delta v = |v|^{s-2} v \ln|v|, \tag{7}$$

Chen and Tian [9] obtained results of global existence, blow-up at $+\infty$ and asymptotic behavior of solutions for the problem (7). In [8], Cao and Zhao considered the following pseudo-parabolic r-Kirchhoff equation

$$v_t - \Delta v_t - M(\|\nabla v\|_r^r)\Delta_r v = |v|^{s-1} v \ln|v|, \tag{8}$$

Using the potential well method, they were able to derive the global existence and finite time blow-up of the solution for problem (8). In other studies [34], they considered the same pseudo-parabolic r-Kirchhoff equation (8) but with the non-local source term $|v|^{s-1} v - \frac{1}{|\Omega|}\int_\Omega |v|^{s-1} v dx$, they successfully demonstrated the existence and nonexistence of global solutions and offered adequate criteria for the finite time blow-up of solutions. When the damped term is not present (i.e $\mu = 0$, Han and Li [21] studied the parabolic Kirchhoff equation

$$v_t - M(\|\nabla v\|_2^2)\Delta v = |v|^{s-1} v,$$

and obtained results of global existence, finite time blow-up and asymptotic behavior of the weak solution, with subcritical, critical and supercritical initial energy. In other studies [22], Han *et al.* considered the same problem treated in [21], and they obtained an upper and a lower bound of the blow-up rate. Later, He *et al.* [23] extended the results of [21, 22] to the parabolic r-Kirchhoff equation

$$v_t - M(\|\nabla v\|_r^r)\Delta_r v = |v|^{s-1} v, \tag{9}$$

and described the impact of the r-Laplacian. On the other hand, in the recent monograph [29], Pan *et al.* studied the following problem

$$v_t - \Delta v_t - div(|\nabla v|^{r(x)-2} \nabla v) = |v|^{s(x)-2} v \ln|v|, \tag{10}$$

which is just the $M(s) = 1$ case of (1)-(3). Using the energy functional and the classical potential well, they were able to derive the global existence and blow-up outcomes of weak solutions with arbitrarily high initial energy to the problem (10). In other studies, Lakshmipriya *et al.* [28] considered the same pseudo-parabolic $r(x)$-Laplacian equation (10) but with the source term $|v|^{s(x)-2} v + |v|^{h-2} v \ln|v|$, the local existence of a weak solution was achieved by employing the Faedo-Galerkin approximation method, alongside the utilization of differential inequality techniques to establish both an upper bound and a lower bound for the blow-up rate. It is worth mentioning some other literature concerning the theory of our type equation, namely, several studies [4, 8, 10, 11, 13, 14, 19, 20, 26, 30, 31, 32, 33].

Motivated by previous research, this study aims to establish an upper bound for blow-up time based on certain conditions of variable exponents and initial data. Additionally, lower bounds on blow-up time will be provided under different conditions for the given problem.

The outline of this paper is as follows. Definitions of $L^{p(.)}(\Omega)$ and $W^{1,p(.)}(\Omega)$ as well as properties are recalled in section 2. The blow-up of solutions to the problem (1)-(3) is studied in section 3 and 4.

2. Function Spaces and Lemmas

Let Ω be a domain of $\mathbb{R}^n$ and $p : \Omega \to [1,\infty)$ be a measurable function. The Lebesgue space $L^{p(.)}(\Omega)$, wherein $p(.)$ is a variable exponent is precisely delineated by its definition.

$$L^{p(.)}(\Omega) = \{v : \Omega \to \mathbb{R} \setminus v \text{ is measurable in } \Omega$$
$$\text{and } \int_\Omega |\lambda v(x)|^{p(x)} dx < \infty \text{ for some } \lambda > 0\}.$$

The Luxemburg-type norm is given by

$$\|v\|_{p(.)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{v(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Variable exponent Lebesgue spaces are similar to classical Lebesgue spaces in various ways. They are Banach spaces, obey the Hölder inequality, and are reflexive if $1 < p(x) < \infty$. The Sobolev space $W^{1,p(.)}(\Omega)$ which features a variable exponent is precisely characterized by its definition.

$$W^{1,p(.)}(\Omega) = \left\{ v \in L^{p(.)}(\Omega) : \nabla v \text{ exists and } \nabla v \in L^{p(.)}(\Omega) \right\}.$$

This is a Banach space with respect to the norm

$$\|v\|_{W^{1,p(.)}(\Omega)} = \|v\|_{p(.)} + \|\nabla v\|_{p(.)}.$$

The space $W_0^{1,p(.)}(\Omega)$ can be characterized as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$.

In the constant exponent case, the space $W_0^{1,p(.)}(\Omega)$ has various definitions. Nevertheless, these definitions align when conditionin (6) is satisfied (See [12, 16]).

Lemma 2.1 (Holder's inequality, [12]) *Let* α *and* β *be elements of the interval* $[1,\infty)$ *such that* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *If* $\varphi \in L^\alpha(\Omega)$ and $\phi \in L^\beta(\Omega)$, *then* $\varphi\phi \in L^1(\Omega)$, *with*

$$\|\varphi\phi\|_1 \le \|\varphi\|_\alpha \|\phi\|_\beta.$$

By taking $\alpha = \beta = 2$, *we obtain the Cauchy-Schwartz inequality*

$$\|\varphi\phi\|_1 \le \|\varphi\|_2 \|\phi\|_2.$$

Lemma 2.2 (Poincare's inequality [12]) *If it is assumed that* $p(.)$ *satisfies* (6), *then,*

$$\|v\|_{p(.)} \le C \|\nabla v\|_{p(.)}, v \in W_0^{1,p(.)}(\Omega),$$

where $C > 0$ *is a constant that depends only on* $p(.)$ *and* Ω.

Lemma 2.3 (Embedding Proprety [12]) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with a smooth boundary* $\partial\Omega$. *If* $q \in C(\overline{\Omega})$ *such that* $q \ge 2$ *and* $q(x) < 2^*$ *in* $\overline{\Omega}$ *with*

$$2^* = \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2, \\ \infty, & \text{if } n \le 2, \end{cases}$$

we can establish a continuous and compact embedding $H_0^1(\Omega) \to L^{q(.)}(\Omega)$. Therefore, it follows that there exists a positive constant C such that

$$\|v\|_{L^{q(.)}(\Omega)} \le C \|v\|_{H_0^1(\Omega)}.$$

Lemma 2.4 ([27, 10]) *For all* $\upsilon \in [1,\infty)$.

$$|\ln \upsilon| \le \frac{\upsilon^\eta}{e\eta},$$

the variable η *represents a positive numerical value.*

3. Upper Bound for Blow-up Time

Firstly, we begin by considering the existence and uniqueness of a local solution for problem (1)-(3). This can be achieved by combining the standard Galerkin's approximation with the Aubin-Lions compactness theorem, as explained in [17]. The proof of local solutions to a $p(x,t)$-Kirchhoff equation, which includes the equation (1) as a specific instance, can be found in [17]. For simplicity, we set $\mu = 1$.

Theorem 3.1 *Let* $\upsilon_0 \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega) \setminus \{0\}$ *be given. Assume that the conditions on* $s(x)$ *and* $r(x)$, *given in Section 1, hold. Then, the problem (1)-(3) has a unique local solution* υ *on* $[0,T)$

$$\begin{cases} \upsilon \in L^{\infty}([0,T]; W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega)), \\ \upsilon_t \in L^2([0,T]; H_0^1(\Omega)), \end{cases}$$

for some $T > 0$, *satisfying* $\upsilon(0) = \upsilon_0$, *and for any* $w \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega)$,

$$(\upsilon_t, w) + (\nabla \upsilon_t, \nabla w) + M(\|\nabla \upsilon\|_{r(x)}^{r(x)})(|\nabla \upsilon|^{r(x)-2} \nabla u, \nabla w)$$
$$= (|\upsilon|^{s(x)-2} \upsilon \ln|\upsilon|, w), \tag{11}$$

where $M(\|\nabla \upsilon\|_{r(x)}^{r(x)}) = a + b\|\nabla \upsilon\|_{r(x)}^{r(x)} = (a + b\int_\Omega \frac{1}{r(x)} |\nabla \upsilon|^{r(x)} dx)$.

Moreover, the following alternatives hold

i) $T = +\infty$ or

ii) $T < +\infty$ and $\lim_{t \to T} \|\upsilon\|_2^2 + \|\nabla \upsilon\|_2^2 = +\infty$.

Remark 3.2 *It is easy to see, under the condition (4) that* $|\upsilon|^{s(x)-2} \upsilon \ln|\upsilon|$, $|\nabla \upsilon|^{r(x)-2} \nabla \upsilon \in L^2(\Omega)$, *hence* $(|\upsilon|^{s(x)-2} \upsilon \ln|\upsilon|, w$ *and* $(|\nabla \upsilon|^{r(x)-2} \nabla \upsilon, \nabla w)$ *make sense in formula (11).*

The decay of the energy of the system (1)-(3) is given in the following lemma:

Lemma 3.3 *For* $\upsilon \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega) \setminus \{0\}$. *The energy functional E of the problem (1)-(3) is a decreasing function. Here*

$$E(t) = -\int_\Omega \frac{|\upsilon|^{s(x)} \ln|\upsilon|}{s(x)} dx + a\int_\Omega \frac{|\Delta \upsilon|^{r(x)}}{r(x)} dx + \frac{b}{2}\left(\int_\Omega \frac{|\nabla \upsilon|^{r(x)}}{r(x)} dx\right)^2$$
$$+ \int_\Omega \frac{|\upsilon|^{s(x)}}{s^2(x)} dx \tag{12}$$

Proof. It is enough to multiply the equation (1) by υ_t and integrate over Ω, to obtain

$$\int_\Omega \upsilon_t \upsilon_t dx - \int_\Omega \Delta \upsilon_t \upsilon_t dx - \int_\Omega M\left(\| \nabla \upsilon \|_{r(x)}^{r(x)}\right) div\left(|\nabla \upsilon|^{r(x)-2} \nabla \upsilon\right) \upsilon_t dx$$
$$= \int_\Omega |\upsilon|^{s(x)-2} \upsilon \ln|\upsilon| \upsilon t dx,$$

where $M\left(\|\nabla \upsilon\|_{r(x)}^{r(x)}\right) = a + b\|\nabla \upsilon\|_{r(x)}^{r(x)} = \left(a + b\int_\Omega \frac{1}{r(x)} |\nabla \upsilon|^{r(x)} dx\right)$.

Then, we use the generalized Green formula and the boundary conditions, to find

$$\int_\Omega \left(|\upsilon_t|^2 + |\nabla \upsilon_t|^2\right) dx + \left(a + b \| \nabla \upsilon \|_{r(x)}^{r(x)}\right) \int_\Omega |\nabla \upsilon|^{r(x)-2} \nabla \upsilon . \nabla \upsilon_t dx$$
$$= \frac{d}{dt} \int_\Omega \left(\frac{|\upsilon|^{s(x)} \ln|\upsilon|}{s(x)} - \frac{|\upsilon|^{s(x)}}{s^2(x)}\right) dx.$$

This implies that

$$\int_\Omega \left(|\upsilon_t|^2 + |\nabla \upsilon_t|^2\right) dx + a\frac{d}{dt} \int_\Omega \frac{|\nabla \upsilon|^{r(x)}}{r(x)} dx + \frac{b}{2}\frac{d}{dt}\left(\int_\Omega \frac{|\nabla \upsilon|^{r(x)}}{r(x)} dx\right)^2$$
$$= \frac{d}{dt} \int_\Omega \left(\frac{|\upsilon|^{s(x)} \ln|\upsilon|}{s(x)} - \frac{|\upsilon|^{s(x)}}{s^2(x)}\right) dx.$$

So

$$E'(t) = -\int_{\Omega}\left(|\upsilon|^2 + |\nabla \upsilon_t|^2\right)dx \le 0. \tag{13}$$

Theorem 3.4 *Assume that (4),(5), and (6) hold. Let* υ *be a solution of (1)-(3) and assume that* $\upsilon_0 \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega) \setminus \{0\}$ *satisfies*

$$\int_{\Omega}\frac{|\upsilon_0|^{s(x)}\ln|\upsilon_0|}{s(x)}dx - a\left(\int_{\Omega}\frac{|\nabla\upsilon_0|^{r(x)}}{r(x)}dx\right) - \frac{b}{2}\left(\int_{\Omega}\frac{|\nabla\upsilon_0|^{r(x)}}{r(x)}dx\right)^2$$

$$-\int_{\Omega}\frac{|\upsilon_0|^{s(x)}}{s^2(x)}dx \ge 0, \tag{14}$$

then the solution υ *blow-up at finite time* $T_{max} > 0$ *in* $H_0^1(\Omega)$*-norm. Additionally, there exists an upper bound for the time as determined by*

$$T_{max} \le \frac{2\left(G(0)\right)^{\left(\frac{2-r_-}{2}\right)}}{(r_- - 2)K}, \tag{15}$$

where K is a suitable positive constant is given later and the constant $G(0) = \|\upsilon_0\|^2_{H_0^1(\Omega)}$.

Proof. Let us define the auxiliary function

$$G(t) = \|\upsilon\|^2_{H_0^1(\Omega)} = \int_{\Omega}\upsilon^2 dx + \int_{\Omega}|\nabla\upsilon|^2 dx \tag{16}$$

Our goal is to show that G satisfies a differential inequality which leads to blow-up in finite time.

Multiply (1) by υ and integrate over Ω to get

$$\int_{\Omega}\upsilon\upsilon_t dx + \int_{\Omega}\nabla\upsilon\nabla\upsilon_t dx = \int_{\Omega}\left(|\upsilon|^{s(x)}\ln|\upsilon| - M\left(\|\nabla\upsilon\|^{r(x)}_{r(x)}\right)|\nabla\upsilon|^{r(x)}\right)dx, \tag{17}$$

where $M\left(\|\nabla\upsilon\|^{r(x)}_{r(x)}\right) = a + b\|\nabla\upsilon\|^{r(x)}_{r(x)} = \left(a + b\int_{\Omega}\frac{1}{r(x)}|\nabla\upsilon|^{r(x)}dx\right)$.

Now differentiate $G(t)$ with respect to t to obtain

$$\begin{aligned}
G'(t) &= 2\int_{\Omega}(\upsilon\upsilon_t dx + \nabla\upsilon\nabla\upsilon_t)dx \\
&= 2\int_{\Omega}\left(|\upsilon|^{s(x)}\ln|\upsilon| - \left(a + b\|\nabla\upsilon\|^{r(x)}_{r(x)}\right)|\nabla\upsilon|^{r(x)}\right)dx \\
&= 2\int_{\Omega}s(x)\left(\frac{|\upsilon|^{s(x)}\ln|\upsilon|}{s(x)} - a\frac{|\nabla\upsilon|^{r(x)}}{r(x)} - \frac{b}{2}\left(\frac{|\nabla\upsilon|^{r(x)}}{r(x)}\right)^2 - \frac{|\upsilon|^{s(x)}}{s^2(x)}\right)dx \\
&\quad + 2\int_{\Omega}s(x)\left(\begin{array}{c} a\frac{|\nabla\upsilon|^{r(x)}}{r(x)} + \frac{b}{2}\left(\frac{|\nabla\upsilon|^{r(x)}}{r(x)}\right)^2 + \frac{|\upsilon|^{s(x)}}{s^2(x)} \\ -\frac{\left(a + b\|\nabla\upsilon\|^{r(x)}_{r(x)}\right)}{s(x)}|\nabla\upsilon|^{r(x)} \end{array}\right)dx.
\end{aligned} \tag{18}$$

By (14) and the fact that $E(t) \le E(0)\left(E'(t) \le 0\right)$ (See Lemma 3.3), we have

$$\int_\Omega s(x)\left[\frac{|v|^{s(x)}\ln|v|}{s(x)}-a\frac{1}{r(x)}|\nabla v|^{r(x)}-\frac{b}{2}\left(\frac{1}{r(x)}|\nabla v|^{r(x)}\right)^2-\frac{1}{s^2(x)}|v|^{s(x)}\right]dx$$

$$\geq \int_\Omega s(x)\left[\frac{|v_0|^{s(x)}\ln|v_0|}{s(x)}-a\frac{|\nabla v_0|^{r(x)}}{r(x)}-\frac{b}{2}\left(\frac{|\nabla v_0|^{r(x)}}{r(x)}\right)^2-\frac{|v_0|^{s(x)}}{s^2(x)}\right]dx$$

$$\geq s_-\int_\Omega\left[\begin{matrix}\frac{|v_0|^{s(x)}\ln|v_0|}{s(x)}-\frac{a|\nabla v_0|^{r(x)}}{r(x)}-\frac{b}{2}\left(\frac{|\nabla v_0|^{r(x)}}{r(x)}\right)^2\\ -\frac{|v_0|^{s(x)}}{s^2(x)}\end{matrix}\right]dx\geq 0 \tag{19}$$

Using (18) and (19), we have

$$\begin{aligned}G'(t)\ \geq\ & 2\int_\Omega s(x)\left(a\frac{|\nabla v|^{r(x)}}{r(x)}+\frac{b}{2}\left(\frac{|\nabla v|^{r(x)}}{r(x)}\right)^2+\frac{|v|^{s(x)}}{s^2(x)}\right.\\ &\left.-\frac{\left(a+b\|\nabla v\|_{r(x)}^{r(x)}\right)}{s(x)}|\nabla v|^{r(x)}\right)dx\\ =\ & 2\int_\Omega\left(as(x)\left(\frac{1}{r(x)}-\frac{1}{s(x)}\right)|\nabla v|^{r(x)}+\frac{b}{2}s(x)\left(\frac{|\nabla v|^{r(x)}}{r(x)}\right)^2\right.\\ &\left.+\frac{|v|^{s(x)}}{s(x)}-b\|\nabla v\|_{r(x)}^{r(x)}|\nabla v|^{r(x)}\right)dx\\ =\ & 2\int_\Omega\left(as(x)\left(\frac{1}{r(x)}-\frac{1}{s(x)}\right)|\nabla v|^{r(x)}+\frac{b}{2}s(x)\left(\frac{|\nabla v|^{r(x)}}{r(x)}\right)^2+\frac{|v|^{s(x)}}{s(x)}\right)dx\\ &-2b\left(\int_\Omega\frac{1}{r(x)}|\nabla v|^{r(x)}dx\right)\left(\int_\Omega|\nabla v|^{r(x)}dx\right)\end{aligned}$$

Using (4), we deduce that

$$\begin{aligned}G'(t)\geq\ & 2\int_\Omega\left(as_-\left(\frac{1}{r_+}-\frac{1}{s_-}\right)|\nabla v|^{r(x)}+\frac{b}{2}s_-\left(\frac{|\nabla v|^{r(x)}}{r_+}\right)^2+\frac{|v|^{s(x)}}{s_+}\right)dx\\ &-2b\frac{1}{r_-}\left(\int_\Omega|\nabla v|^{r(x)}dx\right)^2\end{aligned} \tag{20}$$

Adopting the Cauchy-Schwartz inequality on the last term of the right-hand side of (20), we can easily conclude that

$$\left(\int_\Omega|\nabla v|^{r(x)}dx\right)^2\leq|\Omega|\int_\Omega\left(|\nabla v|^{r(x)}\right)^2dx$$

Then (20) becomes

$$G'(t) \geq 2\int_{\Omega}\left(as_{-}\left(\frac{1}{r_{+}} - \frac{1}{s_{-}}\right)|\nabla v|^{r(x)}\right.$$
$$\left. + \frac{b}{2}\left(\frac{s_{-}}{r_{+}^{2}} - \frac{2|\Omega|}{r_{-}}\right)\left(\left(|\nabla v|^{r(x)}\right)^{2} + \frac{|v|^{s(x)}}{s_{+}}\right)\right)dx$$

It follows from (5) that

$$G'(t) \geq 2\int_{\Omega}\left(as_{-}\left(\frac{1}{r_{+}} - \frac{1}{s_{-}}\right)|\nabla v|^{r(x)} + \frac{|v|^{s(x)}}{s_{+}}\right)dx. \tag{21}$$

As $s_{+} > 0$, we may drop the last term of (21) to get

$$G'(t) \geq C_0 \int_{\Omega} |\nabla v|^{r(x)}\, dx, \tag{22}$$

where $C_0 = 2as_{-}\left(\frac{1}{r_{+}} - \frac{1}{s_{-}}\right) > 0$.

Using the fact that $\|v\|_2 \leq C\|v\|_r$ for all $r > 2$, we have

$$G'(t) \geq C_0\left(\int_{\Omega_{-}} |\nabla v|^{r_{+}}\, dx + \int_{\Omega_{+}} |\nabla v|^{r_{-}}\, dx\right)$$
$$\geq C_1\left(\left(\int_{\Omega_{-}} |\nabla v|^{2}\, dx\right)^{\frac{r_{+}}{2}} + \left(\int_{\Omega_{+}} |\nabla v|^{2}\, dx\right)^{\frac{r_{-}}{2}}\right).$$

Here the sets Ω_ and Ω_{+} are defined as follows:

$$\Omega_{-} = \{x \in \Omega : |\nabla v| < 1\},\ \Omega_{+} = \{x \in \Omega : |\nabla v| \geq 1\}.$$

This implies that

$$\left(G'(t)\right)^{\frac{2}{r_{+}}} \geq C_2\left(\int_{\Omega_{-}} |\nabla v|^{2}\, dx\right) \text{ and } \left(G'(t)\right)^{\frac{2}{r_{-}}} \geq C_3\left(\int_{\Omega_{+}} |\nabla v|^{2}\, dx\right). \tag{23}$$

The Poincare inequality gives $\|\nabla v\|_2^2 \geq \lambda\|v\|_2^2$, where λ is the first eigenvalue of (–Δ). Therefore, we get

$$\|\nabla v\|_2^2 = \frac{1}{1+\lambda}\|\nabla v\|_2^2 + \frac{\lambda}{1+\lambda}\|\nabla v\|_2^2$$
$$\geq \frac{\lambda}{1+\lambda}\|v\|_2^2 + \frac{\lambda}{1+\lambda}\|\nabla v\|_2^2 = \frac{\lambda}{1+\lambda}\|v\|_{H_0^1(\Omega)}^2 \tag{24}$$

It follows from (23) and (24) that

$$\left(G'(t)\right)^{\frac{2}{r+}} + \left(G'(t)\right)^{\frac{2}{r-}} \geq \min\left(C_2, C_3\right)\|\nabla\|v_2^2$$
$$\geq \min\left(C_2, C_3\right)\frac{\lambda}{1+\lambda}\|v\|_{H_0^1(\Omega)}^2$$
$$= C_4 G(t). \tag{25}$$

Since we have $G(t) \geq G(0) > 0$ (because $G'(t) \geq 0$), and from (25), we get

$$\left(G'(t)\right)^{\frac{2}{r+}} \geq \frac{C_4}{2} G(t) \geq \frac{C_4}{2} G(0) \text{ or } \left(G'(t)\right)^{\frac{2}{r-}} \geq \frac{C_4}{2} G(t) \geq \frac{C_4}{2} G(0). \tag{26}$$

This implies that

$$G'(t) \geq C_5 \left(G(0)\right)^{\frac{r+}{2}} \text{ or } G'(t) \geq C_5 \left(G(0)\right)^{\frac{r-}{2}}.$$

Now put $\beta = \min\left\{C_5\left(G(0)\right)^{\frac{r+}{2}}, C_5\left(G(0)\right)^{\frac{r-}{2}}\right\}$, then we get

$$G'(t) \geq \beta, \tag{27}$$

(25) implies that

$$\left(G'(t)\right)^{\frac{2}{r-}}\left(1+\left(G'(t)\right)^{2\left(\frac{1}{r+}-\frac{1}{r-}\right)}\right) \geq C_4 G(t). \tag{28}$$

From (4), we observe that $2\left(\frac{1}{r_+}-\frac{1}{r_-}\right) \leq 0$. Making use (27), we get

$$G'(t) \geq K\left(G(t)\right)^{\frac{r_-}{2}}, \tag{29}$$

where $K = \left(\frac{C_4}{1+\beta^{2\left(\frac{1}{r_+}-\frac{1}{r_-}\right)}}\right)^{\frac{r_-}{2}}$ is a positive constant.

Integrating (29) from 0 to t gives

$$G(t) \geq \frac{1}{\left(\left(G(0)\right)^{1-\frac{r_-}{2}} + \frac{(2-r_-)Kt}{2}\right)^{\frac{2}{r_- -2}}},$$

which implies that $G(t) \to \infty$ as $t \to T_{\max}$ in $H_0^1(\Omega)$, where

$$T_{\max} \leq \frac{2\left(G(0)\right)^{\left(\frac{2-r_-}{2}\right)}}{(r_- -2)K}.$$

Consequently, the solution to the problem (1)-(3) blows up in finite time.

Hence the proof is completed.

4. Lower Bound for Blow-up Time

In this section, we determine a lower bound for the blow-up time of the problem (1)-(3).

Theorem 4.1 *Suppose that the conditions on* $s(x)$, $r(x)$, *and A, given in section 1, hold. Additionally, assume that* $2 < s_+ < \infty$ *if* $n \leq 2$, $2 < s_+ < \frac{2n}{n-2}$ *if* $n > 2$, $\upsilon_0 \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega)$ *and* υ *represents a blow-up solution of problem (1)-(3), then, it is possible to establish a minimum estimate for the blow-up time* T_{min} *in the following manner*

$$T_{\min} \simeq \frac{1}{C\alpha} \frac{1}{(\eta + s_+ - 2)} \left(G(0)\right)^{-\frac{\eta+s_+}{2}+1}, \tag{30}$$

where C,η *are a positive constants,* α *is the optimal constant satisfying the Sobolev embedding inequality* $\|v\|_{L^{s_+}(\Omega)} \le \alpha \|v\|_{H_0^1(\Omega)}$ *and* $G(0) = \|v_0\|^2_{H_0^1(\Omega)}$.

Proof. Consider $G(t)$ as in (16) $G(t) = \|v\|^2_{H_0^1(\Omega)}$.
Multiply (1) by v and perform the process of integration over the domain Ω to get

$$\int_\Omega v v_t dx + \int_\Omega \nabla v \nabla v_t dx = \int_\Omega |v|^{s(x)} \ln|v| dx - \int_\Omega \left(a + b\|\nabla v\|^{r(x)}_{r(x)}\right) |\nabla v|^{r(x)} dx.$$

A direct differentiation of $G(t)$ yields

$$\begin{aligned} G'(t) &= 2\int_\Omega (v v_t + \nabla v \nabla v_t) dx \\ &= 2\left[\int_\Omega |v|^{s(x)} \ln|v| dx - \int_\Omega \left(a + b\|\nabla v\|^{r(x)}_{r(x)}\right)|\nabla v|^{r(x)} dx\right]. \end{aligned}$$

Then

$$G'(t) \le 2\int_\Omega |v|^{s(x)} \ln|v| dx. \tag{31}$$

Defining the sets

$$\Omega_+ = \{x \in \Omega : |v| \ge 1\} \text{ and } \Omega_- = \{x \in \Omega : |v| < 1\}.$$

Thus, we have

$$\begin{aligned} \int_\Omega |v|^{s(x)} \ln|v| dx &= \int_{\Omega_+} |v|^{s(x)} \ln|v| dx + \int_{\Omega_-} |v|^{s(x)} \ln|v| dx \\ &\le \int_{\Omega_+} |v|^{s(x)} \ln|v| dx \end{aligned}$$

due to the negativity of the term $\int_{\Omega_-} |v|^{s(x)} \ln|v| dx$.
Since we have $v^{-\eta} \ln v \le (e\eta)^{-1}$ for all $\eta > 0$ and $v \ge 1$ (See Lemma 2.4), we can deduce

$$\begin{aligned} \int_{\Omega_+} |v|^{s(x)} \ln|v| dx &\le \int_{\Omega_+} |v|^{s_+} \ln|v| dx \\ &\le (e\eta)^{-1} \int_{\Omega_+} |v|^{\eta+s_+} dx \\ &\le C\int_\Omega |v|^{\eta+s_+} dx = C\|v\|^{\eta+s_+}_{\eta+s_+} \end{aligned} \tag{32}$$

Thus, the combination of (31) and (32) implies that

$$G'(t) \le 2C\|v\|^{\eta+s_+}_{\eta+s_+}. \tag{33}$$

Using Sobolev embedding (See Lemma 2.3), we have

$$\|v\|^{\eta+s_+}_{\eta+s_+} \le \alpha \|v\|^{\eta+s_+}_{H_0^1(\Omega)},$$

where α is the corresponding embedding constant. Therefore, (33) becomes

$$G'(t) \le 2C\alpha \left(G(t)\right)^{\frac{\eta+s_+}{2}}.$$

By performing integration on both sides of the last inequality over the interval $(0,T)$, we obtain.

$$\int_{G(0)}^{G(T)} \frac{d\xi}{2C\alpha\xi^{\frac{\eta+s_+}{2}}} \leq T.$$

If υ blow-up in H_0^1-norm, then we establish a lower bound for $T_{\min}$ by the form

$$T_{\min} \geq \int_{G(0)}^{\infty} \frac{d\xi}{2C\alpha\xi^{\frac{\eta+s_+}{2}}},$$

Clearly, the integral is bound since exponents $\eta + s_+ > 2$ and

$$T_{\min} \simeq \frac{1}{C\alpha} \frac{1}{(\eta + s_+ - 2)} \left(G(0)\right)^{-\frac{\eta+s_+}{2}+1}$$

which is the desired result.

References

[1] G.A. Afrouzi, N.T. Chung and S. Shakeri, Existence of positive solutions for Kirchhoff type equations. *Electron. J. Differential Equations*, 180(2013), 8, (2013).

[2] A.B. Al'shin, M.O. Korpusov and A.G. Sveshnikov, Blow-up in nonlinear Sobolev type equations, (Vol. 15). *Walter de Gruyter*, (2011).

[3] S.N. Antontsev, J.I. Díaz, S. Shmarev and A.J. Kassab, Energy methods for free boundary problems: Applications to non linear PDEs and fluid mechanics. *Progress in Nonlinear Differential Equations and Their Applications*, Vol 48. Appl. Mech. Rev, **55(4)** (2002), B74–B75.

[4] N. Barrouk and S. Mesbahi, Generalized result on the global existence of positive solutions for a parabolic reaction diffusion model with a full diffusion matrix. Studia Universitatis Babes-Bolyai, *Mathematica*, **68**, 2 (2023).

[5] K. Bartkowski and P. Górka, One-dimensional Klein-Gordon equation with logarithmic nonlinearities. *Journal of Physics A: Mathematical and Theoretical*, **41(35)**, 355201, (2008).

[6] J.D. Barrow and P. Parsons, Inflationary models with logarithmic potentials. *Physical Review D*, **52(10)**, 5576, (1995).

[7] I. Bialynicki-Birula and J. Mycielski, Wave equations with logarithmic nonlinearities. Bulletin de l'Academie Polonaise des Sciences. *Serie des Sciences, Mathematiques, Astronomiques et Physiques*, **23(4)** (1975), 461–466.

[8] Y. Cao and Q. Zhao, Initial boundary value problem of pseudo-parabolic Kirchhoff equations with logarithmic nonlinearity. *Authorea Preprints*, (2022).

[9] H. Chen and S. Tian, Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity. *Journal of Differential Equations*, **258(12)** (2015), 4424–4442.

[10] T. Cömert and E. Pişkin, Blow-up of weak solutions for a higher-order heat equation with logarithmic nonlinearity. *Miskolc Mathematical Notes*, **24(2)**, (2023), 749–762.

[11] T. Cömert and E. Piskin, Global existence and stability of solutions for Kirchhoff-type parabolic system with logarithmic source term, *Advanced Studies: Euro-Tbilisi Mathematical Journal*, Special Issue (10 - 2022) 153–170.

[12] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents. Springer, (2011).

[13] P. Dai, C. Mu and G. Xu, Blow-up phenomena for a pseudo-parabolic equation with p-Laplacian and logarithmic nonlinearity terms. *Journal of Mathematical Analysis and Applications*, **481(1)**, 123439, (2020).

[14] H. Ding and J. Zhou, Global existence and blow-up for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity. *Journal of Mathematical Analysis and Applications*, **478(2),** (2019), 393–420.

[15] K. Enqvist and J. McDonald, Q-balls and baryogenesis in the MSSM. *Physics Letters B*, **425(3-4)**, (1998), 309-321.

[16] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.* **263**, (2001), 424–446.

[17] Y. Fu and M. Xiang, Existence of solutions for parabolic equations of Kirchhoff type involving variable exponent. *Applicable Analysis*, **95(3)**, (2016), 524–544.

[18] P. Górka, Logarithmic Klein-Gordon equatio. *Acta Physica Polonica B*, **40(1)**, (2009).

[19] M. Ghisi and M. Gobbino, Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: Time-decay estimates. *Journal of Differential Equations*, **245(10)**, (2008), 2979–3007.

[20] A. Hamydy, M. Massar and N. Tsouli, Existence of solutions for p-Kirchhoff type problems with critical exponent. *Electron. J. Differential Equations*, **105**, (2011), 1–8.

[21] Y. Han and Q. Li, Threshold results for the existence of global and blow-up solutions to Kirchhoff equations with arbitrary initial energy. *Computers and Mathematics with Applications*, **75(9)**, (2018), 3283–3297.

[22] Y. Han, W. Gao, Z. Sun, and H. Li, Upper and lower bounds of blow-up time to a parabolic type Kirchhoff equation with arbitrary initial energy. *Computers and Mathematics with Applications*, **76(10)**, (2018), 2477–2483.

[23] Y. He, H. Gao and H. Wang, Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity. *Computers and Mathematics with Applications*, **75(2)**, (2018), 459–469.

[24] M.O. Korpusov and A.G. Sveshnikov, Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, **43(12)**, (2003), 1835–1869.

[25] M.O. Korpusov and A.G. Sveshnikov, Three-dimensional nonlinear evolutionary pseudoparabolic equations in mathematical physics. *II. Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, **44(11)**, (2004), 2041–2048.

[26] D. Liu, On a p-Kirchhoff equation via fountain theorem and dual fountain theorem. *Nonlinear Analysis: Theory, Methods and Applications*, **72(1)**, (2010), 302–308.

[27] J. Li and Y. Han, Global existence and finite time blow-up of solutions to a nonlocal p-Laplace equation. *Mathematical Modelling and Analysis*, **24(2)**, (2019), 195–217.

[28] N. Lakshmipriya, S. Gnanavel, K. Balachandran and Y.K. Ma, Existence and blow-up of weak solutions of a pseudo-parabolic equation with logarithmic nonlinearity. *Boundary Value Problems*, 2022(1) (2022), 1–17.

[29] R. Pan, Y. Gao and Q. Meng, Properties of Weak Solutions for a Pseudoparabolic Equation with Logarithmic Nonlinearity of Variable Exponents. *Journal of Mathematics*, 2023, (2023).

[30] E. Pişkin and G. Butakin, Blow-up phenomena for a $p(x)$-biharmonic heat equation with variable exponent. *Mathematica Moravica*, **27(2)**, (2023), 25–32.

[31] E. Pişkin and G. Butakin, Existence and Decay of solutions for a parabolic-type Kirchhoff equation with variable exponents. *Journal of Mathematical Sciences and Modelling*, **6(1)**, (2023), 32–41.

[32] E. Piskin and T. Cömert, Qualitative analysis of solutions for a parabolic type Kirchhoff equation with logarithmic nonlinearity. *Open Journal of Discrete Applied Mathematics*, **4(2)**, (2021), 1–10.

[33] S. Toualbia, A. Zaraï and S. Boulaaras, Decay estimate and non-extinction of solutions of p-Laplacian nonlocal heat equations. *AIMS Mathematics*, **5(3)**, (2020), 1663–1680.

[34] C. Yang and Z. Qiuting, Initial boundary value problem of a class of mixed pseudo-parabolic Kirchhoff equations. *Electronic Research Archive*, **29(6)**, (2021), 3833–3851.

[35] G. Kirchhoff, Mechanik. *Leipzig: Teubner, Germany*, 1883