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Investigations of a Riemannian manifold with a quarter symmetric metric (QSM) connection to its tangent bundle

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Abstract

The present paper aims to study a quarter symmetric metric connection in the tangent bundle and investigate an induced metric and connection on a submanifold of co-dimension 2 and hypersurface by means of mathematical operators concerning the QSM connection in the tangent bundle TM. Totally geodesic (TG) and totally umbilical (TU) concerning the QSM connection on the submanifold of co-dimension 2 and hypersurface in TM are obtained.

Keywords: Connection, Gauss equation, Weingarten equation, Codazzi equation, Curvature tensor, Mathematical operators, Partial differential equations, Submanifold, Tangent bundle

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1. Introduction

Let M^n (dim=n) be a Riemannian manifold with the Riemannian metric g. If the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ satisfies the following equation:

$$\tilde{T}(X_0, Y_0) = \tilde{\nabla}_{X_0} Y_0 - \tilde{\nabla}_{Y_0} X_0 - [X_0, Y_0] = 0,$$

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$\forall X_0$ and Y_0 on M^n, a linear connection $\tilde{\nabla}$ is said to be symmetric. If not, it is referred to as a non-symmetric connection.

In the view of Schouten and Friedmann [5], a linear connection $\tilde{\nabla}$ on M^n is called semi-symmetric if $\tilde{T}$ satisfies

$$\tilde{T}(X_0, Y_0) = \omega_0(Y_0)X_0 - \omega_0(X_0)Y_0,$$

where ω_0 is a 1-form associated with a vector field U and related by $\omega_0(X_0) = g(X_0, U)$ for all vector field X_0, Y_0 on M^n. $\tilde{\nabla}$ is said to be metric if $\tilde{\nabla}g = 0$. If not, it is referred to as a non-metric connection.

In 1975, Golab [6] introduced the notion of QS connection $\tilde{\nabla}$ on M^n. A linear connection $\tilde{\nabla}$ is said to be a QSM connection if its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(X_0, Y_0) = \omega_0(Y_0)\phi X_0 - \omega_0(X_0)\phi Y_0$$

and $\tilde{\nabla}g = 0$, where ϕ is a (1,1) tensor field on M^n. In 1991, Mukhopadhyay et al. [22] studied the properties of a QSM connection on a Riemannian manifold (M, g) with an almost complex structure ϕ. Sular et al. [29] studied a QSM connection on a Kenmotsu manifold in 2008. For more details about the semi-symmetric and quarter-symmetric connections, see ([1], [2], [7], [11], [18], [19], [21], [24], [25], [30]).

On the other hand, it is conventional to analyze different geometrical connections and structures use natural operations to convert connections and structures on the base manifold to its tangent bundle. Tani [31] presented the idea of prolongations of surfaces to tangent bundle. Khan et al. ([13], [14], [15]) examined the tangent bundle of various manifolds connected to different connections and gave their theories. Motivated by above theory, investigations are made into lifts of submanifolds of Riemannian manifolds that admit a QSM connection.

The paper is organized as follows. Section 2, devoted to basic definitions. Section 3 deals with the study of submanifold of codimension 2 and hypersurface concerning QSM connection in *TM*. In the end, TG and TU submanifolds of codimension 2 and hypersurface concerning such connection in *TM* are investigated.

2. Preliminaries

2.1. Complete and Vertical lifts

Let *TM* be the tangent bundle of *M*. Superscripts *C* and *V* denote the complete and vertical lifts of the tensor fields. The following characteristics of these lifts in the terms of partial differential equations ([33],[26])

$$X_0^V = X_0^i \frac{\partial}{\partial Y_0^i}, \tag{1}$$

$$X_0^C = X_0^i \frac{\partial}{\partial X_0^i} + \frac{\partial X_0^i}{\partial X_0^j} Y_0^j \frac{\partial}{\partial Y_0^i}. \tag{2}$$

The subsequent characteristics of complete and vertical lifts are provided by [33]

$$(f_0 X_0)^V = f_0^V X_0^V, (f_0 X_0)^C = f_0^C X_0^V + f_0^V X_0^C, \tag{3}$$

$$X_0^V f_0^V = 0, X_0^V f_0^C = X_0^C f_0^V = (X_0 f)^V, X_0^C f_0^C = (X_0 f_0)^C, \tag{4}$$

$$\omega^V(f_0^V) = 0, \omega_0^V(X_0^C) = \omega_0^C(X_0^V) = \omega_0(X_0)^V, \omega_0^C(X_0^C) = \omega_0(X_0)^C, \tag{5}$$

$$[X_0, Y_0]^V = [X_0^C, Y_0^V] = [X_0^V, Y_0^C], [X_0, Y_0]^C = [X_0^C, Y_0^C]. \tag{6}$$

$$F_0^V X_0^C = (F_0 X_0)^V, F_0^C X_0^C = (F_0 X_0)^C, \tag{7}$$

$$\nabla^C_{X_0^C} Y_0^C = (\nabla_{X_0} Y_0)^C, \qquad \nabla^C_{X_0^C} Y_0^V = (\nabla_{X_0} Y_0)^V. \tag{8}$$

2.2. Complete and Vertical lifts of $\Im_s^r(M_{n-1}, M_{n+1})$ to TM_{n+1}

If $\overline{f_0}$ is a function on M_{n-1}. The vertical lift $\overline{f_0}^{\bar{V}}$ of $\overline{f_0}$ to TM_{n+1} is given by $\overline{f_0}^{\bar{V}} = \overline{f_0} \circ \pi_{M_{n-1}}$. Let U be neighborhood of p in M_{n+1}. Then the function $\widehat{f_0}$ fits with $\overline{f_0}$ in $U \cup M_{n+1}$ containing p. The complete lift $\widehat{f_0}^{\hat{C}}$ of $\widehat{f_0}$ is given as $\widehat{f_0}^{C} = Y_0^i \partial_i \widehat{f_0}$ in $\pi_{M_{n+1}}^{-1}(U)$. If $\overline{X_0}$ is an element of $\Im_s^r(M_{n-1}, M_{n+1})$. The vertical lift $\overline{X_0}^{\bar{V}}$ to TM_{n+1} is defined by $\overline{X_0}^{\bar{V}} \widehat{f_0}^{C} = (\overline{X_0} \widehat{f_0})^{\bar{V}}$ and complete lift $\overline{X_0}^{\bar{C}}$ to TM_{n+1} is defined as $\overline{X_0}^{\bar{C}} \widehat{f_0}^{C} = (\overline{X_0} \widehat{f_0})^{\bar{C}}$, for each $\hat{f} \in \Im_0^0(M_{n+1})$ along M_{n-1}. Similarly, If $\overline{\omega_0}$ is an element of $\Im_1^0(M_{n-1}, M_{n+1})$. The vertical lift $\overline{\omega_0}^{\bar{V}}$ and complete lift $\overline{\omega_0}^{\bar{C}}$ to TM_{n+1} are defined by $\overline{\omega_0}^{\bar{V}}(\overline{X_0}^{\bar{C}}) = (\overline{\omega_0}(\overline{X_0}))^{\bar{V}}$ and $\overline{\omega_0}^{\bar{C}}(\overline{X_0}^{\bar{C}}) = (\overline{\omega_0}(\overline{X_0}))^{\bar{C}}$ for each $\overline{X_0} \in \Im_1^0(M_{n+1})$ respectively ([27], [28], [23]).

3. Submanifold of codimension 2

Let $M_{n+1}(dim = n+1)$ be a differentiable manifold and $M_{n-1}(dim = n+1)$ be manifold submerged in M_{n+1} by mapping $\tau : M_{n-1} \to M_{n+1}$. The differentaibility $d\tau$ of the submerged τ is shown by B ([3], [32], [4]). Let us assume that M_{n+1} is a Riemannian manifold with $\tilde{g}$ as its metric tensor. Then we infer

$$g(\phi X_0, Y_0) = \tilde{g}(B\phi X_0, \tilde{B} Y_0) \tag{9}$$

$\forall X_0$, Y_0 in M_{n-1}, where the submanifold M_{n-1} is also a Riemannian manifold with g as its metric tensor.

If M_{n-1} and M_{n+1} are both orientable, we can choose mutually orthogonal unit normals N_1 and N_2 defined along M_{n-1} such that

$$\begin{aligned} \tilde{g}(B\phi X_0, N_1) = \tilde{g}(B\phi X_0, N_2) = \tilde{g}(N_1, N_2) \\ \tilde{g}(N_1, N_1) = \tilde{g}(N_2, N_2) = 0 \end{aligned} \tag{10}$$

$\forall X_0$ in M_{n-1}.

A QSM connection $\tilde{\nabla}$ given as [20]

$$\tilde{\nabla}_{\tilde{X}_0} \tilde{Y}_0 = \tilde{\dot{\nabla}}_{\tilde{X}_0} \tilde{Y}_0 + \tilde{\eta}(\tilde{X}_0)\tilde{\phi}\tilde{Y}_0 - \tilde{g}(\tilde{\phi}\tilde{X}_0, \tilde{Y}_0)\tilde{P} \tag{11}$$

where $\tilde{\nabla}$ be Levi-Civita connection, $\tilde{\eta}$ is a 1-form, $\tilde{\phi}$ is a tensor of type (1,1) such that $\tilde{g}(\tilde{\phi}\tilde{X}_0, \tilde{Y}_0) = \tilde{g}(\tilde{X}_0, \tilde{\phi}\tilde{Y}_0)$ and the vector field $\tilde{P}$ given by $\tilde{g}(\tilde{P}, \tilde{X}_0) = \tilde{\eta}(\tilde{X}_0)$.

Let us put

$$\tilde{P} = BP + \lambda N_1 + \mu N_2, \tag{12}$$

λ and μ functions of M_{n-1}.

Let $\dot{\nabla}$ Riemannian connection induced on M_{n-1} form $\tilde{\dot{\nabla}}$ on the enveloping manifold wrt normals N_1 and N_2, we infer

$$\tilde{\dot{\nabla}}_{BX_0} BY_0 = B(\dot{\nabla}_{X_0} Y_0) + h(X_0, Y_0)N_1 + k(X_0, Y_0)N_2 \tag{13}$$

$\forall X_0$, Y_0 in M_{n-1}, where h and k represent II fundamental tensor of M_{n-1}. Correspondingly, if ∇ be connection induced on M_{n-1} from $\tilde{\nabla}$ on M_{n-1}, we infer

$$\tilde{\nabla}_{BX_0} BY_0 = B((\nabla_{X_0} Y_0) + m(X_0, Y_0)N_1 + n(X_0, Y_0)N_2, \tag{14}$$

m and n represent (0,2)-type tensor fields of M_{n-1}.

Let TM_{n-1} and TM_{n+1} be tangent bundle of M_{n-1} and M_{n+1} individually. Let $\tilde{g}$ be a Riemannian metric given in M_{n-1}. The complete lift of $\tilde{g}$ is $\tilde{g}^C$ in TM_{n-1}. Then the induced metric from $\tilde{g}^C$ is denoted by g^C such that

$$g^C((\phi X_0)^C, Y_0^C) = \tilde{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C) \tag{15}$$

$\forall X_0^C, Y_0^C$ in TM_{n-1}.

Use complete lifts on of the equation (10) by mathematical operators, we get

$$\begin{aligned}
\tilde{g}^C(\tilde{B}(\phi X_0)^C, N_1^{\bar{C}}) &= \tilde{g}^C(\tilde{B}(\phi X_0)^C, N_1^{\bar{V}}) = 0\\
\tilde{g}^C(\tilde{B}(\phi X_0)^C, N_2^{\bar{C}}) &= \tilde{g}^C(\tilde{B}(\phi X_0)^C, N_2^{\bar{V}}) = 0\\
\tilde{g}^C(N_1^{\bar{C}}, N_1^{\bar{C}}) &= \tilde{g}^C(N_1^{\bar{V}}, N_1^{\bar{V}}) = 0\\
\tilde{g}^C(N_2^{\bar{C}}, N_2^{\bar{C}}) &= \tilde{g}^C(N_2^{\bar{V}}, N_2^{\bar{V}}) = 0\\
\tilde{g}^C(N_1^{\bar{C}}, N_2^{\bar{C}}) &= \tilde{g}^C(N_1^{\bar{V}}, N_2^{\bar{C}}) = 0\\
\tilde{g}^C(N_1^{\bar{V}}, N_1^{\bar{C}}) &= \tilde{g}^C(N_2^{\bar{V}}, N_2^{\bar{C}}) = 1
\end{aligned} \tag{16}$$

where $N_1^{\bar{V}}, N_1^{\bar{C}}, N_2^{\bar{V}}$ and $N_2^{\bar{C}}$ are complete and vertical lifts of N_1 and N_2 individually in TM_{n-1}.

Use complete lift on the equations (11) and (12) by mathematical operators, we get

$$\begin{aligned}
\tilde{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C &= \overset{\sim}{\dot{\nabla}}{}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C + (\tilde{\eta}(BX_0)(B\phi Y_0))^C - (\tilde{g}(B\phi X_0, by)\tilde{P})^C\\
\tilde{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C &= \overset{\sim}{\dot{\nabla}}{}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C + (\tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V) + (\tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C)\\
&\quad - (\tilde{g}^C(\tilde{B}\phi X_0)^C, \tilde{B}Y_0^C)\tilde{P}^V - (\tilde{g}^C(\tilde{B}\phi X_0)^V, \tilde{B}Y_0^C)\tilde{P}^C
\end{aligned} \tag{17}$$

$\forall X_0^C, Y_0^C$ in TM_{n-1}, where $\overset{\sim}{\dot{\nabla}}{}^C$ denotes complete lift of $\overset{\sim}{\dot{\nabla}}wrt\tilde{g}^C$ defined by $\tilde{g}^C(\tilde{P}^C, \tilde{X}_0{}^C) = (\tilde{\eta}(\tilde{X}_0))^C$ where $\tilde{\eta}^C, \tilde{\phi}^C, \tilde{P}^C$ are complete lifts of 1-form η, tensor of type(1,1) ϕ and vector field $\tilde{P}$.

$$\begin{aligned}
\tilde{P}^C &= \tilde{B}P^C + \lambda N_1^{\bar{C}} + \mu N_2^{\bar{C}}\\
\tilde{P}^V &= \tilde{B}P^V + \lambda N_1^{\bar{V}} + \mu N_2^{\bar{V}}
\end{aligned} \tag{18}$$

where P is a vector field and λ and μ are functions of M_{n-1}. Now, we are going the prove the following theorem:

Theorem 3.1. *The connection $\dot{\nabla}^C$ induced on $T(M_{n-1})$ from $\overset{\sim}{\dot{\nabla}}{}^C$ of a Riemannian manifold with a QSM connection is also a QSM connection.*

Proof: Let $\dot{\nabla}^C$ be the induced connection from $\overset{\sim}{\dot{\nabla}}{}^C$ on $T(M_{n-1})$ from the connection $\overset{\sim}{\dot{\nabla}}{}^C$ on the enveloping manifold concerning the unit normals N_1 and N_2 whose complete and vertical lifts are $N_1^{\bar{C}}, N_1^{\bar{V}}, N_2^{\bar{C}}$ and $N_2^{\bar{V}}$ respectively. Operating complete lift on both sides of equations (13) and (13), we obtain

$$\begin{aligned}
\overset{\sim}{\dot{\nabla}}{}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C &= B(\dot{\nabla}^C_{X_0^C}Y_0^C) + h^C(X_0^C, Y_0^C)N_1^{\bar{V}} + h^V(X_0^C, Y_0^C)N_1^{\bar{C}}\\
&\quad + k^C(X_0^C, Y_0^C)N_2^{\bar{V}} + k^V(X_0^C, Y_0^C)N_2^{\bar{C}}
\end{aligned} \tag{19}$$

where h^V, h^C, k^V and k^C are complete and vertical lifts of h and k respectively of M_{n-1}.

Similarly, if ∇^C be connection induced on $T(M_{n-1})$ from the QSM connection $\tilde{\nabla}^C$ on $T(M_{n-1})$, we have

$$\tilde{\nabla}^{C}_{\tilde{B}X_0^C}\tilde{B}Y_0^C = B(\nabla^{C}_{X_0^C}Y_0^C) + m^C(X_0^C,Y_0^C)N_1^{\bar{V}} + m^V(X_0^C,Y_0^C)N_1^{\bar{C}} + n^C(X_0^C,Y_0^C)N_2^{\bar{V}} + n^V(X_0^C,Y_0^C)N_2^{\bar{C}} \quad (20)$$

where m^V, m^C, n^V and n^C are complete and vertical lifts of m and n, individually of M_{n-1}.

In the view of equations (17), (18), (19) and (20), we have

$$\begin{aligned} B(\nabla^{C}_{X_0^C}Y_0^C) &+ m^C(X_0^C,Y_0^C)N_1^{\bar{V}} + m^V(X_0^C,Y_0^C)N_1^{\bar{C}} + n^C(X_0^C,Y_0^C)N_2^{\bar{V}} + n^V(X_0^C,Y_0^C)N_2^{\bar{C}} \\ &= B(\dot{\nabla}^{C}_{X_0^C}Y_0^C) + h^C(X_0^C,Y_0^C)N_1^{\bar{V}} + h^V(X_0^C,Y_0^C)N_1^{\bar{C}} + k^C(X_0^C,Y_0^C)N_2^{\bar{V}} + k^V(X_0^C,Y_0^C)N_2^{\bar{C}} \\ &+ (\tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V) + (\tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C) - (\tilde{g}^C(\tilde{B}\phi X_0)^C,\tilde{B}Y_0^C)(\tilde{B}P^V + \lambda N_1^{\bar{V}} + \mu N_2^{\bar{V}}) \\ &- (\tilde{g}^C(\tilde{B}\phi X_0)^V,\tilde{B}Y_0^C)(\tilde{B}P^C + \lambda N_1^{\bar{C}} + \mu N_2^{\bar{C}}) \end{aligned} \quad (21)$$

Comparison of tangential and normal vector fields, we get

$$\begin{aligned} \nabla^{C}_{X_0^C}Y_0^C &= \dot{\nabla}^{C}_{X_0^C}Y_0^C + \tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V + \tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C \\ &- \tilde{g}^C(\tilde{B}(\phi X_0)^C,\tilde{B}Y_0^C)P^V - \tilde{g}^C(\tilde{B}(\phi X_0)^V,\tilde{B}Y_0^C)P^C \end{aligned}$$

where λ and μ are choosen such that

$$\begin{aligned} m^C(X_0^C,Y_0^C) &= h^C(X_0^C,Y_0^C) - \lambda\tilde{g}^C(\tilde{B}(\phi X_0)^C,\tilde{B}Y_0^C) \\ m^V(X_0^C,Y_0^C) &= h^V(X_0^C,Y_0^C) - \lambda\tilde{g}^C(\tilde{B}(\phi X_0)^V,\tilde{B}Y_0^C) \\ n^C(X_0^C,Y_0^C) &= k^C(X_0^C,Y_0^C) - \mu\tilde{g}^C(\tilde{B}(\phi X_0)^C,\tilde{B}Y_0^C) \\ n^V(X_0^C,Y_0^C) &= k^V(X_0^C,Y_0^C) - \mu\tilde{g}^C(\tilde{B}(\phi X_0)^V,\tilde{B}Y_0^C) \end{aligned} \quad (22)$$

Thus,

$$\begin{aligned} \nabla^{C}_{X_0^C}Y_0^C - \nabla^{C}_{Y_0^C}X_0^C - [X_0^C,Y_0^C] &= \tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V + \tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C \\ &- \tilde{\eta}^C(\tilde{B}Y_0^C)(\tilde{B}\phi X_0)^V - \tilde{\eta}^V(\tilde{B}Y_0^C)(\tilde{B}\phi X_0)^C. \end{aligned} \quad (23)$$

thus, the connection ∇^C induced on TM_{n-1} is the QSM connection. Hence, the proof is completed.

As an immediate consequence of the above theorem, we have the following corollary:

Let M_{n+1} (dim=$n+1$) be a differentiable manifold and M_n be hypersurface in M_{n+1} by immersion $\tau : M_{n+1} \to M_n$ and by B the mapping induced by τ from $T(M_n)$ to $T(M_{n+1})$, where $T(M_n)$ and $T(M_{n+1})$ denote tangent bundles of manifold M_n and M_{n+1} respectively.

Corollary 3.1. *The connection induced on the hypersurface TM_n from of a Riemannian manifold with a QSM connection concerning the unit normals $N^{\bar{C}}$ and $N^{\bar{V}}$ is also a QSM connection.*

Proof: Suppose $\dot{\nabla}^C$ be the induced connection from $\tilde{\dot{\nabla}}^C$ on the hypersurface TM_n concerning N whose complete and vertical lifts are $N^{\bar{C}}$ and $N^{\bar{V}}$. Then we have,

$$\tilde{\dot{\nabla}}^{C}_{\tilde{B}X_0^C}\tilde{B}Y_0^C = B(\dot{\nabla}^{C}_{X_0^C}Y_0^C) + h^C(X_0^C,Y_0^C)N^{\bar{V}} + h^V(X_0^C,Y_0^C)N^{\bar{C}} \quad (24)$$

where X_0^C, Y_0^C are arbitrary vector fields on TM_n and h is second fundamental tensor of the hypersurface M_n whose complete and vertical lifts are h^C and h^V respectively on $T(M_n)$.

Let ∇^C be connection induced on hypersurface from $\overset{\sim C}{\dot{\nabla}}$ concerning the unit normal N whose complete and vertical lifts are $N^{\bar{C}}$ and $N^{\bar{V}}$.

$$\tilde{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C = B(\nabla^C_{X_0^C}Y_0^C) + m^C(X_0^C, Y_0^C)N^{\bar{V}} + m^V(X_0^C, Y_0^C)N^{\bar{C}} \tag{25}$$

where m^C and m^V are complete and vertical lifts of tensor field m of type (0, 2) on M_n.

From equation (17), we have

$$\begin{aligned} B(\tilde{\nabla}^C_{X_0^C}Y_0^C) = {}& \overset{\sim C}{\dot{\nabla}}_{\tilde{B}X_0^C}\tilde{B}Y_0^C + (\hat{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V) + (\hat{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C) \\ & - (\hat{g}^C(\tilde{B}\phi X_0)^C, \tilde{B}Y_0^C)\tilde{P}^V - (\hat{g}^C(\tilde{B}\phi X_0)^V, \tilde{B}Y_0^C)\tilde{P}^C \end{aligned} \tag{26}$$

Using equations (24) and (25) in the above equation, we get

$$\begin{aligned} B(\nabla^C_{X_0^C}Y_0^C) + m^C(X_0^C, Y_0^C)N^{\bar{V}} + m^V(X_0^C, Y_0^C)N^{\bar{C}} = {}& B(\overset{\bullet}{\nabla}{}^C_{X_0^C}Y_0^C) + h^C(X_0^C, Y_0^C)N^{\bar{V}} + h^V(X_0^C, Y_0^C)N^{\bar{C}} \\ & + (\tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V) + (\tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C) \\ & - (\tilde{g}^C(\tilde{B}\phi X_0)^C, \tilde{B}Y_0^C)\tilde{P}^V - (\tilde{g}^C(\tilde{B}\phi X_0)^V, \tilde{B}Y_0^C)\tilde{P}^C \end{aligned} \tag{27}$$

Making use of equation (18) in equation (27), we get

$$\begin{aligned} B(\nabla^C_{X_0^C}Y_0^C) + m^C(X_0^C, Y_0^C)N^{\bar{V}} + m^V(X_0^C, Y_0^C)N^{\bar{C}} = {}& B(\overset{\bullet}{\nabla}{}^C_{X_0^C}Y_0^C) + h^C(X_0^C, Y_0^C)N^{\bar{V}} + h^V(X_0^C, Y_0^C)N^{\bar{C}} \\ & + (\tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V) + (\tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C) \\ & - (\tilde{g}^C(\tilde{B}\phi X_0)^C, \tilde{B}Y_0^C)(\tilde{B}P^V + \lambda N^{\bar{V}}) \\ & - (\tilde{g}^C(\tilde{B}\phi X_0)^V, \tilde{B}Y_0^C)(\tilde{B}P^C + \lambda N^{\bar{C}}) \end{aligned} \tag{28}$$

Comparison of tangential and normal vector fields, we get

$$\begin{aligned} \nabla^C_{X_0^C}Y_0^C = {}& \overset{\bullet}{\nabla}{}^C_{X_0^C}Y_0^C + \tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V + \tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C \\ & - \tilde{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)P^V - \tilde{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)P^C \end{aligned}$$

$$m^C(X_0^C, Y_0^C) = h^C(X_0^C, Y_0^C) - \lambda\tilde{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)P^V$$
$$m^V(X_0^C, Y_0^C) = h^V(X_0^C, Y_0^C) - \lambda\tilde{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)P^C$$

Thus,

$$\begin{aligned} \nabla^C_{X_0^C}Y_0^C - \nabla^C_{Y_0^C}X_0^C - [X_0^C, Y_0^C] = {}& \tilde{\eta}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^V + \tilde{\eta}^V(\tilde{B}X_0^C)(\tilde{B}\phi Y_0)^C \\ & - \tilde{\eta}^C(\tilde{B}Y_0^C)(\tilde{B}\phi X_0)^V - \tilde{\eta}^V(\tilde{B}Y_0^C)(\tilde{B}\phi X_0)^C. \end{aligned} \tag{29}$$

Hence, the connection ∇^C induced on M_n is QSM connection. The proof is completed.

4. TG and TU submanifold in the tangent bundle

Consider orthonormal set $\{e_1, e_2, \ldots\ldots, e_{n-1}\}$ vector fields on M_{n-1}. Therefore, the function

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}[h(e_i, e_i) + k(e_i, e_i)]$$

is mean curvature of M_{n-1} wrt a Riemannian connection $\overset{\bullet}{\nabla}$ and

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}[m(e_i,e_i)+n(e_i,e_i)]$$

is mean curvature of M_{n-1} wrt∇.

Some definitions:

Definition 4.1. If h and k are zero, M_{n-1} is said to be TG wrt the Riemannian connection $\overset{\bullet}{\nabla}$.

Definition 4.2. M_{n-1} is said to be TU wrt $\overset{\bullet}{\nabla}$ if h and k are propotional to g.

Now, TM_{n-1} is TG and TU wrt the QSM connection ∇^C for m^C, m^V, n^C and n^V are zero and are proportional to g^C respectively.

Theorem 4.1. *The mean curvature of TM_{n-1} wrt the connection $\overset{\bullet}{\nabla}{}^C$ may coincide with that of TM_{n-1} wrt the connection ∇^C it is necessary and sufficient that $\tilde{P}^C$ and $\tilde{P}^V$ are in the tangent space of TM_{n+1}.*

Proof: Taking consideration of equations (22), we have

$$m^C(e_i^C,e_i^C)+n^C(e_i^C,e_i^C)=h^C(e_i^C,e_i^C)+k^C(e_i^C,e_i^C)-(\lambda+\mu)\tilde{g}^C(\tilde{B}(\phi e_i)^C,\tilde{B}e_i^C)$$
$$m^V(e_i^C,e_i^C)+n^V(e_i^C,e_i^C)=h^V(e_i^C,e_i^C)+k^V(e_i^C,e_i^C)-(\lambda+\mu)\tilde{g}^C(\tilde{B}(\phi e_i)^V,\tilde{B}e_i^C)$$

On comparision, we infer

$$m^C(e_i^C,e_i^C)+n^C(e_i^C,e_i^C)=h^C(e_i^C,e_i^C)+k^C(e_i^C,e_i^C)$$
$$m^V(e_i^C,e_i^C)+n^V(e_i^C,e_i^C)=h^V(e_i^C,e_i^C)+k^V(e_i^C,e_i^C)$$

iff $\lambda = \mu = 0$.

In the view of equation (18), it follows that $\tilde{P}^C = BP^C$ and $\tilde{P}^V = BP^V$. Thus the vector fields $\tilde{P}^C$ and $\tilde{P}^V$ are in the tangent space of TM_{n-1}. This completes the proof.

Theorem 4.2. *The submanifold TM_{n-1} is TU wrt the Riemannian connection $\overset{\bullet}{\nabla}{}^C$ iff it is TU wrt the QSM connection ∇^C.*

Proof: The equation (22) readily lead to the proof.

As an immediate consequence of theorem 4.1 and theorem 4.2, we have the following corollaries on hypersurface:

Corollary 4.1. *In order that the mean curvature of TM_{n-1} wrt the connection $\overset{\bullet}{\nabla}{}^C$ may coincide with that of TM_{n-1} wrt the connection ∇^C it is necessary and sufficient that $\tilde{P}^C$ and $\tilde{P}^V$ are in the tangent space of TM_{n+1}.*

Proof: The proof is trivial.

Corollary 4.2. T*he hypersurface TM_n is TU wrt the Riemannian connection $\overset{\bullet}{\nabla}{}^C$ iff it is TU wrt the QSM connection ∇^C.*

Proof: The proof is trivial.

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References

[1] M. B. Ayed, K. E. Mehdi, Multiplicity results for the scalar curvature problem on half spheres. *Discrete and Continuous Dynamical Systems*, 2024, **44(7)**: 1878–1900. https://doi.org/10.3934/dcds.2024013

[2] O. Bahadir, Lorentzian para-Sasakian manifold with quarter-symmetric non-metric connection, *Journal of Dynamical Systems and Geometricc Theories*, **14** (2016), no. 1, 17–33.

[3] L. S. Das, R. Nivas and M. N. I. Khan, On submanifolds of codimension 2 immersed in a hsu–quarternion manifold, *Acta Mathematica Academiae Paedagogicae Nyiregy-haziensis*, **25** (2009), no. 1, 129–135.

[4] S. Elsaeed, O. Moaaz, K. S. Nisar, M. Zakarya, E. M. Elabbasy, Sufficient criteria for oscillation of even-order neutral differential equations with distributed deviating arguments, *AIMS Mathematics* **9(6)**, 2024, 15996–16014.

[5] A. Friedmann and J. A. Schouten, ÄUber die geometrie der halbsymmetrischen Äubertragung, *Math. Zeitschr*, **21** (1924), 211–223.

[6] S. Golab, On semi-symmetric and quarter-symmetric linear connections, Tensor, N. S. **29** (1975), 249–254.

[7] H. A. Hayden, Subspaces of a space with torsion, *Proc. London Math. Soc.,* **34** (1932), 27–50.

[8] M. N. I. Khan, On generalized Ricci-Recurrent Lorentzian Para-Sasakian manifold, *Proceeding of National Academy of Sciences*, **75(II)**, 147–148 (2005).

[9] M. N. I. Khan and J. B. Jun, Lorentzian almost r-para-contact structure in the tangent bundle, *Journal of the Chungcheong Mathematical Society*, **27(1)**, 29–34, 2014.

[10] M. N. I. Khan, Novel theorems for the frame bundle endowed with metallic structures on an almost contact metric manifold, Chaos, Solitons & Fractals, 146, may 2021, 110872.

[11] D. H. Jin and J. W. Lee, Einstein half lightlike submanifolds of a Lorentzian space form with a semi-symmetric metric connection, *Quaestiones Mathematicae*, **37**, (2014), no. 4, 485–505.

[12] M. N. I. Khan, Lifts of hypersurfaces with quarter-symmetric semi-metric connection to tangent bundles, *Afrika Matematika*, **27** (2014), 475–482.

[13] M. N. I. Khan, F. Mofarreh and A. Haseeb, Tangent bundles of P-Sasakian manifolds endowed with a QSM connection, *Symmetry* **15(3)** (2023), 753.

[14] M. N. I. Khan, F. Mofarreh, A. Haseeb and M. Saxena, Certain results on the lifts from an LP-Sasakian manifold to its tangent bundle associated with a QSM connection, *Symmetry* **15(8)** (2023), 1553.

[15] R. Kumar, L. Colney and M. N. I. Khan, Lifts of a semi-symmetric non-metric connection (SSNMC) from statistical manifolds to the tangent bundle, *Results in Nonlinear Analysis* **6(3)** (2023), 5065.

[16] M. N. I. Khan, Lifts of semi-symmetric non-metric connection on a Kähler manifold, *Afrika Matematika*, **27** (2016), no. 3, 345–352.

[17] M. N. I. Khan, Tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold, *Facta Universitatis, Series: Mathematics and Informatics* **35(1)**, (2020) 167–178.

[18] M. N. I. Khan and L. S. Das, Lifts of Hypersurfaces on a Sasakian Manifold with a Quartersymmetric Semimetric Connection (QSSC) to Its Tangent Bundle, *Universal Journal of Mathematics and Applications*, **6(4)** (2023), 170–175.

[19] Y. Liang, On semi-symmetric recurrent-metric connection, Tensor N. S., **55**(1994), 107–112.

[20] R.S. Mishra and S.N. Pandey, On quarter symmetric metric F-connection, Tensor **34** (1980), 1–7.

[21] A. K. Mondal and U. C. De, Some properties of a QSM connection on a sasakian manifold, *Bulletin of Mathematical analysis and applications*, **1** (2009), no. 2, 99–108.

[22] S. Mukhopadhyay, A. K. Roy and B. Barua, Some properties of a QSM connection on a Riemannian manifold, *Soochow J. of Math.* **17** (1991), no. 2, 205–211.

[23] N. Omar, S. Serra-Capizzano, B. Qaraad, F. Alharbi, O. Moaaz, E. M. Elabbasy, More Effective Criteria for Testing the Oscillation of Solutions of Third-Order Differential Equations. *Axioms* 2024, **13**, 139. https://doi.org/10.3390/axioms13030139

[24] R. Prasad and A. Haseeb, Conformal curvature tensor on K-contact manifold with respect to the QSM connection, *FACTA UNIVERSITATIS (NIS) Ser. Math. Inform*, **32** (2017), no. 4, 503–514.

[25] E. Pak, On the pseudo-Riemannian spaces, *J. Korean Math. Soc.* **6** (1969), 23–31.

[26] R. Qaralleh, A. Tallafha and W. Shatanawi, Some Fixed-Point Results in Extended S-Metric Space of type (α, β), *Symmetry*, **15** (2023), 1790. https://doi.org/10.3390/sym15091790.

[27] M. Rahim, K. Shah, T. Abdeljawad, M. Aphane, A. Alburaikan, and H. A. E. W. Khalifa, Confidence levels-based p, q-quasirung orthopair fuzzy operators and its applications to criteria group decision making problem, *IEEE Access*, **1** (2023), 109983–109996. https://doi.org/10.1109/ACCESS.2023.3321876

[28] K. Suwais, N. Taş, N. Özgür and N. Mlaiki, Fixed Point Theorems in symmetric Controlled M-Metric type Spaces, *Symmetry*, **15** (2023), 1665. https://doi.org/10.3390/sym15091665.

[29] S. Sular, C. Ozgur and U. C. De, Quarter symmetric metric connection in a Kenmotsu manifold, *SUT Journal of mathematics* **44** (2008), no. 2, 297–306.

[30] Y.J. Suh, S. K. Chaubey and M. N. I. Khan, Lorentzian manifolds: A characterization with a type of semi-symmetric non-metric connection, *Reviews in Mathematical Physics*, 2023, 2450001.

[31] M. Tani, Prolongations of hypersurfaces of tangent bundles, *Kodai Math. Semp. Rep* **21** (1969), 85–96.

[32] S. Ali and R. Nivas, On submanifolds immersed in a manifold with quarter-symmetric connection, *Riv. Mat. Univ. Parma.* **6(3)** (2000), 11–23.

[33] K. Yano and S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 1973.