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Exploring maximal and minimal open submsets in m-topological spaces: A comprehensive analysis

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Abstract

Topological structures defined in the context of multisets, a set that allows multiple occurrences of objects, are referred to as M-topological spaces. This article introduces the concept of maximal and minimal open submsets in M-topological spaces. The role of whole elements and part elements in maximal open and minimal open submsets and their uniqueness, together with the topological situation in which an open submset becomes both maximal open and minimal open, is analysed. Some conditions for disconnectedness in M-topological spaces are obtained in light of the fact that the existence of a non-empty proper clopen submset is not enough to establish the disconnectedness of an M-topological space.

Key words and phrases: Multiset Multiset topology M-topology maximal open sets minimal open sets connectedness

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1. Introduction

Two basic requirements in any type of mathematical analysis are precision and accuracy. Often, this lacks in many real-world problems, and imprecision, ambiguity, and repetition are abundant there. Deviations from the conventional way of logic has always resulted in fruitful structures in dealing

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with such scenarios. Predominant among them are fuzzy sets, rough sets, and soft sets, where each has its roots in violating one or more orthodox schools of thought [1–3]. Duplicates occur naturally at various stages of information retrieval, and this is the situation where the need for multisets occurs. Multisets are sets like associative containers where we relax the condition of distinctness from the set structure. Multisets (in short, msets) or bag is a collection of objects in which repetition is allowed. Various studies on multisets have been done by a number of authors [4–9].

The branch of topology in mathematics formalises the study of mathematical objects called topological spaces and their qualitative properties, which do not change under certain types of transformations called continuous mappings. Furthermore, multisets rather than classical sets are a more appropriate option in many real-world scenarios where the objects being studied are non-distinct. Hence, topology defined on multisets (M-topology) is very useful in the sense that similarities and dissimilarities among universes that are multisets can be measured more efficiently.

The study of topological structures on multisets can be seen in a series of papers by Girish and John [10, 11, 4, 5] and by Rajish and John [12–14]. Developed by Pawlak [2], rough set theory provides tools for dealing with incomplete or imprecise information. A rough multiset is an extension of rough set theory to the realm of multisets, which are collections where elements can appear multiple times. In their study, Girish et al. [15, 16] investigated Pawlak's rough set theory within the framework of multiset topology. By replacing the traditional universe with multisets, they employoed a multiset-induced topology to generalize the concept of rough multisets, offering a broader and potentially more nuanced understanding. The importance of M-topology is underscored by the recent surge in research, reflected in the numerous and diverse results obtained by various authors [17–21].

In 2001, Nakaoka and N. Oda [22, 23] introduced the concepts of minimal open sets and maximal open sets in general topology. They studied the relations between these concepts and its connection with the similar concepts defined in terms of closed sets. Further, minimal open sets and maximal closed sets appear in spaces which are locally finite, such as the digital line [24, 25].

This paper is an attempt to introduce minimal open submsets and maximal open submsets in M-topological spaces, discuss how they differ in M-topological spaces, and describe the properties satisfied by the submsets of these categories in M-topological spaces. Further, it's relation with the disconnectedness of an m-space is also analyzed.

The results outlined in the paper can be summarised as follows: In Section 2, all the basic definitions and notions needed for further discussion is collected. Section 3 introduces the concepts of minimal open submsets and maximal open submsets in M-topological spaces. It defines the concept of whole core and whole complement of a submset and analyzes their role in M-topology in connection with the maximal and minimal open submsets. Further it discusses the M-topological situations in which there is only one maximal open submset. It studies the role of maximal and minimal open submsets in connectedness of a M-topological space. Establishes a relation with the count of maximum possible maximal open submsets for an M-topological space with the cardinality of core of a maximal open submset. Also, we explore how they behave in both of the two subspace mtopologies of a submset. The properties satisfied by the closure of a maximal open submset and the interior of the complement of a maximal open submsets are also studied. The existence of a submset which is both maximal open and minimal open restricts the M-topology on it and studied all possibilities.

2. Preliminaries

In this section, we give the ineluctable definitions, concepts, and developments discussed in [12], [24], [11], [25] and [26] that are necessary for our study.

Definition 2.1. [10] *For any ordinary set Y, an mset B drawn from the set Y is a function, Count B or* C_B *and is given by* $C_B : Y \to \mathbb{N}$, *where* $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

We denote the number of occurrence of an element y in the mset B by $C_B(y)$. We can also call it as the multiplicity of the element y in B.

If $Y = \{y_1, ..., y_r\}$ and the multiplicity or count of y_i in B is m_i, then we represent mset B as $B = \{m_1/y_1, m_2/y_2, ..., m_r/y_r\}$.

If the number of occurrence of an element in B is zero, we exclude that element in this representation.

Example 2.2. *Let* $Z = \{x, y, z\}$, *then* $B = \{9/x, 8/y\}$ *is an mset drawn from* Z.

Let A and B be two msets drawn from the same ordinary set Z. Then the operations on mset are defined as follows:

- $A = B \Leftrightarrow C_A(z) = C_B(z), \forall z \in Z$.
- $A \subseteq B \Leftrightarrow C_A(z) \leq C_B(z), \forall z \in Z$.
- If $E = A \cup B \Leftrightarrow C_E(z) = max\{C_A(z), C_B(z)\}, \forall z \in Z$.
- If $E = A \cap B \Leftrightarrow C_E(z) = min\{C_A(z), C_B(z)\}, \forall z \in Z$.
- If $E = A \oplus B \Leftrightarrow C_E(z) = C_A(z) + C_B(z), \forall z \in Z$.
- If $E = A \ominus B \Leftrightarrow C_E(z) = max\{C_A(z) - C_B(z), 0\}, \forall z \in Z$, where $\oplus$ is the mset addition and $\ominus$ is the mset subtraction.

Definition 2.3. [10] *The support set or root set of an mset B drawn from an ordinary set Z, denoted by* B^*, *is the subset of Z given by* $B^* = \{z \in Z: C_B(z) > 0\}$.

Definition 2.4. [10] *The ordinary set Z which we consider for the construction of msets is called its domain. We denote the family of all msets drawn from Z such that the multiplicity of each element is not more than w by* $[Z]^w$.

The set of all msets drawn from Z such that there is no limit on the number of occurrences of an element is denoted by $[Z]^\infty$.

Definition 2.5. [10] *If count of every element of the domain Z is zero in an mset, then it is called the null mset or the empty mset. i.e., an mset N is empty if and only if* $C_N(z) = 0, \forall z \in Z$.

Definition 2.6. [12] *Let N be an mset and M be a partial whole submset of M, then* $x \in N$ *is called whole element of N if* $C_N(x) = C_M(x)$ *and it is called a part element of N if* $C_N(x) < C_M(x)$.

The concept of submset in mset theory is defined in terms of the values of count function and hence there are several types of submsets as follows:

Definition 2.7. [10] *If the count of every element of a submset C of D is equal to that of D, i.e., each element has full multiplicity as in D, then C is called whole submset of D.*

Definition 2.8. [10] *If count of atleast one element of the submset C of D is having full multiplicity as in D, then C is called partial whole submset of D.*

Definition 2.9. [10] *If* $C \subseteq D$ *with support sets of C and D are equal, then C is called a full submset of D.*

Definition 2.10. [12] *Let N be an mset and C be a partial whole submset of N, then* $y \in C$ *is called whole element of C, if* $C_C(x) = C_N(x)$ *and it is called a part element of C if* $C_C(x) < C_N(x)$.

Definition 2.11. [10] *(Power Mset) Let C be an mset. The power mset of C,denoted by* $\mathcal{P}(C)$, *is the collection of all submsets of C. That is,* $D \in \mathcal{P}(C)$ *if and only if* $D \subseteq C$.

Definition 2.12. [10] *The power set of C is the support set of* $\mathcal{P}(C)$ *and is denoted by* $\mathcal{P}^*(C)$.

Now we list the definition of M-topology and a few concepts in M-topological spaces that are needed for our work.

Definition 2.13. [10] *Let N be an mset and $\tau \subset \mathcal{P}^*(N)$. Then τ is an M-topology on N if it satisfies the following three conditions:*

(1) $\varnothing, N \in \tau$.
(2) *the operation mset union is closed in τ.*
(3) *the operation finite mset intersection is closed in τ.*

Let N be an M-topological space with M-topology τ, then a submset V of the mset N is said to be open, if V belongs to the collection τ.

Definition 2.14. [11] *Let N be an M-topological space with M-topology τ and $M \subseteq N$. Then the collection $\tau_M = \{M \cap U \,;\, U \in \tau\}$ is an M-topology on M. The M-topology τ_M on the submset M is called open subspace M-topology or subspace M-topology on M. In this case, M is an M-topological subspace of N and the open submsets of M with respect to subspace M-topology on M are obtained by intersecting open submsets of N with M.*

Definition 2.15. [11] *(Closed Submset) A submset F of an M-space N is said to be closed if the mset complement of it is open, i.e., $N \ominus F \in \tau$.*

Definition 2.16. [10] *Let E be a submset of an M-topological space M in $[Y]^w$.*

- *The interior of the submset E is defined as the mset union of all submsets which are open in M and contained in E, and we denote the interior of E by $int(E)$. i.e., $C_{int(E)}(y) = C_{\cup H}(y)$ where the mset union is over all H which are open in M and $H \subset E$.*
- *The closure of the submset E is defined as the mset intersection of all submsets which are closed in M and containing E. We denote the closure of E by $cl(E)$. i.e., $C_{cl(E)}(y) = C_{\cap K}(y)$ where the mset intersection is over all K which are closed in M and $E \subset K$.*

Definition 2.17. [12] *Let A be a submset of an M-topological space N with M-topology τ. The closed subspace M-topology on A is defined by $\tau_c = \{A \ominus (A \cap U^c) : U \text{ is open in } N\}$.*

Definition 2.18. [12] *Let N be an mset and M be a submset of N, then $x \in M$ is called whole element of M if $C_M(x) = C_N(x)$ and it is called a part element of M if $C_M(x) < C_N(x)$.*

Example 2.19. *Let $N = \{10/x, 10/y, 10/z\}$ be an mset with M-topology $\tau = \{N, \varnothing, \{9/x, 8/y, 7/z\}, \{2/y, 2/z\}\}$. Then $M = \{5/x, 3/y, 6/z\}$ is a submset of N and the open subspace M-topology on M is given by*

$$\begin{aligned}\tau_o &= \{M \cap N, M \cap \varnothing, M \cap \{9/x, 8/y, 7/z\}, M \cap \{2/y, 2/z\}\} \\ &= \{M, \varnothing, \{2/y, 2/z\}\}\end{aligned}$$

The closed submsets of N are $N \ominus N = \varnothing$, $N \ominus \varnothing = N$, $N \ominus \{9/x, 8/y, 7/z\} = \{1/x, 2/y, 3/z\}$ and $N \ominus \{2/y, 2/z\} = \{10/x, 8/y, 8/z\}$.

By taking intersection of these closed submsets with M, we get the submsets $\varnothing$, M, $\{1/x, 2/y, 3/z\}$.

Again, by taking complements of the above submsets in M, we get

$$\tau_c = \{M, \varnothing, \{4/x, 1/y, 3/z\}\} \neq \{M, \varnothing, \{2/y, 2/z\}\} = \tau_o.$$

Note that the open subspace M-topology on M is different from Closed sub-space M-topology on M.

Take $E = \{3/x, 3/y, 3/z\}$ which is a submset of N. Then the largest open submset contained in E is $\{2/y, 2/z\}$. Therefore $int(E) = \{2/y, 2/z\}$.

Now, the smallest closed submset containing E is $\{10/x, 8/y, 8/z\}$ and $cl(E) = \{10/x, 8/y, 8/z\}$.

Consider $P = \{10/x, 6/y, 3/z\}$ as a submset of N, then x is a whole element of P and y and z are part elements of P.

3. Maximal and minimal open submsets in M-topological spaces

This section introduces the concepts of maximal open submsets and minimal open submsets in M-topological spaces and also we discuss several properties and characteristics of these sets in M-topological spaces.

Definition 3.1. *Let N be an mset with τ as an M-topology on it. A proper nonempty open submset U of N is called a maximal open submset of N if there is no proper open submsets properly containing U.*

Definition 3.2. *Let N be an mset and τ be an M-topology on it. A nonempty open submset U of N is called a minimal open submset of N if there is no non-empty open submset properly contained in U.*

Example 3.3. *Let $N = \{8/c, 8/d\}$ and $\tau = \{N, \varnothing, \{4/c, 8/d\}, \{8/c, 4/d\}, \{4/c, 4/d\}\}$. Then, τ is clearly an M-topology on N. Moreover, $\{4/c, 8/d\}$ and $\{8/c, 4/d\}$ are maximal open submsets and $\{4/c, 4/d\}$ is a minimal open submset.*

Definition 3.4. *Let A be a subset of an mset N. Then the whole core of A is the submset of all whole elements in A with full multiplicity and it is denoted by $\tilde{A}$.*

Definition 3.5. *Let A be a submset of an mset N. The mset complement of whole core of A in N is called whole complement of A and is denoted by A^w.*

Here both the whole core and the whole complement of a submset are whole submsets. So a submset A of N is a whole if and only if $\tilde{A} = A$. If A is a whole submset, then the complement operation and the whole complement operation coincides. Hence $A^w = A^c$, for some whole submset A.

Theorem 3.6. *If V and U are open submsets of an M-topological space N with V being maximal open, then either $U \subseteq V$ or $V \cup U = N$. Hence part elements of V are whole elements in any other open submsets U not contained in V. In general $V^w \subseteq$* U for any open submset U/⊆ *V.*

Proof. Given that V and U are open submsets. Then $V \cup U$ is an open submset containing V. But since V is maximal open, the only open submsets containing V are V and N. So $V \cup U$ is V or N. If $V \cup U$ is V, then $U \subseteq V$. Hence, the only possibilities are $V \cup U = N$ or $U \subseteq V$. If $y \in V^w$, then y is either a part element of V or not an element of V. In both cases, $C_V(y) < C_N(y)$. If U/⊆ V, then the only possibility is $V \cup U = N$. Therefore, $C_{V \cup U}(y) = C_N(y)$, i.e., $max\{C_V(y), C_U(y)\} = C_N(y)$. Since y is an element of V^w, $C_V(y) < C_N(y)$, and it follows that $C_U(y) = C_N(y)$. Hence, y is a whole element of U and $V^w \subset U$.

Theorem 3.7. *If V and U are open submsets of an M-topological space N with V as a minimal open, then either V contained in U or V and U are disjoint. Hence any two distinct minimal open submsets are always disjoint.*

Proof. By assumption, V and U are open submsets. Then $V \cap U$ is an open submset contained in V. Since V is minimal open, the only open submsets contained in V are V and $\varnothing$. So $V \cup U$ is either V or $\varnothing$. If $V \cap U$ is V, then $V \subseteq U$. Therefore the only possibilities are either $V \cap U = \varnothing$ or $V \subseteq U$. If V and U are two distinct minimal open submsets, then $V \subseteq U$ is not possible. Hence the the only possibility is $V \cap U = \varnothing$.

Theorem 3.8. *Let V be maximal open submset of an M-topological space N and $y \in V^w$. If there is no proper open submsets in which y appears as whole element, then V is the only maximal open submset of N and V contains all proper open submsets of N.*

Proof. Let y be an element of V^w. Suppose U is a proper open submset of N. Then $V \cup U$ is an open submset containing V and is contained in N. By assumption, y is not a whole element of U and hence $C_U(y) < C_N(y)$. Then $y \in V^w \Rightarrow y$ is a part element of V or not an element of V. In either of the cases, $C_V(y) < C_N(y)$. Hence $max\{C_U(y), C_V(y)\} < C_N(y)$ and $V \cup U \neq N$. This follows that the only possibility is $V \cup U = V$. Consequently $U \subseteq V$ and thus every proper open submset U is contained in V. Therefore V is the only maximal open submset.

Theorem 3.9. *If the union $P\tau$ of all proper open submsets of an M-topological space N is not equal to N, then N has one and only one maximal open subm-set.*

Proof. Let $P\tau$ be the union of all proper open submsets of N. If $P\tau \neq N$ and $P\tau$ contains all proper open submsets of N, then there is no proper open submset of N which properly containing $P\tau$. Hence $P\tau$ is the only maximal open submset of N.

Theorem 3.10. *If V is a maximal open submset of N without whole elements ($\tilde{V} = \varnothing$), then V is the only maximal open submset of N. Consequently V is the union $P\tau$ of all proper open submsets of N.*

Proof. Assume that V is a maximal open submset of N and V has no whole elements. Then for every open submset U of N, $V \cup U$ is an open submset containing V and contained in N. Since V is maximal open, either $V \cup U = N$ or $V = V \cup U$.

If $V \cup U = N$, then $max\{C_V(y), C_U(y)\} = C_N(y), \forall y \in N$. Since V has no whole elements, $C_V(y) < C_N(y)$ and hence $C_U(y) = C_N(y), \forall y \in N \Rightarrow U = N$. So U is not proper, hence it cannot be maximal open. Now the only possibility is $V = V \cup U$. Then $U \subseteq V$ and is a proper open submset of N. So every proper open submset U is contained in V and V itself is also a proper open submset. Hence $P\tau = V$ and V is the only maximal open submset.

Corollary 3.11. *If $P\tau = N$ and V is maximal open submset, then V has some whole elements. i.e., $\tilde{V} \neq \varnothing$. If V and U are two distinct maximal open submsets of N, then $\tilde{V} \cup \tilde{U} = N$.*

Proof. By theorem 3.10, if V has no whole elements, V is the only maximal open submset and $P\tau = V$. So V must have some whole elements.

If V and U are two distinct maximal open submsets, then $V \cup U$ is an open submset containing both V and U. Since V and U are maximal open submsets, the only possibility is $V \cup U = N$ and hence $max\{C_V(z), C_U(z)\} = C_N(z), \forall z \in N$. This implies that, $C_V(z) = C_N(z)$ or $C_U(z) = C_N(z), \forall z \in N$ and for every $z \in N$, either $z \in \tilde{V}$ or $z \in \tilde{U}$. Hence $\tilde{V} \cup \tilde{U} = N$.

Definition 3.12. *An M-topological space N is said to be disconnected if there exist two nonempty disjoint whole open submsets H and G such that $H \cup G = N$.*

Lemma 3.13. *If K and L are proper submsets of N and $K \cup L = N$, then each element of N is a whole element of either K or L.*

Proof. Given that $K \cup L = N$. Then, $max\{C_K(z), C_L(z)\} = C_N(z), \forall z \in N$. Hence either $C_K(z) = C_N(z)$ or $C_L(z) = C_N(z), \forall z \in N$. This implies that z is either a whole element of K or a whole element of L. Hence z is either an element of $\tilde{K}$ or an element of $\tilde{L}$ and we get $N = \tilde{K} \cup \tilde{L}$.

Lemma 3.14. *If K and L are proper submsets of N with $K \cup L = N$ and $K \cap L = \varnothing$, then K and L are whole submsets of N and $K = N \ominus L$.*

Proof. We have, $K \cup L = N$ and $K \cap L = \varnothing$. Then, for a particular z in N, $max\{C_K(z), C_L(z)\} = C_N(z)$ and $min\{C_K(z), C_L(z)\} = 0$ and the count in one among them is $C_N(z)$ and in the other one is zero. i.e., for a particular z, it appears as a whole element of one among them and z is not an element of the other one. Therefore K and L are whole submsets of N and complement to each other.

Theorem 3.15. *If L is a minimal open submset and K is a maximal open submset of an M-topological space N, then either $L \subseteq K$ or K is a whole submset of N and the space N is disconnected.*

Proof. If we consider the property of K as a maximal open submset of N, then one among the following is true for any open submset L of N:

(a) $L \cup K = N$ or (b) $L \subseteq K$.
On the otherway, If we take L as a minimal open submset of N, then one of the following is true for any open submset K of N:

(c) $L \cap K = \varnothing$ or (d) $L \subseteq K$.
Now, (a) & (d) $\Rightarrow K = N$, which is not possible. Again, (b) & (c) $\Rightarrow L = \varnothing$, also not possible.
By considering (b) & (d) $\Rightarrow L \subseteq K$.
The only combination remaining is (a) & (c), i.e., $L \cup K = N$ and $L \cap K = \varnothing$. Then $K = N \ominus L$ and K and L are open whole and hence K is a whole clopen submset of N and the space N is disconnected.

Corollary 3.16. *If there exists a minimal open submset L which is not a submset of a maximal open submset K of an M-topological space N, then L and K are complements to each other and the space N is disconnected.*

Proof. Proof is straight forward.

Theorem 3.17. *If a minimal open submset K of an M-topological space N is maximal open also, then either this submset is the only proper nonempty open submset of N or K is a whole submset of N and the space N is disconnected. If K is whole submset, the only proper nonempty open submsets of N are K and* K^c.

Proof. Suppose the minimal open submset K of N is maximal open and L is a nonempty proper open submset of N. Then, $L \cup K$ is an open submset of N containing K. Since K is maximal open, the only open submsets containing K are K and N. this implies that either $L \cup K = K$ or $L \cup K = N$.

If $L \cup K = K$, then we have $L \subseteq K$ and by the minimality of K, the only open submsets contained in K are $\varnothing$ and K. Therefore $L = K$ or $L = \varnothing$. By assumption, L is nonempty and the only possibility is $L = K$. Hence, the only nonempty proper open submset is K.

Now, consider the case $L \cup K = N$. Then, $L \cap K$ is an open submset contained in K and since K is minimal open, the only open submsets contained in K are $\varnothing$ and K. Therefore, either $L \cap K = \varnothing$ or $L \cap K = K$.

when $L \cap K = K \Rightarrow K \subseteq L$. Since K is maximal open, we get $L = K$ or $L = N$. But by assumption, $L \neq N$ and hence the only possibility for L is K.
Also, $L \cap K = \varnothing \Rightarrow L = N \ominus K$, since $L \cup K = N$.

Thus we have two possibilities for L, either $L = K$ or $L = N \ominus K$. This implies that, K and K^c are the only proper nonempty open submsets of N. Moreover, they are whole and clopen submsets of N and the space N is disconnected.

If K is both maximal open and minimal open, then the only possible M-topologies on N are $\{\varnothing, K, N\}$ and $\{\varnothing, K, K^c, N\}$.

Corollary 3.18. *If a minimal open submset K of an M-topological space N is maximal open and F is a closed submset of N, then either* $K = N \ominus F$ *or K is a whole submset of N and* $K = F$.

Proof. Suppose a minimal open submset K of an M-topological space N is maximal open and F is a closed submset of N. Then $N \ominus F$ is an open submset of N. Now, by theorem 3.17, the only possible nontrivial open submsets of N are K and $K^c = N \ominus K$. If K^c is open then K and K^c are whole submsets and complements to each other. This follows that $K = N \ominus F$ or $N \ominus K = N \ominus F$. Hence either $K = N \ominus F$ or $K = F$ with K as a whole submset of N.

Theorem 3.19. *Let N be an M-topological space. If P and Q are two distinct maximal open whole submsets of N with $P \cap Q$ is closed, then the space N is disconnected.*

Proof. Since P and Q are distinct and maximal open submsets, we have $P \cup Q = N$. Also, since P and Q are whole, $P \cap Q$ is a closed whole submset of N. For any two whole submsets E and F, we have, $E \ominus F = E \cap F^c$. Take $K = P \ominus (P \cap Q) = P \cap (P \cap Q)^c$, which is an open submset of N and $L = Q$. Then, K and L are disjoint open sets such that $K \cup L = N$ and hence the space N is disconnected.

Remark 3.20. *In contrast to general topology, M-topology requires that the submset be a whole submset, and if this limitation is removed, the conclusion of the above theorem 3.19 need not be true as shown in the below example:*

Example 3.21. *Let $N = \{8/r, 8/s\}$ and $\tau = \{N, \varnothing, \{4/r, 8/s\}, \{8/r, 4/s\}, \{4/r, 4/s\}\}$, then τ is clearly an M-topology on N. Also, $P = \{4/r, 8/s\}$ and $Q = \{8/r, 4/s\}$ are maximal open submsets and $P \cap Q = \{4/r, 4/s\}$ is a closed submset. Here, N is not disconnected since P and Q are not whole. Actually, it has a clopen proper submset $\{4/r, 4/s\}$, but it is not enough for saying that N is disconnected in M-topology. In general topology, it is enough for the disconnectedness of a space. But, here we cannot find two non-empty open submsets L and K of N such that $N = L \cup K$ and $L \cap K = \varnothing$. Hence N is not disconnected.*

Theorem 3.22. *If V is a maximal open and a whole submset of an M-topological space N, then either V is dense in N or V is a clopen submset of N.*

Proof. Suppose $cl(V) = E$ for some closed submset E of N. Then $K = N \ominus E$ is an open submset. Since V is a whole submset and $V \subseteq E$, we get $V \cap K = \varnothing$.

Now, since V is maximal open and K is open $\Rightarrow K \cup V = V$ or $K \cup V = N$. Suppose $K \cup V = V$, then $K \cup V = V$ and $V \cap K = \varnothing$. So $K = \varnothing$, that is, $N \ominus E = \varnothing$ and $E = N$. Hence $cl(V) = E = N$ and it follows that V is dense in N.

Now, consider $K \cup V = N$, then $K \cup V = N$ and $V \cap K = \varnothing$. This implies that K and V are whole submsets and $V = N \ominus K = N \ominus (N \ominus E) = E$. Hence $cl(V) = E = V$, a clopen submset of N and N is disconnected.

Remark 3.23. *If we remove the restriction that V be a whole submset of N, outcome of the above theorem 3.22 may change as seen in the following example:*

Example 3.24. *Let $N = \{10/c, 10/d, 10/e\}$ be an M-topological space with the M-topology $\tau = \{N, \varnothing, \{3/c, 2/d\}, \{6/c, 4/d\}\}$. Here $V = \{6/c, 4/d\}$ is clearly a maximal open submset which is not whole.*

Closed submsets in this topology are $\varnothing$, N, $\{7/c, 8/d, 10/e\}$ and $\{4/c, 6/d, 10/e\}$. The smallest closed submset containing V is $\{7/c, 8/d, 10/e\}$.
Hence $cl(V) = \{7/c, 8/e, 10/e\} \neq V$ or N.

Corollary 3.25. *If an M-topological space N has a maximal open whole submset which is not a dense submset of N, then the space N is disconnected.*

Proof. Proof is straight forward.

Theorem 3.26. *If a submset H of an M-topological space N is maximal open, then cl(H) is either*

(i) *a full submset of N with $C_{cl(H)}(y) \geq \frac{C_N(y)}{2}$, $\forall y \in N$, or*
(ii) *$cl(H) = H$ or N.*

Proof. Suppose $cl(H) = E$, for some closed submset E of N. Then $K = N \ominus E$ is an open submset of N and it implies that $H \subseteq H \cup K \subseteq N$. Since, H is maximal, $H \cup K = H$ or N.

When $H \cup K = H$,then $K \subseteq H$, i.e. $E^c \subseteq H$ and $H^c \subseteq E = cl(H)$. Hence $H \subseteq cl(H)$ and $H^c \subseteq cl(H)$ implies that $H \cup H^c \subseteq cl(H) = E$. Clearly, $H \cup H^c$ is a full submset and we get $cl(H)$ as a full submset of N.

For a particular $y \in N$, there are only two possibilities: either $max\{C_H(y), C_{H^c}(y)\} = C_H(y)$ or $max\{C_H(y), C_{H^c}(y)\} = C_{H^c}(y)$.

Suppose $max\{C_H(y), C_{H^c}(y)\} = C_H(y)$, then $C_{H^c}(y) \leq C_H(y)$ and $C_{N\ominus H}(y) \leq C_H(y) \Rightarrow C_N(y) - C_H(y) \leq C_H(y) \Rightarrow C_N(y) \leq 2C_H(y) \Rightarrow \frac{C_N(y)}{2} \leq C_H(y)$.

Since $H \cup H^c \subseteq E$, $C_H(y) \leq C_E(y)$ and hence $\frac{C_N(y)}{2} \leq C_E(y)$.

Now, consider the case $max\{C_H(y), C_{H^c}(y)\} = C_{H^c}(y)$. Then, $C_H(y) \leq C_{H^c}(y)$.

Also, $C_H(y) \leq C_{N\ominus H}(y) \Rightarrow C_H(y) \leq C_N(y) - C_H(y) \Rightarrow 2C_H(y) \leq$

$C_N(y) \Rightarrow C_H(y) \leq \frac{C_N(y)}{2}$.

Also, $C_E(y) \geq C_{H^c}(y) = C_N(y) - C_H(y) \geq C_N(y) - \frac{C_N(y)}{2} = \frac{C_N(y)}{2}$, which implies that $C_E(y) \geq \frac{C_N(y)}{2}$. Hence $C_{cl}(H)(y) \geq \frac{C_N(y)}{2}$, $\forall y \in N$.

Finally, when $H \cup K = N$, we get $H^w \subseteq K = E^c$ and $E \subseteq (H^w)^c = \tilde{H} \subseteq H \subseteq cl(H) = E$. So $\tilde{H} = H$ and hence H is whole submset. By theorem 3.22, it follows that $cl(H) = H$ or N.

Corollary 3.27. *If a submset H of an M-topological space N is maximal open and not closed, then cl(H) contains all elements y of N with count greater than or equal to* $\frac{C_N(y)}{2}$.

Theorem 3.28. *Let H be a maximal open submset of an M-topological spae N. If H is not whole, then int(H^c) contains only the part elements y of H with multiplicity less than or equal to* $\frac{C_N(y)}{2}$. *If H is a whole submset, then either H^c is open or int(H^c) = ∅.*

Proof. Suppose H is not a whole submset of N, then $H \cup H^c \neq N$. Let $L = int(H^c)$. Then L is an open submset and H is a maximal open submset $\Rightarrow L \subseteq H$ or $H \cup L = N$.

If $H \cup L = N$, then $H \cup int(H^c) = N$. Also, $H \cup int(H^c) \subseteq H \cup H^c \neq N$. Hence $H \cup int(H^c) \neq N$. So the only possibility is $L \subseteq H$, i.e., $int(H^c) \subseteq H$ and this concludes that $int(H^c) \subseteq H \cap H^c$. Since $H \cap H^c$ contains part elements y of H with count $\leq \frac{C_N(y)}{2}$, $int(H^c)$ contains only the part elements y of H with multiplicity less than or equal to $\frac{C_N(y)}{2}$. If we consider H as a whole submset, then there are two possibilities: $L \subseteq H$ or $H \cup L = N$. If $L \subseteq H$, then $int(H^c) \subseteq H$ and hence $int(H^c) \subseteq H \cap H^c = \emptyset$. On the other way, If $H \cup L = N$, then $H^c \subseteq L \Rightarrow H^c \subseteq int(H^c) \Rightarrow int(H^c) = H^c$. Hence H^c is open.

Theorem 3.29. *If K and L are distinct maximal open submsets of an M-topological space N, then $K^w \nsubseteq L^w$ and $L^w \nsubseteq K^w$.*

Proof. Suppose $K^w \subseteq L^w$. Then, $(\tilde{K})^c \subseteq (\tilde{L})^c \Rightarrow \tilde{L} \subseteq \tilde{K}$. Since K and L are maximal, $K \subseteq L \cup K \subseteq N$ and $L \subseteq L \cup K \subseteq N$. If $y \in K^w \subseteq L^w \subseteq N$, then $y \notin \tilde{K}$ and $y \notin \tilde{L}$. So $C_K(y) < C_N(y)$ and $C_L(y) < C_N(y)$ implies that $K \cup L \neq N$. Therefore the only possibilities are $K = L \cup K$ and $L = L \cup K$. Hence $K = L$.

Theorem 3.30. *If L and K are distinct maximal open submsets of an M-topological spae N and $y \in K^w$, then $y \notin L^w$. Consequently $K^w \subseteq \tilde{L}$ and $K^w \cap L^w = \emptyset$.*

Proof. Given that K and L are distinct and maximal, then $K \cup L = N$ and hence $N = \tilde{K} \cup \tilde{L}$. Also, if $y \in N$ is not an element of $\tilde{K}$, then it belongs to $\tilde{L}$.

Suppose $y \in K^w$. Since K^w, $\tilde{K}$, $(\tilde{K})^c$, L^w, $\tilde{L}$ and $(\tilde{L})^c$ are whole submsets and $y \in (\tilde{K})^c \Rightarrow y \notin \tilde{K} \Rightarrow y \in \tilde{L} \Rightarrow y \notin (\tilde{L})^c \Rightarrow y \notin L^w$. Hence $K^w \subseteq \tilde{L}$.

In a similar manner, if $y \in L^w$ then $y \notin K^w$ and consequently $K^w \cap L^w = \emptyset$.

Corollary 3.31. *If K is a maximal open submset of an M-topological space N such that $K^w = E$, then K is the only maximal open submset for which $K^w = E$.*

Theorem 3.32. *Let K be a maximal open submset of an M-topological space N. If $|(\tilde{K})^*| = n$, then N have at most n + 1 maximal open submsets.*

Proof. Suppose L is a maximal open submset of N different from K, then $K \cup L = N$. If $y \notin \tilde{K}$, then $C_K(y) < C_N(y)$. So $C_L(y) = C_N(y) \Rightarrow y \in \tilde{L} \Rightarrow y \notin L^w$ and hence $L^w \subseteq \tilde{K}$. By corollary 3.31, corresponding to a whole submset there is at most one maximal open submset whose whole complement is the same whole submset. If L is a maximal whole submset different from K, then $L^w \subseteq \tilde{K}$. Therefore, corresponding to every whole submset of K~ there may be at most one maximal open submset. Now by theorem 3.30, whole complements of two distinct maximal open submsets are disjoint, and the problem reduces to finding the maximum number of disjoint nonempty whole submsets of $\tilde{K}$. The maximum is occurred when we take each submset as a submset of $\tilde{K}$ with one element of $\tilde{K}^*$ with full multiplicity as in N. This implies that there exists n such submsets and let $A_1, A_2, ..., A_n$ be such submsets of $\tilde{K}$. Corresponding to each A_i, there may be at most one maximal open submset W_i for which $W_i^w = A_i$. Hence the maximum number of maximal open submsets we can form other than K is n and the maximum number of possible whole submsets of N is $n + 1 = |(\tilde{K})^*| + 1$.

Theorem 3.33. *Let K be a submset of an M-topological space N and H be a maximal open submset of N.* If $K \nsubseteq H$ *and $K \cap H \neq \varnothing$, then the open submset corresponding to H in the open subspace M-topology of K is a maximal open submset of K.*

Proof. Suppose $K \nsubseteq H$, $K \cap H \neq \varnothing$ and H is a maximal open submset. Then, $K \cap H$ is a proper nonempty open submset of K and for every open submset L of N, either $L \subseteq H$ or $H \cup L = N$. Let H' and L' be the open submsets corresponding to H and L respectively, in the open subspace M-topology. So $H' = K \cap H$ and $L' = K \cap L$.

Now, $L \subseteq H \Rightarrow K \cap L \subseteq K \cap H \Rightarrow L' \subseteq H'$ and

$H \cup L = N \Rightarrow K \cap (H \cup L) = K \cap N \Rightarrow (K \cap H) \cup (K \cap L) = K \Rightarrow H' \cup L' = K$.

Every open submset L' of K with respect to open subspace M-topology is of the form $K \cap L$, for some L open in N. Then, for every open submset L' of K, either $L' \subseteq H'$ or $H' \cup L' = K$. This means that there is no proper open submset L' of K properly containing H'. Hence H' is a maximal open submset in K with respect to open subspace M-topology.

Theorem 3.34. *Let K be a submset of an M-topological space N and H be a maximal open submset of N. If $K \nsubseteq \tilde{H}$ and $K \nsubseteq H^c$, then the open submset corresponding to H in the closed subspace M-topology of K is a maximal open submset of K.*

Proof. Since $K \nsubseteq \tilde{H}$ and $\tilde{H}$ is a whole submset $\Rightarrow \exists y \in K$ such that $y \notin \tilde{H} \Rightarrow \exists y \in K \cap H^c \Rightarrow K \cap H^c \neq \varnothing \Rightarrow K \ominus (K \cap H^c)$ is a proper submset of K. Also, by $K \nsubseteq H^c$, we get $K \ominus (K \cap H^c)$ is not empty and obtain the condition that $K \ominus (K \cap H^c)$ is a nonempty proper open submset of K.

Let H'' and L'' be the open submsets corresponding to H and L respectively, in the closed subspace M-topology. Then, $H'' = K \ominus (K \cap H^c)$ and $L'' = K \ominus (K \cap L^c)$. That is, for every open submset L of N, either $L \subset H$ or $H \cup L = N$.

Now, $L \subseteq H \Rightarrow H^c \subseteq L^c \Rightarrow (K \cap H^c) \subseteq (K \cap L^c) \Rightarrow K \ominus (K \cap L^c) \subseteq K \ominus (K \cap H^c) \Rightarrow L'' \subseteq H''$ and $H \cup L = N \Rightarrow H^c \cap L^c = \varnothing \Rightarrow (K \cap H^c) \cap (K \cap L^c) = \varnothing$. By taking the complements, we get $(K \ominus (K \cap H^c)) \cup (K \ominus (K \cap L^c)) = K \Rightarrow H'' \cup L'' = K$. Since every open submset L'' of K with respect to closed subspace M-topology is of the form $K \ominus (K \cap L^c)$ for some L open in N, it follows that for every open submset L'' of K, either $L'' \subseteq H''$ or $H'' \cup L'' = K$. That is, there is no proper open submset L'' of K properly containing H''. Hence H'' is a maximal open submset in K with respect to the closed subspace M-topology.

Theorem 3.35. *Let K be a submset of an M-topological space N and H be a minimal open submset of N. If $K \cap H \neq \varnothing$, then the open submset corresponding to H in the open subspace M-topology of K is a minimal open submset in K.*

Proof. Let L' be an submset of K that is open in the open subspace M-topology τ_o. Then there exists an open submset L of N such that $L' = K \cap L$. Since H is minimal, $H \subseteq L$ or $H \cap L = \varnothing$.

If $H \subseteq L$, then $K \cap H \subseteq K \cap L$, i.e., $K \cap H \subseteq L'$. and if $H \cap L = \varnothing$, then $(K \cap H) \cap (K \cap L) = \varnothing$, i.e., $(K \cap H) \cap L' = \varnothing$. Thus, for every open submset L' of K, either $K \cap H \subseteq L'$ or $(K \cap H) \cap L' = \varnothing$. So there is no nonempty open submset L' properly contained in $K \cap H$ and hence $K \cap H$ is a minimal open submset of K.

Theorem 3.36. *Let K be a submset of an M-topological space N and H be a minimal open submset of N. If $K \nsubseteq H^c$, then the open submset corresponding to H in the closed subspace M-topology of K is a minimal open submset in K.*

Proof. Let L'' be a nonempty submset of K which is open in the closed subspace M-topology τ_c. Then there exists a nonempty open submset L of N such that $L'' = K \ominus (K \cap L^c)$. Since H is minimal, $H \subseteq L$ or $H \cap L = \varnothing$. If $H \subseteq L$, then $L^c \subseteq H^c \Rightarrow (K \cap L^c) \subseteq (K \cap H^c) \Rightarrow K \ominus (K \cap H^c) \subseteq K \ominus (K \cap L^c)$. i.e., $K \ominus (K \cap H^c) \subseteq L''$.

If $H \cap L = \varnothing$, then $H^c \cup L^c = N \Rightarrow (K \cap H^c) \cup (K \cap L^c) = K$. By taking complements in K, $(K \ominus (K \cap H^c)) \cap (K \ominus (K \cap L^c)) = \varnothing$. i.e., $(K \ominus (K \cap H^c)) \cap L'' = \varnothing$.

That is, for every open submset L'' of K, either $(K \ominus (K \cap H^c)) \subseteq L''$ or $(K \ominus (K \cap H^c)) \cap L'' = \varnothing$. So there is no nonempty open submset L'' properly contained in $K \ominus (K \cap H^c)$ and hence $K \ominus (K \cap H^c)$ is a minimal open submset of K.

Theorem 3.37. *If G is a full submset which is also a minimal open submset of an M-topological space N, then N has one and only one minimal open submset. Consequently every nonempty open submset is a full submset of N.*

Proof. Suppose $L \neq \varnothing$ is an open submset of N. Since G is a full submset, it intersects with L and $G \cap L \neq \varnothing$. Also, as G is minimal and $G \cap L \neq \varnothing$, the only possibility is $G \cap L = G$. Then, $G \subseteq L$ and consequently L is a full submset of N. Hence every nonempty open submset of N is a full submset of N.

Lemma 3.38. *Let K be a submset of a multiset N.*

(a) *If $K^c \subseteq K$, then $C_K(y) \geq \frac{C_N(y)}{2}$, $\forall y \in N$, and consequently K is a full submset.*

(b) *If $K \subseteq K^c$, then $C_K(y) \leq \frac{C_N(y)}{2}$, $\forall y \in N$, and consequently K^c is a full submset.*

(c) *If $K^c \cap K = \varnothing$ or $K^c \cup K = N$, then K and K^c are whole submsets of N.*

Proof.

(a) Suppose $K^c \subseteq K$. Then $C_Kc\ (y) \leq C_K(y)$, $\forall y \in N$. Hence $C_N(y) - C_K(y) \leq C_K(y) \Rightarrow C_K(y) \geq \frac{C_N(y)}{2}$, $\forall y \in N$. Since count of every element y in K is greater than $\frac{C_N(y)}{2}$, every element of K has nonzero count and K is a full submset of N.

(b) In a similar way, we can prove *b*)

(c) If $K \cap K^c = \varnothing$, then $\forall y \in N$, $min\{C_K(y), C_N(y) - C_K(y)\} = 0 \Rightarrow C_K(y) = 0$ or $C_N(y) - C_K(y) = 0$. For every $y \in N$, either $C_K(y) = 0$ or $C_K(y) = C_N(y)$. So K is a whole submset and consequently K^c is also a whole submset. The condition $K \cap K^c = \varnothing$ is equivalent to say that $K^c \cup K = N$.

Theorem 3.39. *If a submset H of an M-topological space N is minimal open, then one among the following is satisfied by H:*

(a) $int(H^c) \subseteq int(\widetilde{(H^c)})$.

(ii) $C_H(y) \leq \frac{C_N(y)}{2}$, $\forall y \in H$, *H^c is a full submset and* $H \subseteq int(H^c)$.

Proof. Let $G = int(H^c)$. Since H is minimal open, $H \cap G = \emptyset$ or $H \subseteq G$. If $H \cap G = \emptyset$, then $\forall y \in G$, $C_H(y) = 0 \Rightarrow y \in H^c$ with full multiplicity. Hence $G \subseteq \widetilde{(H^c)}$. Consequently, $int(H^c) \subseteq int(\widetilde{(H^c)})$. If $H \subseteq G$, then $H \subseteq int(H^c) \subseteq H^c$. Therefore $C_H(y) \leq \frac{C_N(y)}{2}$, $\forall y \in N$.

Now, $H \subseteq int(H^c) \subseteq H^c \Rightarrow H \subseteq (int(H))^c \subseteq H^c \Rightarrow H \subset int(H^c) \cap (int(H^c))^c$ and $int(H^c) \cup (int(H^c))^c \subseteq H^c$. Hence $H \subseteq int(H^c) \cap cl(H)$ and H^c is a full submset of N.

Corollary 3.40. *If a submset H of an M-topological space N is minimal open and* $int(\widetilde{H^c}) = \emptyset$, *then H is the only minimal open submset of N.*

Proof. Suppose $int(\widetilde{(H^c)}) = \emptyset$. Let L be an open submset of N. Then either $H \cap L = \emptyset$ or $H \subseteq L$.

If $H \cap L = \emptyset$, then $L \subseteq \widetilde{(H^c)}$ and $int(\widetilde{(H^c)}) = \emptyset$ implies that $L = \emptyset$.

Thus for every nonempty open submset L, $H \subseteq L$ and hence H is the only minimal open submset of N.

Theorem 3.41. *If a submset H of an M-topological space N is minimal open, then one among the following is satisfied by H:*

(i) *Part element of cl(H) is not an element of H and hence every element* $y \in H$ *appears as a whole element of cl(H).*

(ii) $C_H(y) \leq \frac{C_N(y)}{2}$, *$H^c$ is a full submset and* $H \subseteq int(H^c)$

Proof. Suppose H is a minimal open submset and $L = (cl(H))^c$. Then L is open and either $H \subseteq L$ or $H \cap L = \emptyset$. If $H \cap L = \emptyset$, then $H \cap (cl(H))^c = \emptyset$. Assume that y is a part element of $cl(H)$ which implies that $y \in ((cl(H))^c$. Since $H \cap (cl(H))^c = \emptyset$, $y \notin H$. Hence part element of $cl(H)$ is not an element of H. So every $y \in H$ appears as a whole element of $cl(H)$ and we get $(H^c)^w \subseteq \widetilde{cl(H)}$.

If $H \subseteq L$, then $H \subseteq (cl(H))^c \Rightarrow cl(H) \subseteq H^c$ and hence $H \subseteq cl(H) \subseteq H^c$. By taking complements, $H \subseteq (cl(H))^c \subseteq H^c$. So $H \subseteq cl(H) \cap (cl(H))^c$ and $cl(H) \cup (cl(H))^c \subseteq H^c$ and it follows that H^c is a full submset with count greater than or equal to half of the full multiplicity. Hence elements of H has multiplicity less than or equal to half of the full multiplicity.

Also, since $(cl(H))^c = int(H^c)$, $H \subseteq int(H^c) \cap cl(H)$.

Corollary 3.42. *Let H be a full submset of an M-topological space N which is also minimal open. If there is an element* $y \in H$ *such that* $C_H(y) > \frac{C_N(y)}{2}$, *then H is dense in N.*

Proof. Given that H is minimal open submset and it has an element y with count greater than $\frac{C_N(y)}{2}$. By theorem 3.41, every elements in H appears as a whole element in $cl(H)$. Since H is a full submset, $cl(H)$ contains all elements of N with full multiplicity. Hence $cl(H) = N$ and H is dense in N.

Theorem 3.43. *If H and G are minimal open submsets and* $H \cup G$ *is a full submset of an M-topological space N, then H and G are the only minimal open submsets of N and for every open submset L of N, either* $H \subseteq L$ *or* $G \subseteq L$.

Proof. Let L be an open submset in N. Since H and G are minimal open submsets, $\emptyset \subseteq H \cap L \subseteq H$ and $\emptyset \subseteq G \cap L \subseteq G$. Therefore, the only possibilities are:

(i) either $H \cap L = \emptyset$ or $H \subseteq L$.

(ii) either $G \cap L = \emptyset$ or $G \subseteq L$.

Suppose $H \cap L = \varnothing$. Since $H \cup G$ is a full submset, $L \cap (H \cup G) \neq \varnothing \Rightarrow (L \cap H) \cup (L \cap G) \neq \varnothing \Rightarrow \varnothing \cup (L \cap G) \neq \varnothing \Rightarrow L \cap G \neq \varnothing$.

Then, G is minimal open and $L \cap G \neq \varnothing \Rightarrow G \subseteq L$.

In a similar way, if we assume $G \cap L = \varnothing$, we get $H \subseteq L$ and for every nonempty open submset L, either $H \subseteq L$ or $G \subseteq L$. Hence H and G are the only minimal open submsets of N.

Theorem 3.44. *Let* U *be the collection of all minimal open submsets in an M-topological space N. Then the elements of* U *are mutually disjoint. If* $\cup_{U \in \mathcal{U}} U$ *is a full submset, then for every nonempty open submset G of N, there exists a minimal open submset* $U \in$ U *such that* $U \subseteq G$.

Proof. Let U_1 and U_2 be two minimal open submsets of M. Then $U_1 \cap U_2$ is an open submset contained in both U_1 and U_2 and either $U_1 \cap U_2 = \varnothing$ or $U_1 = U_2$ which implies that elements of U are mutually disjoint.

Let G be an non-empty open submset of N. Since $\cup_{U \in \mathrm{U}} U$ is a full submset, there is a minimal open submset U in U which intersects with G, i.e., $U \cap G \neq \varnothing$. As U is minimal open, the only possibility is $U \cap G = U$. Hence $U \subseteq G$.

Theorem 3.45. *Let U be the collection of all minimal open submsets in an M-topological space N. If* $\cup_{U \in U} U$ *is a full submset of N and S is a nonempty closed partial whole submset of N, then* $int(S) \neq \varnothing$. *In other words, there exists a minimal open submset U in U such that* $U \subseteq S$.

Proof. Let S be a closed partial whole submset of N. Then S has a whole element y. Since $\cup_{U \in U} U$ is a full submset, there exists a minimal open submset U in U which also contains y. Now, since the minimal open submsets are mutually disjoint, U is the only minimal open submset which contains y. As U is minimal and S^c is open, either $U \cap S^c = \varnothing$ or $U \subseteq S^c$. But $U \subseteq S^c$ is not possible, since $y \notin S^c$. Therefore the only possibility is $U \cap S^c = \varnothing$ which follows that $z \in U \Rightarrow z \notin S^c \Rightarrow z \in S$. Hence $U \subseteq S$ and $int(S) \neq \varnothing$.

Theorem 3.46. *Let* $H \neq \varnothing$ *be open and a whole submset of an M-topological space N. Then the following three conditions are equivalent:*

(a) *H is a minimal open submset of N.*
(b) $H \subseteq cl(T)$ *for any nonempty submset T of H.*
(c) $cl(H) = cl(T)$ *for any nonempty submset T of H.*

Proof. (a) $\Rightarrow$ (b).
Assume (a). Let T be any nonempty submset of H. Suppose $cl(T) = F$. Then $T \subseteq F$ and F is a closed submset of N. So $N \ominus F$ is an open submset. By the minimality of H, $H \cap (N \ominus F)$ is $\varnothing$ or H. Now we claim that $H \cap (N \ominus F) = H$ is not possible. since T is nonempty submset of H and $T \subset F$, F contains an element y of H with non-zero count less than or equal to the count of it in H. Then, $C_{N \ominus F}(y) < C_N(y) = C_H(y)$ and hence $C_{H \cap (N \ominus F)}(y) \neq C_H(y)$. So $H \cap (N \ominus F) = H$ is not possible and the only possibility is $H \cap (N \ominus F) = \varnothing$. In multiset theory, $C \ominus D = C \cap D^C$ if one among them is a whole submset. Therefore, $H \cap (N \ominus F) = H \cap F^c = H \ominus F = \varnothing$.

If $H \ominus F = \varnothing$, then $H \subseteq F = cl(T)$.

(b) $\Rightarrow$ (c).

Assume (b). i.e., $H \subseteq cl(T)$ for any nonempty subset T of H. So $cl(H) \subseteq cl(cl(T)) = cl(T)$. Since $T \subseteq H \Rightarrow cl(T) \subseteq cl(H)$ and this clearly follows that $cl(H) = cl(T)$.

(c) $\Rightarrow$ (a).

Assume (c), i.e., $cl(H) = cl(T)$ for any nonempty subset T of H. Suppose L is an open submset such that $\varnothing \subseteq L \subseteq H$. Now we need to prove that $L = \varnothing$ or H. Since $L \subseteq H \subseteq N \Rightarrow H \ominus L \subseteq N \ominus L$. If $H \ominus L = \varnothing$, then $H \subseteq L$ and hence $L = H$. If $H \ominus L \neq \varnothing$, then $H \ominus L$ is a nonempty submset of H. Therefore by assumption, $cl(H \ominus L) = cl(H)$. But $N \ominus L$ is a closed set containing $H \ominus L$. So $cl(H) = cl(H \ominus L) \subset N \ominus L$. Hence $H \subseteq cl(H) \subseteq N \ominus L$. Suppose there is an element $y \in L \subseteq H$. Then, $0 \neq C_L(y) \leq C_H(y) = C_N$

(y). Now, consider $C_N \ominus_L(y) = Max\{C_N(y) - C_L(y), 0\} = C_N(y) - C_L(y) < C_N(y)$. Therefore $H \subseteq N \ominus L$ is impossible, since $C_H(y) = C_N(y)$. Hence existence of such an element $y \in L$ is not possible and $L = \varnothing$ implies that H is a minimal open submset.

Remark 3.47. *In general topology, the above theorem 3.46 is true for any open submset. But in M-topology we need the restriction that V is whole. It need not be true, if V is not a whole submset of N as shown in the following example:*

Example 3.48. *Let* $N = \{10/c, 10/d, 10/e\}$ *be an mset with M-topology,* $\tau = \{M, \varnothing, \{6/c, 4/d, 6/e\}, \{3/c, 2/d, 6/e\}\}$. *Here,* $H = \{3/c, 2/d, 6/e\}$ *is clearly a minimal open submset which is not a whole submset. Then,* $T = \{2/c, 2/d\} \subseteq H$.
Closed submsets in this topology are $\varnothing$, N, $\{4/c, 6/d, 4/e\}$, $\{7/c, 8/d, 4/e\}\}$. *The smallest closed submset containing T is* $\{4/c, 6/d, 4/e\}$. *The closure of T is* $cl(T) = \{4/c, 6/d, 4/e\}$. Hence $H \nsubseteq cl(T))$ *and* $cl(H) = N \neq cl(T)$.

Theorem 3.49. *Let V be a minimal open whole submset of an M-topological space N. Then any non-empty submset T of V is a pre-open submset of N.*

Proof. By theorem 3.46, $V \subseteq cl(T)$ and hence $int(V) \subset int(cl(T))$. Since V is an open submset, we have $T \subseteq V = int(V) \subseteq int(cl(T))$. Hence T is pre-open.

4. Conclusion

In this study, we introduced and thoroughly examined the novel concepts of minimal open subsets and maximal open subsets within the framework of M-topology. Our analysis delved into their intricate interplay with fundamental M-topological concepts such as the whole core, whole complement, closure, interior, and connectedness. Furthermore, we explored their behavior within the context of both subspace M-topologies of a given subset. One significant contribution of this research is the establishment of a result that establishes a connection between the maximum achievable number of maximal open subsets and the cardinality of the core of a maximal open subset. This result not only adds depth to our understanding of M-topology but also offers valuable insights into the structural properties of maximal open subsets in this context. As a future direction, all the investigations that have been done in this work can be carried out by introducing the concept of minimal closed submset and maximal closed submset. It is also possible to extend these discussions to explore the behavior of minimal and maximal closed submsets in subspace M-topologies of a given subset, discussing various properties arising from different combinations.

M-topology's reach extends beyond mathematical exploration, holding significant promise for applications in various scientific disciplines. Its potential includes detecting and locating of mutations in DNA and RNA structures, uncovering similarities and discrepancies within gene sequences, and potentially opening doors to further breakthroughs in diverse scientific fields.

References

[1] L.A. Zadeh, *Fuzzy sets, Inform. and control*. 8 (1965) 338–353.
[2] Z. Pawlak, Rough sets, *International Journal of Computer and Information Science*. 11 (1982) 341–356.
[3] S.J. John, *Soft Sets: Theory and Applications*, Springer, Cham, 2021.
[4] K. P. Girish, S. J. John, Relations and functions in multiset context, *Information Sciences*. 179(6) (2009) 758–768.
[5] K. P. Girish, S. J. John, Rough multisets and information multisystems, *Advances in Decision Sciences*. 2011.
[6] W. D. Blizard, Multiset theory, *Notre Dame J. Formal Logic*. 30 (1989) 36–66.
[7] D. Singh, J. N. Singh, Some combinatorics of multisets, *International Journal of Mathematical Education in Science and Technology*. 34 (2003) 489–499.
[8] W.D.Blizard, The development of multiset theory, *Modern Logic*. 1 (1991) 319-352.
[9] W.D. Blizard, Real-valued multisets and fuzzy sets, *Fuzzy Sets and Systems*. 33 (1989), 77-97.

[10] Girish K.P., Sunil Jacob John, Multiset topologies induced by multiset relations, *Information Sciences*. 188(0) (2012) 298–313.
[11] Girish K.P., Sunil Jacob John, On multiset topologies, *Theory and Applications of Mathematics and Computer Science*. 2(1) (2012) 37–52.
[12] Rajish Kumar P., Sunil Jacob John, On redundancy, separation and connectedness in multiset topological spaces, *AIMS Mathematics*. 5(3) (2020) 2484–2499.
[13] Kumar, P. Rajish, Sunil Jacob John, Sovan Samanta, Compactness and subspace M-topologies, *Soft Computing*. 26(9) (2022) 4115-4122.
[14] Kumar, P. Rajish, Sunil Jacob John, Role of interior and boundary in M-topology, AIP conference proceedings. 2336(1) (2021).
[15] Girish KP, John SJ, Rough multiset and its multiset topology, Transactions on Rough Sets XIV. (2011) 62–80.
[16] Girish KP, John SJ, On rough multiset relations, *International Journal of Granular Computing, Rough Sets and Intelligent Systems*. 3(4) (2014) 306–326.
[17] Shravan, Karishma, Binod Chandra Tripathy, Multiset mixed topological space, *Soft computing*. 23(20) (2019) 9801–9805.
[18] Mahanta, Sougata, S. K. Samanta, Compactness in multiset topology, *Int. J. Math. Trends Tech.* 47 (2017) 275–282.
[19] El-Sharkasy, M. M., Wafaa M. Fouda, Mohamed S. Badr, Multiset topology via DNA and RNA mutation, *Mathematical Methods in the Applied Sciences.* 41(15) (2018) 5820–5832.
[20] Shravan, Karishma, Binod Chandra Tripathy, Multiset ideal topological spaces and local functions, *Proyecciones (Antofagasta).* 37(4) (2018) 699–711.
[21] Das, Rakhal, Binod Chandra Tripathy, Neutrosophic multiset topological space, *Neutrosophic sets and systems*. 35 (2020) 142–152.
[22] Fumie Nakaoka, Nobuyuki Oda, Some applications of minimal open sets, *International journal of mathematics and mathematical sciences*. 27(8) (2001) 471–476.
[23] Fumie Nakaoka, Nobuyuki Oda, Some properties of maximal open sets, *International Journal of Mathematics and Mathematical Sciences*. 21 (2003) 1331–1340.
[24] Fumie Nakaoka, Nobuyuki Oda, Minimal closed sets and maximal closed sets, *International Journal of Mathematics and Mathematical Sciences*. (2006) Article ID 18647.
[25] P.M. Mahalakshmi, P. Thangavelu, M-Connectedness in M-topology, *International Journal of Pure and Applied Mathematics*. 106(8)(2016) 21–25.
[26] A Ghareeb, Redundancy of multiset topological spaces, *Iranian Journal of Fuzzy Systems.* 14(4) (2017) 163–168.