Results in Nonlinear Analysis 7 (2024) No. 2, 64–78
https://doi.org/10.31838/rna/2024.07.02.007
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Numerical solutions for the time fractional Black-Scholes model governing European option by using double integral transform decomposition method

Wurood R. Abd AL-Hussein[a]*, Mohammed G.S. AL-Safi[b], Rand Muhaned Fawzi[c]

[a]*Department of Mathematics and Computer Applications, College of Sciences, Al-Nahrain University, Jadriya, Baghdad, IRAQ;*
[b]*Department of Accounting, Al-Esraa University College, Baghdad, IRAQ;*
[c]*Department of Mathematics, College of education for pure science ibn Al-Haitham, University of Baghdad, Baghdad, Iraq.*

Abstract

In today's world, financial derivatives are an essential component of every single financial transaction. When it comes to investing in options, the Black-Scholes model (BSM) for pricing options provides a risk-free opportunity analysis. This research paper presents a novel approach to obtaining analytical solutions for the time fractional Black-Scholes model (TFBSM), which is a mathematical model used to describe the behaviour of European option (EO). The solutions are derived with the Double Sumudu-Elzaki Transform technique (DSET). The incorporation DSET into a semi-analytical framework, precisely the Adomain decomposition method (ADM). The precision and efficacy of the suggested method are demonstrated through the selection of numerical examples carried out using Matlab R2015a. In conclusion, this study employs the TFBSM and the aforementioned numerical technique to price various EO.

Key words and phrases: Adomain decomposition, Black-Scholes equation, Double Sumudu-Elzaki transform, Fractional calculus.

Mathematics Subject Classification (2010): 35R11, 26A33, 65R10

Email addresses: *worood.riad@nahrainuniv.edu.iq* (Wurood R. Abd AL-Hussein)*; *dr.mohammed.ghazi@esraa.edu.iq* (Mohammed G. S. AL-Safi); *rand.m.f@ihcoedu.uobaghdad.edu.iq* (Rand Muhaned Fawzi)

Received November 29, 2023; Accepted February 14, 2024; Online March 20, 2024

1. Introduction

Finances play a significant part in every field in today's global and technology society. Financial derivatives have become influential tools for firms and large investors in recent decades. various contracts have been present in various markets for some time, but their utilization grew significantly in the 1970s and they are now a fundamental component of markets. Financial derivatives exist in several forms such as Over the Counter transaction, options, swaps, forwards, and futures. Contracts that seem intriguing and fascinating tend to be more complex, leading to a higher chance of loss. When beginning to invest in options, investors should understand the key factors that influence an option's value, including the current market rate, financial value, expiration date, volatility, interest rates, and cash dividends. Options, like other investments, have similarities in certain instances. Understanding the factors that determine their pricing is essential for their effective use. Various models, such as the Binomial model and trinomial model, are used for risk-free analysis. Implementing speculative pricing models like the one mentioned serves the purposes of maintaining a risk-neutral portfolio and identifying and trading price discrepancies. The BSM is the most frequently cited and renowned model among these. Fischer and Myron [1] designed the acclaimed and efficient model in 1973. They chose to analyse a simple case: a EO on a dividend-free stock. They limited their investigation to conditions that made the mathematical problem easier to solve, as referenced in [2, 3]. An American option can be executed by the buyer at any time before the maturity date, but a EO can only be executed by the buyer on the date of maturity. Fractal assembly has introduced fractional Brownian motion (BM) instead of the tradition BM in the classical model, including fractional calculus (FC) and FPDEs into finance. Due to fractional BM not being a semi-martingale, the Itô theory of stochastic integrals cannot be applied directly. Replacing the Itô integral with a pathwise Riemann–Stieltjes integral results in a model of option values that permits arbitrage, as Rogers [4] shows. Arbitrage opportunities exist in the TFBSM inside a complete, frictionless setting. Researchers have changed the BSM more and more to a fractional order because fractional-order derivatives and integrals are better at showing how substances remember and transmit information (Bjork and Hult [5]; Meerschaert and Sikorskii [6]. Utilising fractional order process modelling is a method to manage excessive volatility in the stock market. Aghili [7] use the combination of exponential operators and special functions is a potent tool for solving space fractional Black-Scholes equations. The TFBSM is a particular case of the bifractional BSM introduced by Liang et al. [8]. In his paper, Cartea [9] shows how a PIDE with a non-local operator in time-to-maturity can be used to show how European-style derivatives are valued. Leonenko et al. [10] studied the explanation of fractional Pearson diffusions governed by a time-fractional diffusion equation. The explanations were then used to enhance the BSM. Because of the memory properties of fractional derivatives, finding a precise solution to these issues is very challenging, leading many academics to seek methods to approximate them. Closed-form and numerical solutions were used by Orland and Taglialatela [11] to estimate the implied volatility for the options. The findings were illustrated with the presentation of the computational results. In their study, Ouafoudi and Gao [12] employed both the Homotopy Perturbation Method (HPM) and a modified version of HPM, along with the Sumudu transform, to obtain solutions for the BSM. The answers were represented as convergent power series, and each component was calculated regularly. Farhadi and Erjaee [13] suggested the time-fractional derivative be applied to solve the BSM. Sawngtong et al. [14] investigated an analytical approach to solving the BSM involving two assets. The LTHPM method was used within the context of the Liouville-Caputo fractional derivative. In their work, Yavuz and Ozdemir [15] presented the use of CFADM and CFMHPM to tackle the fractional BSM. Jena and Chakraverty [16] introduced a new technique known as the RPST to calculate the analytical solution for the TFBSM problem. The described method was used to tackle the problem of pricing EO, considering the beginning situation. Prathumwan and Trachoo [17] employed the LHPM as a computational approach to obtain an approximate solution for the PDE that governs the EO, involving two assets. Golbabai et al. [18] describe the numerical solution of the TFBSM with BC's for a problem of EO involved with the method of RBFs. The LADM was first introduced by

Sumiati [19] as a computational approach for solving the BSM. The BSM was solved by Golbabai and Nikan [20] through the approximation solution of the TFBSM of order $0 < \varepsilon \leq 1$ governing EO based on the moving least-squares (MLS) method. The fractional differential equation (FDE) appears more and more frequently in research areas and engineering applications. Research topics and engineering applications are becoming increasingly prevalent. Many researchers have put forth diverse numerical methodologies for solving differential equations. Nikan et al. [21] used the local meshless method for the numerical simulation of the TFBSE. Nikan et al. [22] used a noval meshless numerical procedure, the radial basis function-generated finite difference (RBF-FD), to approximate the TFCM involving two fractional temporal derivatives. Mohammed et al. [23] used the double integral transform with VIM to solve nonlinear PDEs. Mohammed et al. [24] solved non-linear PDEs using SETDM. In this study, we employ the features of the Sumudu and Elzaki transforms to examine the following aspects. The BSM for calculating an option's value is described by the following equation:

$$\frac{\partial^{\epsilon}\omega}{\partial t^{\epsilon}}+\frac{\sigma^2 z^2}{2}\frac{\partial^2\omega}{\partial z^2}+r(t)z\frac{\partial\omega}{\partial z}-r(t)\omega=0,(z,t)\in R^{+}\times(0,T) \tag{1}$$

Where $0 < \varepsilon \leq 1$, ω(z,t) denotes the price of a European call option, which is dependent on the asset price and the time t. K represents the exercise price, T represents the maturity of the option, $r(t)$ represents the risk-free interest rate, and σ(z,t) represents the volatility function of the underlying asset. Let $\omega_c(z,t)$ and The values of the European call and put options are denoted by the notation $\omega_P(z,t)$. The corresponding payoff functions can be expressed as follows:

$$\left.\begin{aligned}\omega_c(z,t)&=\max(z-E,0)\\ \omega_P(z,t)&=\max(E-z,0)\end{aligned}\right\} \tag{2}$$

Eq. (2) defines the max function as yielding the highest value in its input. $\omega_c(z,t)$ denotes the European call option value, whereas $\omega_P(z,t)$ represents the European put option value. The maturity date price of the option is represented by E, whereas the mathematical function max(z,0) generates a higher value between z and 0. When an option asset is purchased, the owner does not instantly get the right to sell or buy it. The call function, represented by $\omega_c(z,t)$, signifies the right to purchase the asset, whereas the function $\omega_P(z,t)$ signifies the right to sell the asset.

The primary purpose of this study is to solve TFBSM utilizing the Double Sumudu and Elzaki transform decomposition technique (DSETDM).

2. Preliminaries

This section lays out the primary ideas of the FC theory that was used in this research.

Definition 2.1 *[25] A real function ω(z), z > 0 is said to be in the space $C\mu$, $\mu \in \mathbb{R}$, if there exists a real number p,($p > \mu$), such that $\omega(z) = z^p\omega_1(z)$, where $\omega_1(z) \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $\omega^{(m)} \in C\mu, m \in \mathbb{N}$.*

Definition 2.2 *[4] The following is the definition of the Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ for a function ω that is a member of the set $C\mu$, where $\mu \geq 1$:*

$$I^{\varepsilon}\omega(z)=\begin{cases}\frac{1}{\Gamma(\varepsilon)}\int_0^z(z-t)^{\varepsilon-1}\omega(t)dt,\varepsilon>0,z>0\\ I^0\omega(z)=\omega(z),\varepsilon=0\end{cases}$$

The function $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,$ *is known as the gamma function when* $z > 0$.

Moreover, the fractional integral of Riemann-Liouville has the following characteristics:

for $\omega \in C_{\mu}$, $\mu \geq -1, \epsilon, \tau \geq 0$ *and* $\gamma \geq 1$:

(1) $I^{\epsilon} I^{\tau} \omega(z) = I^{\epsilon+\tau} \omega(z)$,

(2) $I^{\epsilon} I^{\tau} \omega(z) = I^{\epsilon} I^{\tau} \omega(z)$,

(3) $I^{\epsilon} Z^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\epsilon+\gamma+1)} z^{\epsilon+\gamma}$.

Definition 2.3 *[26] The definition of the fractional derivative* $\omega(z)$ *in the Caputo sense is:*

$$D_*^{\epsilon}\omega(z) = \frac{1}{\Gamma(m-\varepsilon)} \int_0^z (z-t)^{m-\epsilon-1} \omega^{(m)}(t) dt,$$

for $m-1 < \epsilon < m$, $\epsilon \in N$, $z > 0$, $\omega \in C_{-1}^{m}$.

The operator D_*^{ϵ} *has the following fundamental characteristic*

$$D_*^{\epsilon} z^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(1-\epsilon+\gamma)} z^{\gamma-\epsilon},$$

$$D_*^{\epsilon} I^{\epsilon} \omega(z) = \omega(z),$$

$$I^{\epsilon} D_*^{\epsilon} \omega(z) = \omega(z) - \sum_{k=0}^{m-1} w^k(0^+) \frac{z^k}{k!}, z > 0.$$

Definition 2.4 *[25] The Mittag-Leffler function* $E_{\in}(z)$ *with* $\in > 0$*, in the entire complex plane is defined as:*

$$E_{\in}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m \in +1)}, z \in \mathbb{C}, \Re e\left(\in\right) > 0.$$

3. The DSETDM and Its Modification

This section introduces the DSET approach and its essential characteristics.

3.1. Basic Concepts

Definition 3.1 *[27] The Sumudu Transform ST of the function* $\omega(z)$ *for all* $z \geq 0$ *is defined as:*

$S_z(\omega(z)) = \frac{1}{\rho} \int_0^{\infty} \omega(z) e^{-\left(\frac{z}{\rho}\right)} dz = \overline{\omega}(\rho), \rho \in (p_1, p_2)$, *where the operator* S_z *is called the Sumudu transform operator*.

Definition 3.2 *[28] The Elzaki Transform ET of the function* $\omega(t)$ *for all* $t \geq 0$ *is defined as:*

$Et(\Phi(t)) = \tau \int_0^{\infty} \Phi(t) e^{-\left(\frac{t}{\tau}\right)} dt = \overline{w}(\tau), \tau \in (p_1, p_2)$, *where the operator* E_t *is called the Elzaki transform operator*.

These functions are of exponential order and consider functions in the set G as stated by:

$$G = \left\{ \omega(\tau) : \exists Q, p_1, p_2 > 0, \left|\omega(\delta)\right| < Q e^{\frac{|\delta|}{pi}}, if \delta \in (-1)^i \times \left[0, \infty\right) \right\}.$$

Definition 3.3 *[24] The DSET of* $S_Z E_t[\omega(z,t)] = \bar{\omega}(\eta,\xi)$ *is defined as:*

$$S_Z E_t[\omega(z,t)] = \bar{\omega}(\eta,\xi) = \frac{\xi}{\eta}\int_0^\infty \int_0^\infty \omega(z,t) e^{-\left(\frac{z}{\eta}+\frac{t}{\xi}\right)} dz dt.$$

The clear demonstration of the linearity of the DSET is evident in the subsequent relationship, as depicted below:

$$S_z E_t\left[\rho\omega(z,t)+\tau\chi(z,t)\right] = \frac{\xi}{\eta}\int_0^\infty \int_0^\infty e^{-\left(\frac{z}{\eta}+\frac{t}{\xi}\right)}\left[\rho\omega(z,t)+\tau\chi(z,t)\right]dzdt,$$

$$=\frac{\rho\xi}{\eta}\int_0^\infty \int_0^\infty e^{-\left(\frac{z}{\eta}+\frac{t}{\xi}\right)}\omega(z,t)dzdt + \frac{\tau\xi}{\eta}\int_0^\infty \int_0^\infty e^{-\left(\frac{z}{\eta}+\frac{t}{\xi}\right)}\chi(z,t)dzdt,$$

$$\rho S_Z E_t[\omega(z,t)] + \tau S_Z E_t[\chi(z,t)] = \rho\bar{\omega}(\eta,\xi) + \tau\bar{\chi}(\eta,\xi).$$

Definition 3.4 *[24] The inverse of DSET, i.e. IDSET* $S_Z E_t^{-1}[\bar{\omega}(\eta,\xi)] = \omega(z,t)$ *is defined by:*

$$S_Z E_t^{-1}[\bar{\omega}(\eta,\xi)] = \omega(z,t) = \tfrac{1}{2\pi i}\int_{\upsilon-i\infty}^{\upsilon+i\infty} \frac{1}{\gamma} e^{-\frac{z}{\eta}} d\eta \cdot \tfrac{1}{2\pi i}\int_{\chi-i\infty}^{\chi+i\infty} \delta e^{-\frac{t}{\xi}} \bar{\omega}(\eta,\xi) d\xi.$$

3.2. Basic Derivative Properties of the DSET:

$$S_z E_t\left[\frac{\partial\omega(z,t)}{\partial z}\right] = \frac{1}{\eta}\bar{\omega}(\eta,\xi) - \frac{1}{\eta}(\omega(0,t)),$$

$$S_z E_t\left[\frac{\partial^2\omega(z,t)}{\partial z^2}\right] = \frac{1}{\eta^2}\bar{\omega}(\eta,\xi) - \frac{1}{\eta^2}E_t(\omega(0,t)) - \frac{1}{\eta}E_t\left(\frac{\partial\omega(0,t)}{\partial z}\right),$$

$$S_z E_t\left[\frac{\partial\omega(z,t)}{\partial t}\right] = \frac{1}{\xi}\bar{\omega}(\eta,\xi) - \xi S_z(\omega(z,0)),$$

$$S_z E_t\left[\frac{\partial^2\omega(z,t)}{\partial t^2}\right] = \frac{1}{\xi^2}\bar{\omega}(\eta,\xi) - S_z(\omega(z,0)) - \xi S_z\left(\frac{\partial\omega(z,0)}{\partial t}\right),$$

$$S_z E_t\left[\frac{\partial^2\omega(z,t)}{\partial z\partial t}\right] = \frac{1}{\eta\xi}\bar{\omega}(\eta,\xi) - \frac{1}{\eta\xi}E_t(\omega(z,0)) - S_z\left(\frac{\partial\omega(z,0)}{\partial z}\right),$$

$$S_z E_t\left[\frac{\partial^m\omega(z,t)}{\partial z^m}\right] = \eta^{-m}\bar{\omega}(\eta,\xi) - \sum_{k=0}^{m-1}\eta^{-m+k}E_t\left(\frac{\partial^k\omega(0,t)}{\partial z^k}\right),$$

$$S_z E_t\left[\frac{\partial^n\omega(z,t)}{\partial t^n}\right] = \xi^{-n}\bar{\omega}(\eta,\xi) - \sum_{j=0}^{n-1}\xi^{-n+j+2}S_z\left(\frac{\partial^j\omega(z,0)}{\partial t^j}\right)$$

$$S_z E_t\left[\frac{\partial^\upsilon\omega(z,t)}{\partial z^\upsilon}\right] = \eta^{-\upsilon}\bar{\omega}(\eta,\xi) - \sum_{k=0}^{m-1}\eta^{-\upsilon+k}E_t\left(\frac{\partial^k\omega(0,t)}{\partial z^k}\right)$$

$$S_z E_t\left[\frac{\partial^\mu\omega(z,t)}{\partial t^\mu}\right] = \xi^{-\mu}\bar{\omega}(\eta,\xi) - \sum_{j=0}^{n-1}\xi^{-\mu+j+2}S_z\left(\frac{\partial^j\omega(z,0)}{\partial t^j}\right)$$

The features of the DSET and the existence condition are described in [29].

3.3. Solution by DSETM

To apply the DSET with the ADM method to solve the TFBSM, it is imperative to convert the unbounded domain into a finite interval using truncation. In this analysis, we truncate the variable z's range in Eq. 1 to a finite interval denoted as (B_d, B_u). The model under consideration is formulated in the following manner:

$$D_t^{\epsilon}\omega(z,t) = a\frac{\partial^2\omega(z,t)}{\partial z^2} + rb\frac{\partial\omega(z,t)}{\partial z} - c\omega(z,t) + f(z,v) \quad 0 < \epsilon \leq 1 \tag{3}$$

where $a = \frac{1}{2}\sigma^2 > 0, b = r - a, c = r > 0$, with $f(z,v)$ is source term with initial conditions and boundary condition :

$$\begin{cases} \omega(z,0) = \psi(z), \\ \omega(B_d,t) = p(t), \omega(B_u,t) = q(t) \end{cases} \tag{4}$$

applying the DSET on Eq. 3, we obtain the following:

$$S_zE_t\left[D_t^{\epsilon}\omega(z,t)\right] = S_zE_t\left[a\omega_{zz}(z,t) + rb\omega_z(z,t) - c\omega(z,t) + f(z,v)\right] \tag{5}$$

using the I.Cs (4) and the single ST in Eq. 5, we obtain the following:

$$\bar{\omega}(\eta,\xi) = \sum_{j=0}^{n-1}\xi^{j+2}S_z\left(\frac{\partial^j\omega(z,0)}{\partial j}\right) + \xi^{\epsilon}S_zE_t[f(z,v)] + \xi^{\epsilon}S_zE_t\left[a\omega_{zz}(z,t) + rb\omega_z(z,t) - c\omega(z,t)\right] \tag{6}$$

taking the inverse DSET $S_zE_t^{-1}(\bar{\omega}(\eta,\xi))$ of Eq.6, we have the following:

$$\omega(z,t) = \Omega(z,t) + S_zE_t^{-1}\left[\xi^{\epsilon}S_zE_t\left[a\omega_{zz}(z,t) + rb\omega_z(z,t) - c\omega(z,t)\right]\right] \tag{7}$$

where $\Omega(z,t) = \sum_{j=0}^{n-1}\xi^{j+2}\ S_z\left(\frac{\partial^j\omega(z,0)}{\partial j}\right) + \xi^{\epsilon}S_zE_t[f(z,v)]$.

Now, use the iterative approach by assuming the following:

$$\omega(z,t) = \sum_{i=0}^{\infty}\omega_i(z,t) \tag{8}$$

Substituting Eq. 8 in Eq. 7, we get:

$$\sum_{i=0}^{\infty}\omega_i(z,t) = \Omega(z,t) + S_zE_t^{-1}\left[\xi^{\epsilon}S_zE_t\left[a\sum_{i=0}^{\infty}(\omega_n)_{zz} + rb\sum_{i=0}^{\infty}(\omega_n)_z - c\sum_{i=0}^{\infty}\omega_i(z,t)\right]\right],$$

following that, we obtain the recurrence relations as follows:

$$\begin{cases} \omega_0(z,t) = \Omega(z,t), \\ \omega_1(z,t) = S_zE_t^{-1}\left[\xi^{\varepsilon}S_zE_t\left[a(\omega_0)_{zz} + rb(\omega_0)_z - c\omega_0\right]\right. \\ \vdots \\ \omega_{n+1}(z,t) = S_zE_t^{-1}\left[\xi^{\varepsilon}S_zE_t\left[a(\omega_n)_{zz} + rb(\omega_n)_z - c\omega_n\right]\right., n \geq 1 \end{cases} \tag{9}$$

So, the approximate solution ω(*z,t*) is given by:

$$\omega(z,t) = \lim_{n\to\infty}\omega_n(z,t). \tag{10}$$

4. Numerical Examples

Three examples are provided in this part to demonstrate the precision of our proposed numerical system using the ADM technique combined with the DSET. The current pricing strategy for barrier options using a TFBSM model was used, considered one of the most intriguing models in the financial sector. Numerical simulations were performed using Matlab R2015a software on a 12th Gen Intel(R) Core(TM) i7-1255U 1.70 GHz CPU processor with 8 Gbyte RAM.

Example 4.1 *Consider the TFBSM equation as follows [25-30]*

$$\frac{\partial^{\epsilon}\omega}{\partial t^{\epsilon}}=\frac{\partial^2\omega}{\partial z^2}+(\mathrm{k}-1)\frac{\partial\omega}{\partial \mathrm{z}}-\mathrm{k}\omega, 0<\epsilon\leq 1 \tag{11}$$

With the I.C.

$$\omega(z,0)=max\left(e^z-1,0\right) \tag{12}$$

it is seen that the system of equations under consideration is characterized by two dimensionless parameters, namely k = $\frac{2r}{\sigma^2}$, which reflects the balance between interest rates and stock return variability, and the dimensionless time until expiry, $\frac{1}{2}\sigma^2\mathrm{T}$.

Using the DSET on each side of Eq. 11 where S_z is a single ST, the following result is obtained:

$$S_zE_t\left[\frac{\partial^{\epsilon}\omega}{\partial t^{\epsilon}}\right]=S_zE_t\left[\frac{\partial^2\omega}{\partial z^2}+(k-1)\frac{\partial\omega}{\partial z}-k\omega\right] \tag{13}$$

$$\xi^{-\epsilon}\bar{\omega}(\eta,\xi)-\sum_{j=0}^{n-1}\xi^{-\epsilon+j+2}S_z\left(\frac{\partial^j\omega(z,0)}{\partial t^j}\right)=S_zE_t\left[\frac{\partial^2\omega}{\partial z^2}+(k-1)\frac{\partial\omega}{\partial z}-k\omega\right]$$

from Eq. 13 and I.C. 12 we have

$$\bar{\omega}(\eta,\xi)=\xi^2\max\left(\frac{\xi^2}{1-\eta}-\xi^2,0\right)+\xi^{\epsilon}S_zE_t\left[\omega_{zz}+(k-1)\omega_z-k\omega\right] \tag{14}$$

Taking the inverse DSET $S_zE_t^{-1}(\bar{\omega}(\eta,\xi))$ on Eq.14, gives:

$$\omega(\mathrm{z},\mathrm{t})=\max\left(\mathrm{e}^{\mathrm{z}}-1,0\right)+\mathrm{S_zE_t^{-1}}\left[\xi^{\epsilon}\mathrm{S_zE_t}\left[\omega_{\mathrm{zz}}+(\mathrm{k}-1)\omega_{\mathrm{z}}-\mathrm{k}\omega\right]\right], \tag{15}$$

now, use the iterative approach by assuming the following:

$$\omega(\mathrm{z},\mathrm{t})=\sum_{\mathrm{n}=0}^{\infty}\omega_{\mathrm{n}}(\mathrm{z},\mathrm{t}) \tag{16}$$

Substituting Eq. 16 in Eq. 15 we obtain the following:

$$\sum_{n=0}^{\infty}\omega_n(z,t)=\max(e^z-1,0)+S_zE_t^{-1}\left[\xi^{\epsilon}S_zE_t\left[\sum_{n=0}^{\infty}(\omega_n)_{zz}+(k-1)\sum_{n=0}^{\infty}(\omega_n)_z-k\sum_{n=0}^{\infty}\omega\right]\right] \tag{17}$$

Finally, by using two parties of Eq.17, we obtain the repeated algorithm as the following:

$$\omega_0(\mathrm{z},\mathrm{t})=\max\left(\mathrm{e}^{\mathrm{z}}-1,0\right),$$

$$\omega_1(\mathrm{z},\mathrm{t})=\mathrm{S_zE_t^{-1}}\left[\xi^{\epsilon}\mathrm{S_zE_t}\left[(\omega_0)_{\mathrm{zz}}+(\mathrm{k}-1)(\omega_0)_{\mathrm{z}}-\mathrm{k}\omega_0\right]\right]$$

$$=\mathrm{S_zE_t^{-1}}\left[\xi^{\epsilon}\mathrm{S_zE_t}\left[\mathrm{k}\left\{-\max\left(\mathrm{e}^{\mathrm{z}}-1,0\right)+\max\left(\mathrm{e}^{\mathrm{z}},0\right)\right\}\right]\right]$$

$$= S_z E_t^{-1}\left(k\left[-\max\left(\frac{\xi^{\epsilon+2}}{1-\eta}-\xi^2,0\right)+\max\left(\frac{\xi^{\epsilon+2}}{1-\eta},0\right)\right]\right),$$

$$= \frac{t^\epsilon}{\Gamma(\epsilon+1)}k\left\{-\max\left(e^z-1,0\right)+\max\left(e^z,0\right)\right\},$$

$$\omega_2(z,t) = S_z E_t^{-1}\left[\xi^\epsilon S_z E_t\left[(\omega_1)_{zz}+(k-1)(\omega_1)_z-k\omega_1\right]\right]$$

$$= S_z E_t^{-1}\left[\xi^\epsilon S_z E_t\left[\frac{t^\epsilon}{\Gamma(\epsilon+1)}k\left\{\max\left(e^z-1,0\right)-\max\left(e^z,0\right)\right\}\right]\right]$$

$$= S_z E_t^{-1}\left(k\left[\max\left(\frac{\xi^{2\epsilon+2}}{1-\eta}-\xi^2,0\right)-\max\left(\frac{\xi^{2\epsilon+2}}{1-\eta},0\right)\right]\right)$$

$$= \frac{t^{2\epsilon}}{\Gamma(2\epsilon+1)}\left(-k^2\right)\left\{-\max\left(e^z-1,0\right)+\max\left(e^z,0\right)\right\},$$

Hence,

$$\omega(z,t) = \max\left(e^z-1,0\right)+\frac{t^\epsilon}{\Gamma(\epsilon+1)}k\left\{-\max\left(e^z-1,0\right)+\max\left(e^z,0\right)\right\}$$
$$+\frac{t^{2\epsilon}}{\Gamma(2\epsilon+1)}\left(-k^2\right)\left\{-\max\left(e^z-1,0\right)+\max\left(e^z,0\right)\right\}+\cdots, \tag{18}$$

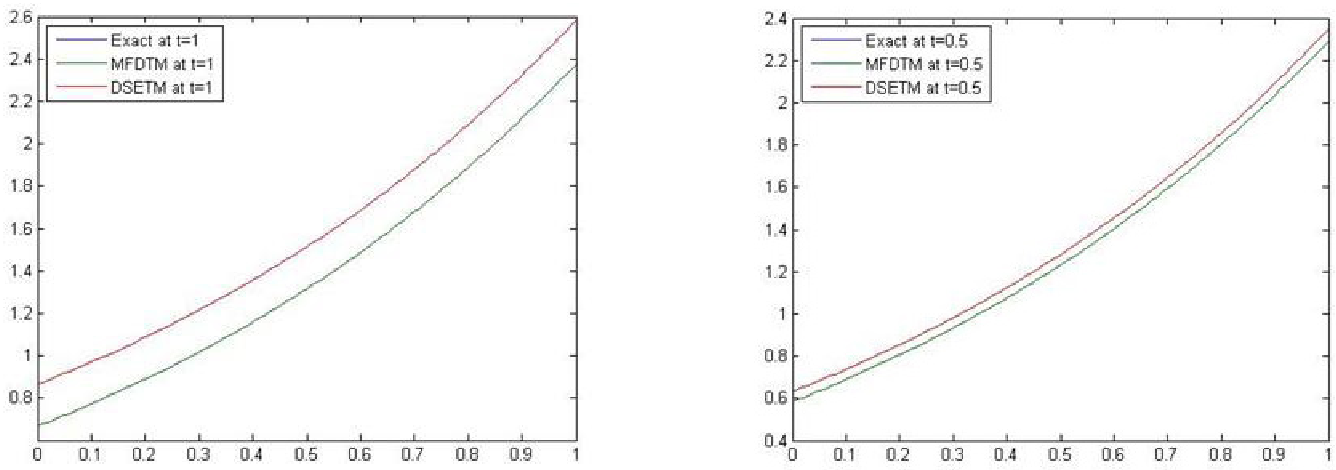
also, Eq. 18 can be expressed as:

$$\omega(z,t) = \max\left(e^z-1,0\right)E_\epsilon\left(-kt^\epsilon\right)+\max\left(e^z,0\right)\left(1-E_\epsilon\left(-kt^\epsilon\right)\right), \tag{19}$$

The symbol E_ϵ denotes the Mittag-Leffler function. The solution of Eq. 11 can be expressed as a series using the Mittag-Leffler function, as shown in Eq.19. This series solution converges to the exact solution presented in Eq. 20 when the parameter ϵ is set to 1 .

$$\omega(z,t) = \max\left(e^z-1,0\right)e^{-kt}+\max\left(e^z,0\right)\left(1-e^{-kt}\right) \tag{20}$$

The comparison between the exact solution (ES) for various values of fractional order ϵ for fixed t and k and the approximate solution (AS) derived using the approach FDTM described in [17] and DSETDM is shown in Figures 1(a)-1(b).



(a) (b)

Figure 1: (a) in the case t = 1, k = 2 , (b) in the case t = 0.5, k = 2

Table 1 displays a comparison between the AS found by the FDTM method as shown in [25], and the current results achieved using the DSETDM method. for various fractional order values.

Table 1: The comparison of the AS of the suggested method and the solution generated by the FDTM described in [25], $t = 1$, when $z = \epsilon = 1$.

z	Exact	Our Method	Method in [25]	Absolute Error
0.1	1.89955	1.89955	1.89955	2.78143e – 14
0.2	2.04796	2.04796	2.04796	1.01696e – 12
0.3	2.16947	2.16947	2.16947	8.65445e – 11
0.4	2.26895	2.26895	2.26895	2.01685e – 09
0.5	2.35040	2.35040	2.35040	2.31143e – 08
0.6	2.41709	2.41709	2.41709	1.69107e – 07
0.7	2.47168	2.47168	2.47169	9.07724e – 07
0.8	2.51639	2.51638	2.51641	3.88444e – 06
0.9	2.55298	2.55297	2.55307	1.39815e – 05
1.0	2.58295	2.58290	2.58318	4.39055e – 05

Example 4.2 *Consider the TFBSM equation [25,31]*

$$\frac{\partial^{\epsilon}\omega}{\partial t^{\epsilon}} = 0.06\omega - 0.08(2+\sin x)^2 z^2 \frac{\partial^2\omega}{\partial z^2} - 0.06z\frac{\partial\omega}{\partial z}, 0 < \epsilon \le 1 \tag{21}$$

With the I.C.

$$\omega(z,0) = max\left(z - 25e^{-0.06}, 0\right) \tag{22}$$

Using the DSET on each side of Eq. 21 where S_z is a single ST, gives:

$$S_z E_t\left[\frac{\partial^{\epsilon}\omega}{\partial t^{\epsilon}}\right] = S_z E_t\left[0.06\omega - 0.08(2+\sin x)^2 z^2 \frac{\partial^2\omega}{\partial z^2} - 0.06z\frac{\partial\omega}{\partial z}\right]$$

$$\xi^{-\epsilon}\bar{\omega}(\eta,\xi) - \sum_{j=0}^{n-1}\xi^{-\epsilon+j+2} S_z\left(\frac{\partial^j\omega(z,0)}{\partial t^j}\right) = S_z E_t\left[0.06\omega - 0.08(2+\sin x)^2 z^2 \frac{\partial^2\omega}{\partial z^2} - 0.06z\frac{\partial\omega}{\partial z}\right] \tag{23}$$

from Eq. 23 and I.C. 22 we have:

$$\bar{\omega}(\eta,\xi) - \xi^2 S_z(\omega(z,0)) = \xi^{\epsilon} S_z E_t\left[0.06\omega - 0.08(2+\sin x)^2 z^2 \frac{\partial^2\omega}{\partial z^2} - 0.06z\frac{\partial\omega}{\partial z}\right]$$

$$\bar{\omega}(\eta,\xi) = \xi^2 \max\left(\eta - 25e^{-0.06}, 0\right) + \xi^{\epsilon} S_z E_t\left[0.06\omega - 0.08(2+\sin x)^2 z^2 \frac{\partial^2\omega}{\partial z^2} - 0.06z\frac{\partial\omega}{\partial z}\right] \tag{24}$$

Taking the inverse DSET $S_z E_t^{-1}(\bar{\omega}(\eta,\xi))$ on Eq.24, gives:

$$\omega(z,t) = \max\left(z - 25e^{-0.06}, 0\right) + S_z E_t^{-1}\left[\xi^{\epsilon} S_z E_t\left[0.06\omega - 0.08(2+\sin x)^2 z^2 \frac{\partial^2\omega}{\partial z^2} - 0.06z\frac{\partial\omega}{\partial z}\right]\right] \tag{25}$$

Now, use the iterative approach by assuming the following:

$$\omega(z,t)=\sum_{n=0}^{\infty}\omega_n(z,t) \tag{26}$$

substituting Eq. 26 in Eq. 25 we obtain the following:

$$\begin{aligned}\sum_{n=0}^{\infty}\omega_n(z,t)=\max\left(z-25e^{-0.06},0\right)\\ +S_zE_t^{-1}[\xi^{\epsilon}S_zE_t[0.06\sum_{n=0}^{\infty}\omega_n\\ -0.08(2+\sin x)^2z^2\sum_{n=0}^{\infty}(\omega_n)_{zz}\\ -0.06\sum_{n=0}^{\infty}(\omega_n)_z]].\end{aligned} \tag{27}$$

Finally, by using two parties of Eq.27, we obtain the repeated algorithm as the following:

$$\omega_0(z,t)=\max\left(z-25e^{-0.06},0\right)$$

$$\omega_1(z,t)=S_zE_t^{-1}\left[\xi^{\epsilon}S_zE_t\left[0.06\omega_0-0.08(2+\sin x)^2z^2(\omega_0)_{zz}-0.06z(\omega_0)_z\right]\right]$$

$$=S_zE_t^{-1}\left[\xi^{\epsilon}S_zE_t\left[0.06\max\left(z-25e^{-0.06},0\right)-0.06z\right]\right],$$

$$=S_zE_t^{-1}\left[0.06\xi^{\epsilon+2}\left(\max\left(\eta-25e^{-0.06},0\right)\right)-0.06\eta\xi^{\epsilon+2}\right]$$

$$=\frac{0.06}{\Gamma(\epsilon+1)}t^{\epsilon}\left(\max\left(z-25e^{-0.06},0\right)-z\right),$$

$$\omega_2(z,t)=S_zE_t^{-1}\left[\xi^{\epsilon}S_zE_t\left[0.06\omega_1-0.08(2+\sin x)^2z^2(\omega_1)_{zz}-0.06z(\omega_1)_z\right]\right],$$

$$=S_zE_t^{-1}\left[\xi^{\varepsilon}S_zE_t\left[\frac{0.06}{\Gamma(\varepsilon+1)}t^{\varepsilon}[0.06\max(z-25e^{-0.06},0)-0.06z]\right]\right],$$

$$=\frac{(0.06)^2}{\Gamma(2\epsilon+1)}t^{2\epsilon}\left(\max\left(z-25e^{-0.06},0\right)-z\right),$$

Therefore,

$$\begin{aligned}\omega(z,t)=\max\left(z-25e^{-0.06},0\right)+\frac{0.06}{\Gamma(\epsilon+1)}t^{\epsilon}\left(\max\left(z-25e^{-0.06},0\right)-z\right)\\ +\frac{(0.06)^2}{\Gamma(2\epsilon+1)}t^{2\epsilon}\left(\max\left(z-25e^{-0.06},0\right)-z\right)+\cdots.\end{aligned} \tag{28}$$

Hence, the exact solution of Eq. 21 in a closed form when $\epsilon = 1$, is given by Eq. 29

$$\omega(z,t)=\max\left(z-25e^{-0.06},0\right)e^{0.06t}+z\left(1-e^{0.06t}\right). \tag{29}$$

The absolute error, for example 4.2 in the case $\epsilon = 1$, $t = 10$, given in Figure 2., while Table 2 displays the comparison between the AS acquired by the method described in reference [25], with present results DSETDM for $\epsilon =1$, $t = 10$.

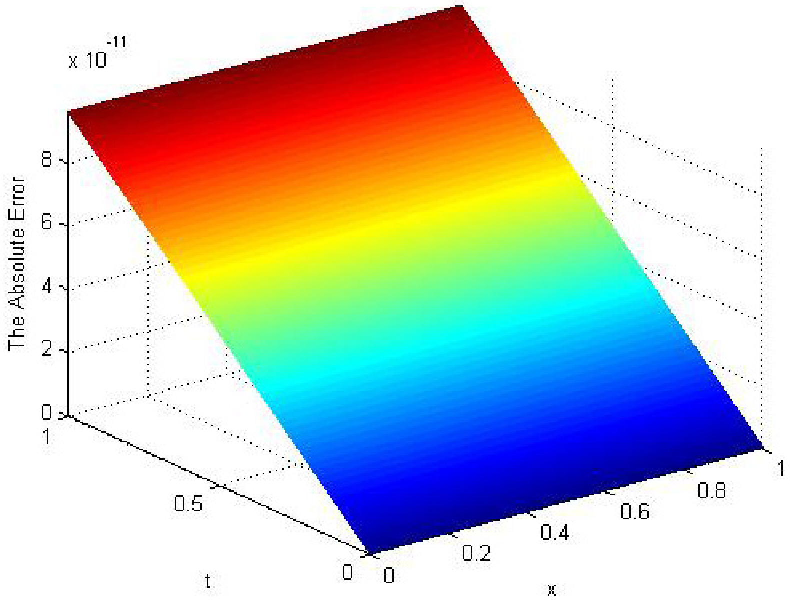


Figure 2: Absolute error, for example 4.2 in the case $\epsilon = 1$, $t = 10$.

Table 2: The comparison of the AS of the suggested method and the solution generated by the FDTM described in [25], when $t = 10$ and $\epsilon = 1$

z	Exact	Our Method	Method in [25]	Absolute Error
0.1	-0.09043	-0.09043	-0.08160	1.05217e – 11
0.2	-0.17264	-0.17264	-0.16320	2.00869e – 11
0.3	-0.25486	-0.25486	-0.24480	2.96521e – 11
0.4	-0.33707	-0.33707	-0.32640	3.92172e – 11
0.5	-0.41928	-0.41928	-0.40800	4.87824e – 11
0.6	-0.50149	-0.50149	-0.48960	5.87824e – 11
0.7	-0.58370	-0.58370	-0.57120	6.79127e – 11
0.8	-0.66592	-0.66592	-0.65280	7.74779e – 11
0.9	-0.74813	-0.74813	-0.73440	8.70430e – 11
1.0	-0.82211	-0.82211	-0.81600	9.56517e – 11

Example 4.3 *Consider the TFBSM equation [25,31]*

$$\frac{\partial^{\epsilon}\omega}{\partial t^{\epsilon}} = r\omega - \frac{\sigma^2}{2} z^2 \frac{\partial^2\omega}{\partial z^2} - (r-\tau) z \frac{\partial\omega}{\partial z}, \quad 0 < \epsilon \le 1 \tag{30}$$

with the I.C.

$$\omega(z - 0.41928, 0) = max(Az - B, 0) \tag{31}$$

Using the DSET on each side of Eq. 30 where S_z is a single ST, gives:

$$S_z E_t\left[\frac{\partial^{\epsilon}\omega}{\partial t^{\epsilon}}\right] = S_z E_t\left[r\omega - \frac{\sigma^2}{2} z^2 \frac{\partial^2\omega}{\partial z^2} - (r-\tau) z \frac{\partial\omega}{\partial z}\right]$$

$$\xi^{-\epsilon}\,\overline{\omega}(\eta,\xi) - \sum_{j=0}^{n-1} \xi^{-\epsilon+j+2} S_z\left(\frac{\partial^j \omega(z,0)}{\partial t}\right) = S_z E_t\left[r\omega - \frac{\sigma^2}{2} z^2 \frac{\partial^2\omega}{\partial z^2} - (r-\tau) z \frac{\partial\omega}{\partial z}\right] \tag{32}$$

from Eq. 32 and I.C. 31 we have:

$$\bar{\omega}(\eta,\xi)-\xi^2 S_z\left(\omega(z,0)\right)=\xi^{\epsilon} S_z E_t\left[r\omega-\frac{\sigma^2}{2}z^2\frac{\partial^2\omega}{\partial z^2}-(r-\tau)z\frac{\partial\omega}{\partial z}\right]$$

$$\bar{\omega}(\eta,\xi)=\xi^2\max(A\eta-B)+\xi^{\epsilon} S_z E_t\left[r\omega-\frac{\sigma^2}{2}z^2\frac{\partial^2\omega}{\partial z^2}-(r-\tau)z\frac{\partial\omega}{\partial z}\right] \tag{33}$$

taking the inverse DSET $S_z E_t^{-1}\left(\bar{\omega}(\eta,\xi)\right)$ on Eq.33, gives:

$$\omega(\mathrm{z},\mathrm{t})=\max(\mathrm{Ax}-\mathrm{B},0)+\mathrm{S_z E_t^{-1}}\left[\xi^{\epsilon} S_z \mathrm{E_t}\left[\mathrm{r}\omega-\frac{\sigma^2}{2}\mathrm{z}^2\frac{\partial^2\omega}{\partial \mathrm{z}^2}-(\mathrm{r}-\tau)\mathrm{z}\frac{\partial\omega}{\partial \mathrm{z}}\right]\right] \tag{34}$$

Now, use the iterative approach by assuming the following:

$$\omega(\mathrm{z},\mathrm{t})=\sum_{\mathrm{n}=0}^{\infty}\omega_{\mathrm{n}}(\mathrm{z},\mathrm{t}), \tag{35}$$

substituting Eq. 35 in Eq. 34 we obtain the following:

$$\sum_{n=0}^{\infty}\omega_n(z,t)=\max(Ax-B,0)\ +S_z E_t^{-1}\left[\xi^{\epsilon} S_z E_t\left[r\sum_{n=0}^{\infty}\omega_n-\frac{\sigma^2}{2}z^2\sum_{n=0}^{\infty}(\omega_n)_{zz}-(r-\tau)\mathrm{z}\sum_{\mathrm{n}=0}^{\infty}(\omega_{\mathrm{n}})_{\mathrm{z}}\right]\right]. \tag{36}$$

Finally, by using two parties of Eq.36, we obtain the repeated algorithm as the following:

$$\omega_0(\mathrm{z},\mathrm{t})=\max(\mathrm{Az}-\mathrm{B},0)$$

$$\begin{aligned}
\omega_1(\mathrm{z},\mathrm{t})&=\mathrm{S_z E_t^{-1}}\left[\xi^{\epsilon}\mathrm{S_z E_t}\left[\mathrm{r}\omega_0-\frac{\sigma^2}{2}\mathrm{z}^2(\omega_0)_{\mathrm{zz}}-(r-\tau)\mathrm{z}(\omega_0)_{\mathrm{z}}\right]\right],\\
&=\mathrm{S_z E_t^{-1}}\left[\xi^{\epsilon}\mathrm{S_z E_t}\left[\mathrm{r}\max(\mathrm{Az}-\mathrm{B},0)-(\mathrm{r}-\tau)\max(A,0)z\right]\right],\\
&=\mathrm{S_z E_t^{-1}}\left[\mathrm{r}\max(\mathrm{A}\eta-\mathrm{B},0)\xi^{\epsilon+2}-(\mathrm{r}-\tau)\max(A,0)\eta\xi^{\epsilon+2}\right]\\
&=\frac{\mathrm{t}^{\epsilon}}{\Gamma(\epsilon+1)}\left[\mathrm{r}\max(\mathrm{Az}-\mathrm{B},0)-(\mathrm{r}-\tau)z\,max(A,0)\right]
\end{aligned}$$

$$\omega_2(\mathrm{z},\mathrm{t})=\mathrm{S_z E_t^{-1}}\left[\xi^{\epsilon}\mathrm{S_z E_t}\left[\omega_1-\frac{\sigma^2}{2}\mathrm{z}^2(\omega_1)_{\mathrm{zz}}-(\mathrm{r}-\tau)\mathrm{z}(\omega_1)_{\mathrm{z}}\right]\right]$$

$$=\mathrm{S_z E_t^{-1}}\left[\xi^{\epsilon}\mathrm{S_z E_t}\left[\frac{t^{\epsilon}}{\Gamma(\epsilon+1)}\left(\mathrm{r}^2\max(\mathrm{Az}-\mathrm{B},0)-2r(r-\tau)z\,max(A,0)+(r-\tau)^2 z\,max(A,0)\right)\right]\right]$$

$$=\frac{t^{2\epsilon}}{\Gamma(2\epsilon+1)}\left[r^2\max(Az-B,0)-2r(r-\tau)z\max(A,0)+(r-\tau)^2 z\max(A,0)\right]$$

Therefore,

$$\omega(\mathrm{z},\mathrm{t})=\max(\mathrm{Az}-\mathrm{B},0)$$

$$+\frac{\mathrm{t}^{\epsilon}}{\Gamma(\epsilon+1)}\left[\mathrm{r}\max(\mathrm{Az}-\mathrm{B},0)-(\mathrm{r}-\tau)z\,max(\mathrm{A},0)\right]$$

$$+\frac{t^{2\epsilon}}{\Gamma(2\epsilon+1)}[r^2\max(Az-B,0)-2r(r-\tau)z\max(A,0)$$
$$+(r-\tau)^2 z\max(A,0)]+\cdots. \quad (37)$$

Hence, the exact solution of Eq. 30 in a closed form is given by Eq. 29 when $\epsilon = 1$, so Eq. 37 becomes
$\omega(z,t)=\max(Az-B,0)e^{rt}-\max(A,0)z(e^{rt}-e^{-\tau t})$.

Figures 3(a)-3(b) presents the Absolute error, for example 4.3 in the case (a), $\epsilon = 1$, $t = 10$, B = 10, $\tau = 0.2$, $r = 0.25$, (b), $\epsilon = 0.95$, $t = 10$, B = 10, $\tau = 0.2$, r = 0.25, while Table 3 displays the comparison between the AS acquired by the method described in reference [25], with present method DSETM for $\epsilon = 1$, t = 10.

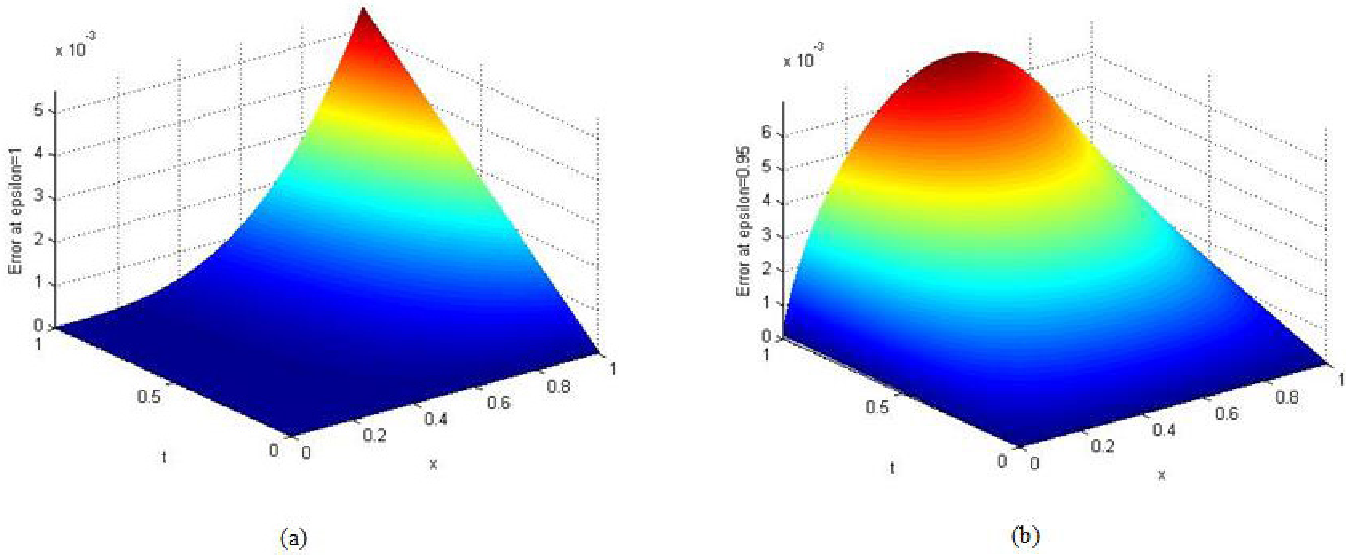


Figure 3: Absolute error, for example 4.3. (a) in the case $\epsilon = 1$, $t = 10$, (b) the case $\epsilon = 0.95$, $t = 10$.

Table 3: The comparison of the approximate solution obtained by the solution FDTM in [25] when $t = \epsilon = 1$ and different values of z, $r = 0.25$, $\tau = 0.2$ and B = 10.

z	Exact	Our Method	Method in [25]	Absolute Error
0.1	-0.02755	-0.02786	-0.01519	3.15410e – 04
0.2	-0.05260	-0.05320	-0.03039	6.02147e – 04
0.3	-0.07765	-0.07854	-0.04558	8.88883e – 04
0.4	-0.10019	-0.09808	-0.06077	1.17562e – 03
0.5	-0.12775	-0.12921	-0.07597	1.46235e – 03
0.6	-0.15279	-0.15454	-0.09116	1.74909e – 03
0.7	-0.17484	-0.17988	-0.10635	2.03583e – 03
0.8	-0.20289	-0.20522	-0.12155	2.32256e – 03
0.9	-0.22794	-0.23055	-0.13674	2.60930e – 03
1.0	-0.25044	-0.25335	-0.15193	2.86736e – 03

5. Conclusion

This paper aimed to demonstrate the operations and methodology of DSET in predicting the behaviour of PDEs used in economics. In this instance, a well-known TFBSM model with economic significance was used. Three specific instances of TFBSM were examined. The put and call options varied in each instance, thus the ADM approach paired with the DSET was used to provide analytical solutions for three chosen situations of TFBSM. The results indicated that DSET is a very dependable technique for solving FPDEs. This document provides a visual depiction and analysis of each case. R2015a is used to generate all the figures. One notable observation from the graphs is that as time approaches infinity, the disparities between the solutions at various values increase or decrease based on the distinct parameter settings used in each scenario. The option price fluctuates significantly over time. There is significant variety, even in the smallest proportion of time. These examples show that finding numerical solutions to TFBSM is straightforward when utilizing DSET. Hence, DSET is the most direct approach for solving FPDEs such as TFBSM. In the future, firm's share prices may be analyzed using this approach and TFBSM to understand their real-life implications.

References

[1] F. Black and M. Scholes, The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy*. 1973 May-Jun; 637–654.

[2] R.C. Merton, Theory of Rational option pricing. *Bell Journal of Economics*, 1973 March 1; 141–183.

[3] J.M. Manale and F.M. Mahomed, A simple formula for valuing American and European all and put options, in: J. Banasiak (Ed.), Proceeding of the Hanno Rund Workshop on the Differential Equations University of Natal, Durban, South Africa, 2000; 210–220.

[4] L. C. G. Rogers, Arbitrage with Fractional Brownian Motion, *Mathematical Finance*. 1997 Jan 1; 7(1): 95-105.

[5] T. Bjork and H. Hult, A note on Wick products and the fractional Black-Scholes model. *Finance and Stochastics*. 2005 Apr1; **9(2)**: 197–209.

[6] M. M. Meerschaert and A. Sikorskii, Stochastic Models for Fractional Calculus, 2011.

[7] A. Aghili, Fractional Black-Scholes equation. *International Journal of Financial Engineering*. 2017 Mar 1; **04(01)**:1750004.

[8] J.-R. Liang, J. Wang, W.-J. Zhang, W.-Y. Qiu, and F.-Y. Ren, The solution to a bifractional Black-Scholes-Merton differential equation, *International Journal of Pure and Applied Mathematics*. 2010; **58(1)**: 99–112. 2010.

[9] Á. Cartea, Derivatives pricing with marked point processes using tick-by-tick data. *Quantitative-Finance*, 2013 Jan 1; **13(1)**, 111–123.

[10] N. N. Leonenko, M. M. Meerschaert, and A. Sikorskii, Fractional Pearson diffusions. *Journal of Mathematical Analysis and Applications*. 2013 Jul 1: **403(2)**: 532–546.

[11] G.Orlando and G. Laglialatela, A review on implied volatility calculation. *Journal of Computer Applied and Mathematics*. 2017 Mar 1; **320**: 202–220.

[12] M. Ouafoudi and F. Gao, Exact solution of fractional Black-Scholes European option pricing equations. *Applied Mathematics*. 2018 Jan 1; **9(1)**:86–100.

[13] S.M. Farhadi and G.H. Erjaee, A new version of Black-Scholes equation presented by time-fractional derivative. Iranian Journal of Science and Technology. *Transaction A-Science*. 2017 May 9; **42(4)**:2159–2166.

[14] P. Sawngtong, K. Trachoo, W Sawngtong and B.Wiwattanapataphee, The Analytical Solution for the Black-Scholes equation with Two Assets in the Liouville-Caputo Fractional Derivative Sense. *Mathematics*. 2018 Jul 25: **6(8)**:129.

[15] M. Yavuz and N. Ozdemir, A different approach to the European option pricing model with new fractional operator. *Mathematical Modeling of Natural Phenomena*. 2018 Apr 1; **13(1)**:12.

[16] R.M. Jena and S. Chakraverty, A new iterative method based solution for fractional Black-Scholes option pricing equations(BSOPE). *SN Applied Sciences*. 2018 Dec 13: **1(1)**:95.

[17] D. Prathumwan and K. Trachoo, Application of the Laplace homotopy perturbation method to the Black-Scholes model based on a European put option with two assets. *Mathematics*. 2019 Mar 27;**7(4)**:310.

[18] A. Golbabai, O. Nikan, and T. Nikazad, Numerical analysis of time fractional BlackScholes European option pricing model arising in financial market. *Computational and Applied Mathematics*. 2019 Oct 5;**38(4)**.

[19] I. Sumiati, Black-Scholes equation solution using Laplace- Adomian decomposition method. *IAENG International Journal Computer Science*. 2019 Nov 1; **46(4)**:707–712.

[20] A. Golbabai and O. Nikan, A Computational Method Based on the Moving Least-Squares Approach for Pricing Double Barrier Options in a Time-Fractional Black-Scholes Model. *Computational Economics*. 2019 Feb 2; **55(1)**:119–141.

[21] O. Nikan, Z. Avazzadeh, and J. A. Tenreiro Machado, Localized kernel-based meshless method for pricing financial options underlying fractal transmission system. *Mathematical Methods in the Applied Sciences*. 2021 Nov 20.

[22] O. Nikan, A. Golbabai, J. A. T. Machado, and T. Nikazad, Numerical approximation of the time fractional cable model arising in neuronal dynamics. *Engineering with Computers*. 2020 May 25;**38(1)**:155–173.

[23] M.G.S. AL-Safi, R.M. Fawzi, and W.R. AL-Hussein, Numerical Solutions for the Nonlinear PDEs of Fractional Ã— rder by using a new double integral transform with variational iteration method. *Baghdad Science Journal*. 2023 Jun 20;**20**(3(Suppl.)):1087–1098.

[24] M.G.S AL-Safi, A.A. Yousif and M.S. Abbas, Numerical investigation for solving nonlinear partial differential equation using Sumudu-Elzaki transform decomposition method. *International Journal of Nonlinear Analysis and Applications*. 2022 Jan 1;**13(1)**:963–973.

[25] F.S. Khan, M. Sultana, M. Khalid, F. Zaidi and K. Nonlaopon, Forecasting the behavior of fractional Black-Sholes option pricing equation by Laplace perturbation iteration algorithm. *Alexandria Engineering Journal*. 2023 Jan 1;**62**:85–97.

[26] M.G.S. AL-Safi, W.R. Abd AL-Hussein and R.M. Fawzi, Numrical and Analytical Solutions of Space*Time Fractional Partial differential Equations by Using a New Double Integral Transform method. *Iraqi Journal of Sciences*. 2023 July 1; **64(4)**:1935–1947.

[27] M.G.S. AL-Safi, W.R. Abd AL-Hussein and A.G.N. AL-Shammari, A new approximate solution for the Telegraph equation of space fractional order derivative by using Sumudu method. *Iraqi Journal of Sciences*. 2018 July 1; **59(3A)**:301–311.

[28] T.M. Elzaki, “Application of new transform Elzaki transform” to partial differential equations, *Global Journal Pure and Applied Mathematics*. 2011 Jan 1;**7(1)**:65–70.

[29] M.S. Abbas, M.G.S. Al-Safi, and W.R.A. AL-Hussein, A computational methods for solving partial differential equations and integral differential equations by using SumuduElzaki transform.AIP Conference Procedings [Internet]. 2023 Jan 1;, Available from: https://pubs.aip.org/aip/acp/article-abstract/2834/1/080119/2926042/A-computational-methods-for-solving-partial?redirectedFrom=fulltexthttps://pubs.aip.org/aip/acp/article-abstract/2834/1/080119/2926042/A-computational-methods-for-solving-partial?redirectedFrom=fulltext.

[30] G. Hariharan, An efficient wavelet based approximation method to time fractional BlackScholes European option pricing problem arising in financial market. *Applied Mathematical Sciences*. 2013 May 1;**7**:3445–3456.

[31] A.S.V Ravi Kanth and K .Aruna, Solution of time fractional Black-Scholes European option pricing equation arising in fractional market. *Nonlinear Engineering*. 2016 Jan1; **5(4)**:269–276.