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Weighted inequalities for the traces of functions represented by the generalized riesz potentials

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Abstract

The theory of potentials has wide applications in singular integral operators and harmonic analysis. In this context, the Riesz potential inclusion theorems play an important role. Generalized Riesz potentials associated with the Laplace-Bessel differential operator are studied. The article investigates the properties of functions given in the form of these potentials. Local integral Ω_p characteristics are used in the terms, and inequalities are established based on evaluations made in these terms. The weight functions used in the inequalities are treated as monotonic functions.

Key words and phrases: Generalized Riesz potential, weight spaces, locally summable function, function trace, harmonic analysis.

Mathematics Subject Classification (2010): 46A03 General theory of locally convex spaces

1. Introduction

The study of integral operators in terms of characteristics of type Ω_p, takes its origin from the works [1, 2], where the operators of classic Fourier harmonic analysis are considered.

In these studies, the starting point is to establish estimates connecting these characteristics of the image with the same characteristics of the prototype of the operator from a certain class.

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These estimates make it possible to prove completely new theorems about the properties of harmonic analysis operators from a certain class in spaces introduced in terms of these characteristics (see. [3]).

In this work, similar studies are carried out for operators of harmonic Fourier-Bessel analysis.

The paper was devoted to establishing weighted inequalities for traces of function represented by the generalized Riesz potentials based on the estimates obtained in terms of $\Omega^{\iota(_s x)}_{p,\mu_{s,k_s}}$ characteristics of locally summable functions.

Due to the generality of the approach, the results obtained in this paper also contain the case of classic Fourier harmonic analysis.

2. Some Designations and Preliminaries

Let R^n be a Euclidean space of dimension n and m, $k \geq 0$, the integers, $n = m + k \geq 1, R^+_{m+k,k} = \{(x_1,...,x_{m+k}) \in R^{m+k} : x_{m+i} > 0, i = 1,...,k\}, R^+_{m+0,0} \equiv R^m$.

$T^y_{\gamma_{n,k}}(u(x)) = c_\nu \int_0^\pi ... \int_0^\pi u(x' - y', (x_{m+1}, y_{m+1}),...,(x_{m+k}, y_{m+k})_{\alpha_k}) \prod_{i=1}^{m+k} \sin^{\gamma_{m+i}-1} \alpha_i d\alpha_i,\ x \in R^+_{m+k,k}, x', y' \in R^m$ be a generalized shift operator (GSO) generated by the Laplace-Bessel operator (see [4]):

$$\Delta_{B_{m+k,k}}(x) = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} + \sum_{j=m+1}^{m+k} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j} \right), \gamma_{m+1} > 0,...,\gamma_{m+k} > 0,$$

$(x_{m+i}, y_{m+i})_{\alpha_i} = \sqrt{x^2_{m+i} - 2x_{m+i} y_{m+i} \cos\alpha_i + y^2_{m+i}}, i = 1,...,k, C_\nu$ be a normalizing factor. In what follows, we assume $\gamma_{n,k} = (0,...,0,\gamma_{m+1},...,\gamma_{m+k}) \in R^+_{m+k,k}, |\gamma_{n,k}| = \sum_{i=1}^{k} \gamma_{m+i}, y^{\gamma_{n,k}} = y^{\gamma_{m+1}}_{m+1} \dots y^{\gamma_{m+k}}_{m+k}, d\mu_{n,k}(y) = y^{\gamma_{n,k}} dy$, if $y \in R^+_{m+k,k}$.

The designation $\gamma_{n,k}$, n indicates the dimension of this vector, while k the amount of its positive coordinates. If $k = 0$, then $\gamma_{n,k} = 0 \in R^m, T^y_{\gamma_{n,k}} f(x) = f(y - x)$ is a normal shift and $d\mu_{n,k}(y) = dy \equiv dy_1 ... dy_n$.

When $n = m + k \geq 2$ and $s \in \{1, ..., n - 1\}$, we split the space $R^+_{n,k}$ by the direct sum of the space R^+_{s,k_s} of the points $_s x = (x_{n_1},...,x_{n_s})$ (the coordinates $x_{n1}, ..., x_{ns}$ are fixed and by the same token the integers m_s, k_s are determined so that $0 \leq m_s \leq m$, $0 \leq k_s \leq k$, and $m_s + k_s = s$, $s' = n - s$, $k'_s = k - k_s$), and the space R^+_{s',k'_s} of the points $_s x'$ so that $x = \uparrow(_s x, _s x') \in R^+_{n,k}$.

Note that in the same value of the parameters s, m_s, k_s the expansion $x = \uparrow(_s x, _s x')$ is determined unambiguously.

When $G \subseteq R^+_{n,k} (G' \subseteq R^+_{s,k_s})$ is a measurable set and $p \geq 1$

$$L_{p,\gamma n,k}(G) = \Big[f - measurable. : \|f : L_{p,\gamma n,k}(G)\| = \left(\int_G |f(y)|^p \, d\mu_{\gamma n,k}(y) \right)^{1/p} < +\infty \Big] - \Big(L_{p,\mu_{s,k_s}}(\omega : G') = \Big[f - \text{measurable.} : \|f : L_{p,\mu_{s,k_s}}(G')\| = \left(\int_{G'} |f(y)\omega(y)|^p d\mu_{s,k_s}(y) \right)^{1/1p} < +\infty \Big] \Big)$$

is a space of functions summable to the pth power (with weight $w(y)$) on the set $G(G')$.

3. Main Part

If $1 \leq p < +\infty$ and $\alpha = \alpha_{n,k} = n + |\gamma_{n,k}|, R_{\gamma_{n,k}}(p,\alpha)$ is a class of generalized Riesz potentials of the operators of the form $I^\omega_{\gamma_{n,k}}(f)(x) = \int_{R^+_{m+k,k}} T^y \left(|f(x)| \right) \omega\left(|y| \right) |y|^{-(m+k+|\gamma_{n,k}|)} d\mu_{n,k}(y)$, such that:

1) $I^\omega_{\gamma_{n,k}}(f)(x)$ exists for almost all $x \in R^+_{m+k,k}$, where $f \in L_{p,\gamma_{n,k}}(R^+_{m+k,k})$;
2) there exists such $\alpha_\omega \in (0, \alpha / p)$ that

$$\omega(t) \sim t^{\alpha_\omega}, \text{i.e.} \exists c > 0, c^{-1}t^{\alpha_\omega} \le \omega(t) \le ct^{\alpha_\omega}\ t > 0.$$

We introduce the designations

$$a_{s,k_s} = s + |\gamma_{s,k_s}|, \omega_{p,a_s,k_s}(t) = \omega(t)t^{-a_{s,k_s}/p}.$$

We have

Lemma 3.1 (5) *Let* $k \ge 0, s \in \{1,...,m+k-1\}, 0 \le k_s \le k, 0 \le m_s \le m, k_s + m_s = s, 1 \le p < \infty$ and $I^{\omega}_{\gamma_{n,k}} \in R_{\gamma_{n,k}}(p,\alpha_{n,k})$. *Then there exists such* $C_1 > 0$ *that for any function,* $f \in L^{loc}_{1,\gamma_{n,k}}(R^+_{n,k})$ *and any* $x = \uparrow({}_s x, {}_s x') \in R^+_{n,k}$ *we have the inequality.*

$$I^{\omega}_{\gamma_{n,k}}(f)(x) \le C_1 I^{\omega_{p,\alpha_{n-s,k_s'}}}_{\gamma_{s,k_s}}(f_{p,s})({}_s x), \tag{1}$$

where $f_{p,s}({}_s x) = \|f({}_s x, \cdot)\|_{L_{p,\gamma_{n-s,k'_s}}(R^+_{n-s,k'_s})}, {}_s x \in R^+_{s,k_s}$ and

$$I^{\omega_{p,\alpha_{n-s,k_s'}}}_{\gamma_{s,k_s}}(f_{p,s})({}_s x) = \int_{R^+_{s,k_s}} T^{sy}(\omega_{p,\alpha_{n-s,k_s'}}(|{}_s x|)|{}_s x|^{-(s+|\gamma_{s,k_s}|)}) f_{p,s}({}_s y) d\mu_{s,k_s}(y).$$

This lemma allows to establish some estimates in terms of Ω- characteristics of functions for generalized Riesz potentials $I^{\omega}_{\gamma_{n,k}} \in R_{\gamma_{n,k}}(p,\alpha_{n,k})$.

Let ${}_s x = \left(x_{j_1},...,x_{j_{m_s}}, x,...,x_{m+i_{k_s}}\right)$ be chosen. We take $t \in \{1,...,s-1\}$. Expand the space R^+_{s,k_s} in direct sum of the space $R^+_{t,(k_s)_t}$ of the points ${}_t({}_s x) = (({}_s x)_{s_1},...,({}_s x)_{s_t})$ (here $1 \le s_1 < ... < s_t \le n$) and the space $R^+_{s-t,(k_s-(k_s)_t)}$ of the points ${}_t({}_s x)'$ so that, ${}_s x = \uparrow({}_t({}_s x), {}_t({}_s x)') \in R^+_{s,k_s}$.

When $t \in \{1,...,s\}, A_{p,\gamma_{s,k_s}}({}_t({}_s x))$ denotes the totality of all functions measurable on the set R^+_{s,k_s} and belonging to $L_{p,\gamma_{s,k_s}}\left(\{x \in R^+_{s,k_s} : |{}_t({}_s x)| \ge \xi\}\right)$, for each number $\xi > 0$ and $\beta_{p',t} = (t + |\gamma_{t,(k_s)_t}|)/p'$.

For the function $u \in A_{p,\gamma_{s,k_s}}({}_t({}_s x))$ we introduce the characteristics

$$\Omega^{{}_t({}_s x)}_{p,\mu_{s,k_s}}(u,\xi) = \left\{\int_{\{x \in R^+_{s,k_s} : |{}_t({}_s x)| \ge \xi\}} |u(x)|^p d\mu_{s,k_s}(x)\right\}^{1/p}, \xi > 0.$$

and the set

$$J_{p,\mu_{s,k_s}}({}_t(x_s)) = \left\{u \in A_{p,\gamma_{s,k_s}}({}_t({}_s x)) : \int_0^{\xi} \tau^{\beta_{p',t}-1} \Omega^{{}_t({}_s x)}_{p,\mu_{s,k_s}}(u,\tau) d\tau < +\infty, \forall \xi > 0\right\}$$

And, by definition the function $f({}_s x, {}_s x')$, $x = ({}_s x, {}_s x') \in R^+_{m+k,k}$ belongs to the set $J^{n-s,s}_{p,\mu_{n,k}}({}_t(x_s))$, if for almost all ${}_s x \in R^+_{s,k_s}$ the integral $\int_{R_{n-s,k_s'}} |f({}_s x, {}_s x')|^p d\mu({}_s x')$ converges and $f_{p,s}({}_s x) \in J_{p,\mu_{s,k_s}}({}_t(x_s))$.

Theorem 3.2 (6) *Let* $1 \le p < +\infty, n = m+k \ge 2, s \in \{1,...n-1\}, I^{\omega}_{\gamma_{n,k}} \in R_{\gamma_{n,k}}(p,\alpha_{n,k})$,

$$\frac{\alpha_{n-s,k_s'}}{p} < \alpha_\omega < \frac{\alpha_{n,k}}{p} \tag{2}$$

and these exist such $q_s > 1$ that,

$$\frac{1}{p} - \frac{1}{q_s} = \frac{\alpha_\omega - (\alpha_{n-s,k_s'}/p)}{\alpha_{s,k_s}} \tag{3}$$

Then, if $t \in \{1, ..., s\}$, the following estimate is valid

$$\Omega^{{}_t({}_s x)}_{q_s,\mu_{s,k_s}}(I^{\omega}_{\gamma_{n,k}}(f(\cdot, {}_s x')), \xi) \le C$$

$$\xi^{-\beta_{p',t}} \int_0^{\xi} \tau^{\beta_{p',t}-1} \Omega^{{}_t({}_s x)}_{p,\mu_s,k_s}(f_{p,s},\tau) d\tau, \xi > 0, (\Omega)$$

where $\beta_{p',t} = (t + |\gamma_{t,(k_s)_t}|)/p'; f_{p,s}(x) = \|f(x,\cdot)\|_{L_{p,\gamma_{n-s,k_s'}}(R^+_{n-s,k_s'})}, x \in R^+_{s,k_s}; C > 0$ is a constant independent of f and ξ.

The obtained estimates allow to prove theorems completely new in content on generalized Riesz potentials in the spaces in introduced terms of Ω characteristics in weight $L_{p,\gamma}$ spaces (see [7,9]).

By definition, the non-negative function $\varphi(t), 0 < t < \infty$, belongs to the set N if for small $\varepsilon > 0$, for almost all $t \in (0,\varepsilon)\, \varphi(t) > 0$, and $\forall \varepsilon > 0$ the integral $\int_0^\varepsilon \varphi(t)dt$ converges.

Let $1 \le p < \infty$ and $\varphi \in N$. We introduce the spaces

$$l^{t\,(_s x)}_{p,\mu_{s,k_s}}(\varphi) = \left\{ u \in J_{p,\mu_{s,k_s}}(_t(_s x)) : \left\| u : l^{t\,(_s x)}_{p,\mu_{s,k_s}}(\varphi) \right\|^p \overset{df}{=} \int_0^{+\infty} \left(\Omega^{t\,(_s x)}_{p,\mu_s,k_s}(u,\xi) \right)^p \varphi(\xi) d\xi < +\infty \right\},$$

$$l^{t\,(_s x)}_{p,\mu_{n,k}}(\varphi) = \left\{ u \in J^{n-s,s}_{p,\mu_{n,k}}(_t(x_s)) : \left\| u : l^{t\,(_s x)}_{p,\mu_{n,k}}(\varphi) \right\|^p \overset{df}{=} \left\| u(_s x,\cdot) \right\|_{L_{p,\gamma_{n-s,k'_s}}(R^+_{n-s,k'_s})} : l^{t\,(_s x)}_{p,\mu_{s,k_s}}(\varphi) \right\| < +\infty \right\}.$$

4. Proof of Theorem 2

Due to (φ,ψ) the integral converges under the conditions of Lemma 2, therefore $l^{t\,(_s x)}_{p,\mu_{n,k}}(\varphi) \subset J^{n-s,s}_{p,\mu_{n,k}}(_t(x_s))$ and consequently Theorem 1 holds.

Assume

$$\bar{u}(\xi) = \xi^{-\beta_{p',t}} \psi^{1/q_s}(\xi) \text{ and } \bar{v}(\xi) = \varphi^{1/p}(\xi)\xi^{(1-\beta_{p',t})},$$

then

$$\psi(\xi) = \left(\bar{u}(\xi)\xi^{\beta_{p',t}} \right)^{q_s} \text{ and } \varphi(\xi) = \left(\bar{v}(\xi)\xi^{(\beta_{p',t}-1)} \right)^p.$$

Then by means of the estimate (Ω), applying the Hardy theorem on a maximum function we get (see [7-8]):

$$\left(\int_0^\infty (\Omega^{t\,(_s x)}_{q_s,\mu_{s,k_s}}(I^\omega_{\gamma_{n,k}}(f(\cdot,_s x')),\xi))^{q_s} \psi(\xi) d\xi \right)^{1/q_s}$$

$$\le c\left(\int_0^\infty (\xi^{-\beta_{p',t}} \psi^{1/q_s} \int_0^\xi \tau^{\beta_{p',t}-1} \Omega^{t\,(_s x)}_{p,\mu_{s,k_s}}(f_{p,s},\tau)d\tau)^{q_s} d\xi \right)^{1/q_s}$$

$$\le c\left(\int_0^\infty (\bar{u}(\xi) \int_0^\xi \tau^{\beta_{p',t}-1} \Omega^{t\,(_s x)}_{p,\mu_{s,k_s}}(f_{p,s},\tau)d\tau)^{q_s} d\xi \right)^{1/q_s}$$

$$\le c\left(\int_0^\infty \bar{v}^p (\tau^{\beta_{p',t}-1} \Omega^{t\,(_s x)}_{p,\mu_{s,k_s}}(f_{p,s},\tau))^p d\xi \right)^{1/p} = c\left(\int_0^\infty \varphi(\xi) (\Omega^{t\,(_s x)}_{p,\mu_{s,k_s}}(f_{p,s},\tau))^p d\xi \right)^{1/p}$$

Theorem 2 is proved.

In the case $t = 1$, Theorem 2 for the Riesz potentials associated with Laplace-Bessel differential operator in the appropriate setting was considered in [8,10].

5. Conclusions

Theorem 5.1 *Let* $1 \le p < +\infty$, $n = m + k \ge 2$, $s \in \{1,...n-1\}$,

$I^\omega_{\gamma_{n,k}} \in R_{\gamma_{n,k}}(p,\alpha_{n,k})$, $t \in \{1,...s\}$ and conditions (2) and (3) of Theorem 1 be fulfilled.

Then, if

$$\omega^p(\xi) = \int_0^\xi \varphi(t)dt, \tilde{\omega}(\xi) = \int_0^\xi \psi(t)dt \varphi, \psi \in N_1$$

$$\sup_{0<t} \left(\int_t^d \xi^{-\beta_{p',t} q_s} \psi(\xi) d\xi \right)^{\frac{1}{q}} \left(\int_0^t \varphi^{-\frac{1}{p}}(\xi) \xi^{-(1-\beta_{p',t})p'} \right)^{\frac{1}{p'}} < \infty,$$

then there exists such $C > 0$ that for any function $f \in L_{p,\gamma_{n,k}}\left(\tilde{\omega}\left(\left|{}_t({}_s x)\right|\right), R^+_{m+k,k}\right)$

and for almost all $x = \uparrow({}_s x, {}_s x') \in R^+_{m+k,k}$ there exists $(I^{\omega}_{\gamma_{n,k}})(x)$ and we have the estimate

$$\left\| I^{\omega}_{\gamma_{n,k}}(f)(\cdot, {}_s x') : L_{q_s,\gamma_{s,k_s}}\left(\omega\left(\left|{}_t({}_s x)\right|\right), R^+_{s,k_s}\right)\right\| \le C \left\| f : L_{p,\gamma_{n,k}}\left(\tilde{\omega}\left(\left|{}_t({}_s x)\right|\right), R^+_{m+k,k}\right)\right\|.$$

Lemma 5.2 *Let* $\varphi \in N$ *and* $\omega^p(\tau) = \int\limits_0^{\tau} \varphi(\xi) d\xi \ \ \tau > 0$.

Then

$$l^{{}_t({}_s x)}_{p,\mu_{s,k_s}}(\varphi) = L_{p,\mu_{s,k_s}}\left(\omega\left(\left|{}_t({}_s x)\right|\right) : R^+_{s,k_s}\right), l^{{}_t({}_s x)}_{p,\mu_{n,k}}(\varphi) = L_{p,\mu_{s,k_s}}\left(\omega\left(\left|{}_t({}_s x)\right|\right) : R^+_{n,k}\right)$$

then appropriate norms are equal.

Proof.

$$\int\limits_0^{\infty} (\Omega^{{}_t({}_s x)}_{p,\mu_{s,k_s}}(u,\xi))^p \varphi(\xi) d\xi = \int\limits_0^{\infty} \int_{R_{s,k_s}} \left| u\left({}_t({}_s y), {}_t({}_s y)'\right)\right| \chi_{[0,\xi]}\left(\left|{}_t({}_s y)\right|\right) d\mu_{s,k_s} \varphi(\xi) d\xi$$

$$= \int_{R^+_{s,k_s}} \left| u\left({}_t({}_s y), ({}_s y)'\right)\right|^p \left(\int_0^{\infty} \left(\chi_{[0,\xi]}\left(\left|{}_t({}_s y)\right|\right) \varphi(\xi) d\xi\right)\right) d\mu_{s,k_s}({}_s y)$$

$$= \int_{R^+_{s,k_s}} \left| u\left({}_t({}_s y), {}_t({}_s y)'\right)\right|^p \left(\int_0^{\left|{}_t({}_s y)\right|} \varphi(\xi) d\xi\right) d\mu_{s,k_s}({}_s y)$$

$$= \int_{R^+_{s,k_s}} \left| u\left({}_t({}_s y), {}_t({}_s y)'\right)\right|^p \omega^p\left(\left|({}_s y)\right|\right) d\mu_{s,k_s}({}_s y).$$

The second statement of the lemma is proved in the same way. □

Lemma 5.3 *Let* $\varphi \in N$ *and the integral* $\int_0^{\xi} \left| \varphi^{1/p}(\xi) \xi^{(1-\beta_{p',t})} \right|^{-p'} d\xi < \infty$ *converge.*

Then $l^{{}_t({}_s x)}_{p,\mu_{s,k_s}}(\varphi) \subset J_{p,\mu_{s,k_s}}\left({}_t(x_s)\right)$ *and* $l^{{}_t({}_s x)}_{p,\mu_{n,k}}(\varphi) \subset J^{n-s,s}_{p,\mu_{n,k}}\left({}_t(x_s)\right)$.

Proof.

$$\int_0^{\xi} \tau^{\beta_{p',t}-1} \Omega^{{}_t({}_s x)}_{p,\mu_{s,k_s}}\left(f_{p,s}, \tau\right) d\tau = \int_0^{\xi} \Omega^{{}_t({}_s x)}_{p,\mu_{s,k_s}}\left(f_{p,s}, \tau\right) \left[\varphi(\tau)\right]^{1/p} \left[\varphi(\tau)\right]^{-1/p} \tau^{\beta} p', t^{-1} d\tau$$

$$\le \left(\int_0^{\xi} \left(\Omega^{{}_t({}_s x)}_{p,\mu_{s,k_s}}\left(f_{p,s}, \tau\right)\right)^p \varphi(\tau) d\tau\right)^{1/p} \left(\int\limits_0^{\xi} \left(\left[\varphi(\tau)\right]^{-1/p} \tau^{\beta} p', t^{-1}\right)^{p'} d\tau\right)^{1/p'} < +\infty.$$

This proves $l^{{}_t({}_s x)}_{p,\mu_{s,k_s}}(\varphi) \subset J_{p,\mu_{s,k_s}}\left({}_t(x_s)\right)$.

The second statement of the lemma is proved in the same way. □

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