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All metric fixed point theorems hold for quasi-metric spaces

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Abstract

Our aim in this article is to show that all metric fixed point theorems hold for quasi-metric spaces (X, δ) (without symmetry). In fact, we show some well-known theorems on metric spaces hold for quasi-metric spaces from the beginning. We check this fact for the Banach contraction principle, the Covitz-Nadler fixed point theorem, the Rus-Hicks-Rhoades fixed point theorem, and others. In these theorems the concepts of continuity and completeness can be replaced by orbital continuity and T-orbital completeness for a selfmap T, respectively.Consequently, we improve and generalize the basic known theorems in the metric fixed point theory.

Key words and phrases. Banach contraction, fixed point, fixed point theorem, metric space, quasi-metric, maximal element.

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1. Introduction

The notion of distance is very important for human life and it was axiomatically formulated as metric by Fréchet in the early 1900s. It opened a new method for mathematical analysts. A large number of generalizations, modifications, improvements and extensions of the metric concept have appeared in different fields due to its fundamental role in analytic sciences and applications.

The metric fixed point theory is originated from the Banach contraction principle in 1922. It was extended to multimaps by Nadler [1] in 1969 and Covitz-Nadler [2] in 1970. Moreover, the Banach contraction was extended to hundreds of contractive type conditions; see Rhoades [3], Park [4], [5] and many others.

Let(X, d) be a complete metric space and $f : X \to X$ a map satisfying $d(fx, f^2x) \leq \alpha, d(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. The fixed point theorems due to Rus [6] in 1973 and Hicks-Rhoades [7] in 1979 on such maps were not popular than the Banach contraction. Recently, in [8], [9] we noticed that it has an interesting long history and has a large number of examples. Moreover, the Rus-Hicks-Rhoades theorem is closely related to the Banach contraction principle in 1922, its multi-valued versions due to Nadler [17] in 1969 and Covitz-Nadler [2] in 1970.

The aim of our previous works [8], [9] were to trace such history of the Rus-Hicks-Rhoades theorem, and to collect its grown-up versions or equivalents or closely related theorems. Such theorems are too many and could be called its relatives.

Among hundreds of extensions of metric spaces, a quasi-metric is the one not necessarily symmetric. In fact, a quasi-metric δ satisfies all axioms of a metric except the symmetry $\delta(x, y) = \delta(y, x)$ for all x, y in the space. In our [9], we collected almost all facts about the Rus-Hicks-Rhoades theorem in quasi-metric spaces.

Our aim in this article is to show that all fixed point theorems on metric spaces also hold for quasi-metric spaces from the beginning. In fact, we will check this fact by giving direct proofs for the Banach contraction principle in 1922, results due to Nadler [1] in 1969, Covitz-Nadler [2] in 1970, Rus-Hicks-Rhoades in 1980s, and some others.

This article is organized as follows: Section 2 is preliminaries for basic facts on quasi-metric spaces, and we show that all metric fixed point theorems hold for quasi-metric spaces. In Section 3, we give a generalization of the RHR theorem for quasi-metric spaces. Section 4 deals with the Banach contraction principle as a corollary of the BHR theorem. In Section 5, we investigate the Berinde type maps [2] as examples of the RHR maps.

Section 6 is to show that some Caristi type maps and other examples are RHR maps. In Section 7, the Covitz-Nadler fixed point theorem and others are easy consequences of our RHR theorem on quasi-metric spaces. Section 8 deals with some other results on quasi-metric spaces. Finally, in Section 9, we give some conclusion.

2. Preliminaries

We recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $\delta : X \times X \to [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $\delta(x, y) = \delta(y, x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

A *metric* on a set X is a quasi-metric satisfying

(c) (symmetry) $\delta(x,y) = \delta(y,x)$ for all $x, y \in X$.

The convergence and completeness in a quasi-metric space (X, δ) are defined as follows:

Definition 2.2. ([10], [11])

(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} \delta(x_n, x) = \lim_{n\to\infty} \delta(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $n > m > N$.
(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m > n > N$.
(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3. ([10], [11])

(1) (X, δ) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, δ) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, δ) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let (X, δ) be a quasi-metric space and $T : X \to X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, T(x), \cdots, T^n(x), \cdots\}.$$

The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n(x) = x_0 \Rightarrow \lim_{n\to\infty} T^{n+1}(x) = T(x_0)$$

for any $x \in X$.

Note that every complete metric space is T-orbitally complete for all maps $T : X \to X$. There exists a T-orbitally complete metric space but it is not complete. Moreover, there exists an orbitally continuous map but it is not continuous.

Every quasi-metric induces a metric, that is, if (X, δ) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{\delta(x, y), \delta(y, x)\}$$

is a metric on X; see Jleli et al. [11].

The following was given in [12]:

Theorem 2.5. *A selfmap $T : X \to X$ of a quasi-metric space (X, δ) has a fixed point $z \in X$ if and only if z is a fixed point of the selfmap T of the induced metric space (X, d).*

Proof. If $z = T(z)$ in (X, δ), then

$$d(z, T(z)) = \max\{\delta(z, T(z)), \delta(T(z), z)\} = 0,$$

and hence $d(z, T(z)) = 0$. The converse is true for $d = \delta$.

From this, all metric fixed point theorems are true for quasi-metric spaces. This is a rather surprising fact in the one hundred year history of the metric fixed point theory since Banach in 1922.

The following Rus-Hicks-Rhoades theorem is given in our previous work [8], [9] as a generalization of the Banach contraction principle:

Theorem 2.6. *Let T be a selfmap of a complete metric space (X, d) satisfying*

$$d(T(x), T^2(x)) \leq \alpha, d(x, T(x)) \qquad \text{for every } x \in X,$$

where $0 < \alpha < 1$. *Then T has a fixed point.*

In our works [8], [9] we showed that the Rus-Hicks-Rhoades theorem has a large number of closely related theorems. A map satisfying the condition in Theorem 2.6 is called a Rus-Hicks-Rhoades map or simply an RHR map.

In the sequel, we demonstrate that some typical metric fixed theorems can be directly extended for quasi-metric spaces.

3. Main Results

In this section, we give a generalization of the Rus-Hicks-Rhoades theorem for a quasi-metric spaces (X, δ) with an orbitally continuous selfmap $T : X \to X$ such that X is T-orbitally complete.

Lemma 3.1. *Let (X, δ) be a quasi-metric space*, $T : X \to X$ a selfmap, and $\varphi : X \to [0, \infty)$ any function such that

$$\delta(x, T(x)) \leq \varphi(x) - \varphi(T(x)) \quad \text{for some } x \in M.$$

Then $\{T^n(x)\}$ is a right-Cauchy sequence.

Proof. If we fix $x \in X$ and take $m > n \in \mathbb{N}$, we obtain

$$\delta(T^n(x), T^{m+1}(x)) \leq \sum_{i=n}^{m} \delta(T^i(x), T^{i+1}(x)) \leq \varphi(T^n(x)) - \varphi(T^{m+1}(x)).$$

(Notice that the last inequality comes from cancelation in the telescoping sum.) In particular by taking $n = 1$ and letting $m \to \infty$ we conclude that

$$\sum_{i=1}^{\infty} \delta(T^i(x), T^{i+1}(x)) \leq \varphi(T(x)) < \infty.$$

This implies that $\{T^n(x)\}$ is a right-Cauchy sequence.

The following is the main theorem of this section:

Theorem 3.2. *Let (X, δ) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$\delta(T(x), T^2(x)) \leq k, \delta(x, T(x)) \qquad \text{for every } x \in X,$$

where $0 < k < 1$.

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \leq \frac{k^n}{1-k} \delta(x, T(x)), \, n = 1, 2, \cdots.$$

(ii) *$T : X \to X$ is orbitally continuous at $x_0 \in X$ in* (i) *if and only if x_0 is a fixed point of T.*

We prove this by analyzing a typical proof of the Banach contraction principle given by Art Kirk ([13], Theorem 2.2).

Proof. Step 1. For each $x \in X$, $\{T^n(x)\}$ is right Cauchy:

Adding $\delta(x, T(x))$ to both sides of the inequality $\delta(T(x), T^2(x)) \leq k\, \delta(x, T(x))$ yields

$$\delta(x,T(x)) + \delta(T(x),T^2(x)) \leq \delta(x,T(x)) + k\, \delta(x,T(x))$$

which can be rewritten

$$\delta(x,T(x)) - k\, \delta(x,T(x)) \leq \delta(x,T(x)) - \delta(T(x),T^2(x)).$$

This in turn is equivalent to

$$\delta(x, T(x)) \leq (1-k)^{-1}[\delta(x, T(x)) - \delta(T(x),T^2(x))].$$

Now define the function $\varphi : X \to [0, \infty)$ by setting $\varphi(x) = (1-k)^{-1}\delta(x, T(x))$, for $x \in X$.
This gives us the basic inequality

$$\delta(x,T(x)) \leq \varphi(x) - \varphi(T(x)), \qquad x \in X.$$

Therefore $\{T^n(x)\}$ is a right-Cauchy sequence by Lemma 3.1.

Step 2. T-orbital completeness:

Since X is T-orbitally complete, for any $x \in X$ there exists $x_0 \in M$ such that

$$\lim_{n\to\infty} T^n(x) = x_0.$$

Step 3. Orbital continuity at x_0:

If T is orbitally continuous at x_0, then

$$x_0 = \lim_{n\to\infty} T^n(x) = \lim_{n\to\infty} T^{n+1}(x) = T(x_0).$$

Thus x_0 is a fixed point of T. Conversely, if x_0 is fixed, then clearly T is orbitally continuous at x_0.

Step 4. Convergence for $\{T^n(x)\}$:

The last part of Kirk's original proof in [13] is added for completeness. Returning to the inequality

$$\delta(T^n(x),T^{m+1}(x)) \leq \varphi(T^n(x)) - \varphi(T^{m+1}(x)),$$

upon letting $m \to \infty$ we see that

$$\delta(T^n(x),x_0) \leq \varphi(T^n(x)) = (1-k)^{-1}\delta(T^n(x),T^{n+1}(x)).$$

Since $(1-k)^{-1}\delta(T^n(x),T^{n+1}(x)) \leq \dfrac{k^n}{1-k}\delta(x,T(x))$, we obtain

$$\delta(T^n(x),x_0) \leq \frac{k^n}{1-k}\delta(x,T(x)).$$

This provides an estimate on the rate of convergence for the sequence $\{T^n(x)\}$ which depends only on $\delta(x,T(x))$.

Remark 3.3. (1) In the quasi-metric space, the condition (a) is consistent with

$$\delta(T(x), T^2(x)) \leq k\, \delta(x, T(x))$$

when x is a fixed point of T. The triangle inequality (b) is used in Step 2.

(2) In the above lengthy proof, the symmetry (c) of a metric is not used.

Example 3.4. Let $X = [0, 2] \subset \mathbb{R}$ with the usual metric δ and

$$T(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ x & \text{if } x \in [1,2]. \end{cases}$$

Then $\delta(T(x), T^2(x)) \le k\, \delta(x, T(x))$ for some $k < 1$. In fact,

$$0 = \delta(1, 1) \le k\, \delta(x, 1) \text{ for } x \in [0, 1],$$

$$0 = \delta(x, x) \le k\, \delta(x, x) \text{ for} \} x \in [1, 2].$$

This example has 'many' fixed points of T.

From the proof of Theorem 3.2, we have the following:

Theorem 3.5. *Let* (X, δ) *be a quasi-metric space and let* $T : X \to X$ *be a map satisfying*

$$\delta(x,T(x)) \le \varphi(x) - \varphi(T(x)), \qquad x \in X,$$

for a real-valued function $\varphi : X \to [0, \infty)$ *such that*

$$\varphi(\mathrm{x}) = (1 - \mathrm{k})^{-1}\delta(x, T(x)) \textit{ with } 0 < \mathrm{k} < 1.$$

(i) *If X is T-orbitally complete, then, for each* $x \in X$, *there exists a point* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \le \frac{k^n}{1-k}\delta(x, T(x)), \; n = 1, 2, \cdots.$$

(ii) $T : X \to X$ *is orbitally continuous at* $x_0 \in X$ *in* (i) *if and only if* x_0 *is a fixed point of T.*

This is a particular form of the Caristi type fixed point theorem.

4. The Banach Type Theorems

The following extends the standard Banach contraction principle formulated by Art Kirk ([13], Theorem 2.2):

Theorem 4.1. *Let* (X, δ) *be a quasi-metric space and let* $T : X \to X$ *be a contraction, that is,*

$$\delta(T(x), T(y)) \le k\, \delta(x, y) \textit{ for every } x, y \in \mathrm{X},$$

with $0 < k < 1$. *If* (X, δ) *is T-orbitally complete, then T has a unique fixed point* $x_0 \in X$. Moreover, for each $x \in X$,

$$\lim_{n\to\infty} T^n(x) = x_0$$

and, in fact, for each $x \in X$,

$$\delta(T^n(x), x_0) \le \frac{k^n}{1-k}\delta(x, T(x)), \; n = 1, 2, \cdots.$$

Proof. All things follow from Theorem 3.2 except the following in the proof of Kirk [13]:

Step 5. Uniqueness of fixed point:

In order to see that x_0 is the only fixed point of T, suppose $T(y) = y$. Then by what we have shown in Theorem 3.2

$$x_0 = \lim_{n\to\infty} T^n(y) = y.$$

This completes the proof.

Moreover, under the hypothesis of Theorem 4.1, Steps 1–4 of the proof of Theorem 3.2 hold. Therefore, Theorem 3.2 implies all conclusions of the Banach principle except the uniqueness of the fixed point.

5. The Berinde Type Theorems

Let (M, d) be a complete metric space, and $T : M \to M$ be a following type of RHR maps for $y = Tx$. Then T has a fixed point.

(1) Kannan [14] in 1969: There exists $r \in [0, 1/2)$ such that for all $x, y \in M$,

$$d(Tx, Ty) \le r[d(x,Tx) + d(y,Ty)].$$

(2) Chatterjea [15] in 1972: There exists $r \in [0, 1/2)$ such that for all $x, y \in M$,

$$d(Tx, Ty) \le r[d(x,Ty) + d(y,Tx)].$$

(3) Zamfirescu [16] in 1972: There exist real numbers a, b, c satisfying $0 \le a < 1$, $0 < b$, $c < 1/2$ such that for each pair $x, y \in M$, at least one of the following is true:
(z_1) $\mathrm{d}(Tx,Ty) \le a\, d(x,y)$;
(z_2) $d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)]$;
(z_3) $d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)]$.

Then T is a Picard operator, that is,

$(p1)$ T has a unique fixed point $p \in X$;
$(p2)$ the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \qquad n = 0,1,2, \cdots$$

converges to p, for any $x_0 \in M$.

(4) Ćirić [17] in 1971: The map T having $h \in [0,1)$ such that

$$d(Tx,Ty) \le h \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}$$

for all $x, y \in M$ is a Picard operator.

(5) Berinde [18] in 2004: A map $T : M \to M$ is a weak contraction if there exist $r \in (0,1)$ and $L \ge 0$ satisfying

$$d(Tx,Ty) \le r\, d(x,y) + Ld(y,Tx) \text{ all } x,y \in X.$$

It is easy to check that all examples **(1)–(5)** are RHR maps and **(1)–(4)** are the Berinde type weak contraction **(5)**; see Berinde [18].

Therefore, Theorem 3.2(i) and the following hold for all maps in the class **(5)**:

Theorem 5.1. *Let (X, d) be a quasi-metric space and let $T : X \to X$ be a Berinde type weak contraction.*

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$d(T^n(x), x_0) \le \frac{r^n}{1-r} d(x, T(x)), \ n = 1,2,\cdots.$$

(ii) *Each $x_0 \in X$ in* (i) *is a fixed point of T.*

Proof. (i) Since the Berinde type weak contraction is a RHR map, (i) follows from Theorem 3.2(i).

(ii) In view of Theorem 3.2(ii), we have to show $T: X \to X$ is orbitally continuous at $x_0 \in X$; that is,

$$\lim_{n\to\infty} T^n(x) = x_0 \Rightarrow \lim_{n\to\infty} T^{n+1}(x) = T(x_0)$$

for any $x \in X$. Since T is a Berinde type weak contraction, we have

$$d(T^{n+1}(x), T(x_0)) \le r, d(T^n(x), x_0) + L, d(x_0, T^{n+1}(x)).$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(T^{n+1}(x), T(x_0)) = 0.$$

This implies the conclusion.

Remark 5.2. In Theorem 3.2(ii), we considered the orbital continuity of T. However, we showed that the Berinde type weak contraction is orbitally continuous.

6. Some Caristi Type Theorems

Here we add a new theorem as follows:

Theorem 6.1. *Let T be a selfmap of a quasi-metric space (X, δ) which is T-orbitally complete. Suppose $\varphi : X \to [0, \infty)$ is a function such that for* all $x \in X$,

$$\delta(Tx, T^2x) \le \varphi(x) - \varphi(Tx).$$

If T is orbitally continuous, then T has a fixed point.

Proof. By Lemma 3.1, $\{T^n x\}$ is a right-Cauchy sequence for each $x \in X$. Since X is T-orbitally complete, it converges to some $x_0 \in X$. Moreover, T is orbitally continuous, we have some

$$x_0 = \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^{n+1} x = T(x_0).$$

Hence any orbital limit is a fixed point.

Example 6.2. (1) Theorem 6.1 is comparable to Lemma 3.1 and can be applied to many results in this article.

(2) Pant et al. ([19], Theorem 2.1) in 2021 considered the map $f: X \to X$ satisfying

$$d(fx, fy) \le [\varphi(x) - \varphi(fx)] + [\varphi(y) - \varphi(\mathrm{f}y)]$$

for all $x, y \in X$, where (X, d) is a complete metric space.

The Rus-Hicks-Rhoades theorem can be extended to the following consequence of the Caristi type fixed point theorem:

Theorem 6.3. *Let T be a selfmap of a complete quasi-metric space (M, δ) satisfying*

$$\delta(T(x), T^2(x)) \le \alpha\, \delta(x, \mathrm{T}(x)) \text{ for every } x \in M,$$

where $0 < \alpha < 1$. *Then T has a fixed point and the statement* (i) *of Theorem 3.2 holds.*

Remark 6.4. In Step 1 of the proof of Theorem 3.2, we obtained the basic inequality

$$\delta(x, T(x)) \le \varphi(x) - \varphi(T(x)),\ x \in M.$$

From this, by applying the Caristi fixed point theorem, we can obtain fixed points. This leads to Step 4 directly. Therefore, the orbital continuity in Theorem 3.2(ii) is redundant in this case. This is why we did not assume the orbital continuity of maps in Theorem 6.3.

The following consequence of Theorem 6.3 extends Suzuki ([24], Theorem 2):

Corollary 6.5. *Let* (M, δ) *be a complete quasi-metric space and let* T *be a selfmap on* M.

(1) *For any function* θ *from* $[0, 1)$ *onto* $[0, 1]$, *there exists* $r \in [0, 1)$ *such that*

$$\theta(r)\,\delta(x, Tx) \le \delta(x, y) \quad \textit{implies} \quad \delta(Tx, Ty) \le r\,\delta(x, y)$$

for all $x, y \in M$. *Then there exists a unique fixed point* z *of* T. *Moreover* $\lim_n T^n x = z$ *for all* $x \in M$.

(2) *There exists* $r \in (0, 1)$ *such that every selfmap* T *on* M *satisfying the following has a fixed point:*

$$\frac{1}{10{,}000}\delta(x,Tx) \le \delta(x,y) \ \textit{ implies } \delta(Tx,Ty) \le r\delta(x,y)$$

for all $x, y \in M$.

Proof. For $y = Tx$, we have $\delta(Tx, T^2x) \le r,\delta(x, Tx)$ for all $x \in M$. Then we can apply Theorem 6.3.

The following is motivated from Chandra-Joshi-Joshi [21] in 2022.

Let (M, δ) be a quasi-metric space, and $T : M \to M$. Then for all $x, y \in X$, we denote

$$m(Tx, Ty) = a\,\delta(x, y) + b\,\max\{\delta(x, Tx),\delta(y, Ty)\} + c[\delta(x, Ty) + \delta(y, Tx)],$$

where a, b and c are non-negative reals such that $a + b + 2c = r$ with $r \in [0,1)$. Now, we consider the following generalized contractive condition

$$\theta(r)\,\min\{\delta(x, Tx),\delta(x, Ty)\} \le \delta(x, y) \text{ implies } \delta(Tx, Ty) \le m(Tx, Ty),$$

where $\theta : [0, 1) \to (1/2, 1]$ is as defined in Corollary 6.5. It is remarkable that this condition is a generalization of the condition (22) and several other conditions mentioned in the Transaction Paper of Billy E. Rhoades [3].

The following improves Chandra-Joshi-Joshi ([21], Theorem 4):

Corollary 6.6. *Let* (M, δ) *be a complete quasi-metric space, and* $T : M \to M$. *Assume that there exists* $r \in [0, 1)$ *such that the requirement* (1) *of Corollary 6.5 is satisfied for each* $x, y \in M$. *Then* T *has a unique fixed point* $z \in M$. *Moreover,* $\lim_{n\to\infty} T^n x = z$ for all $x \in M$.

As in the original proof in [21], we have

$$\delta(Tx, T^2x) \le r,\delta(x, Tx), \quad \forall\, x \in M.$$

Therefore the conclusion follows from Theorem 3.2.

7. Equivalents of the Covitz-Nadler Theorem

In our previous works [4], [5] in 2023, we showed that many results in metric fixed point theory are closely related to the Rus-Hicks-Rhoades theorem. In this section, we show some of them holds for quasi-metric spaces.

Let (X, δ) be a quasi-metric space and Cl(X) denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in$ Cl(X), set

$$H(A, B) = \max\{\sup\{\delta(a,B) : a \in A\},\ \sup\{\delta(b, A) : b \in B\}\},$$

where $\delta(a,B) = \inf\{\delta(a,b) : b \in B\}$. Then H is called a generalized Hausdorff quasi-metric since it may have infinite values.

Based on our well-known 2023 Metatheorem, we obtained the following in [3], [4], [12], [22]:

Theorem H. *Let* (X, δ) *be a complete quasi-metric space, and* $0 < h < 1$. *Then the following equivalent statements hold:*

(α) *For a multimap* $T : X \to \mathrm{Cl}(X)$, *there exists an element* $v \in X$ such that $H(T(v), T(w)) > h,\delta(v,w)$ for any $w \in X\backslash\{v\}$.

(β) *If* $\mathfrak{I}$ *is a family of maps* $f: X \to X$ *such that, for any* $x \in X\backslash\{f(x)\}$, *there exists a* $y \in X\backslash\{x\}$ *satisfying* $\delta(f(x),f(y)) \leq h,\delta(x,y)$, *then* $\mathfrak{I}$ *has a common fixed element* $v \in X$, *that is,* $v = f(v)$ *for all* $f \in \mathfrak{I}$.

(γ) *If* $\mathfrak{I}$ *is a family of maps* $f: X \to X$ *satisfying* $\delta(f(x),f^2(x)) \leq h,\delta(x,f(x))$ *for all* $x \in X\backslash\{f(x)\}$, *then* $\mathfrak{I}$ *has a common fixed element* $v \in X$, *that is,* $v = f(v)$ *for all* $f \in \mathfrak{I}$.

(δ) *Let* $\mathfrak{I}$ *be a family of multimaps* $T : X \multimap X$ *such that, for any* $x \in X\backslash T(x)$, *there exists* $y \in X\backslash\{x\}$ *satisfying* $H(T(x),T(y)) \leq h,\delta(x,y)$. *Then* $\mathfrak{I}$ *has a common fixed element* $v \in X$, *that is,* $v \in T(v)$ *for all* $T \in \mathfrak{I}$.

(ϵ) *If* $\mathfrak{I}$ *is a family of multimaps* $T : X \to X$ *satisfying* $H(T(x),T(y)) \leq h\ \delta(x,y)$ *for all* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$, *then* $\mathfrak{I}$ *has a common stationary element* $v \in X$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{I}$.

(η) *If* Y *is a subset of* X *such that for each* $x \in X \setminus Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $H(T(x), T(z)) \leq h\ \delta(x,z)$ *for a* $T : X \multimap X$, *then there exists a* $v \in X \cap Y = Y$.

Proof. The equivalency follows from our 2023 Metatheorem in [5]. Recall that (β)–(ϵ) are equivalent to (β1)-(ϵ1) for the case $\mathfrak{I}$ is a singleton, respectively. Note that (β1) holds by Theorem 4.1. Then Theorem H holds.

A more general form of Nadler's theorem [1] was established by Covitz-Nadler [2] in 1970 for metric spaces. Note that (δ1) and (ϵ1) improves the well-known theorems of Nadler [1] and Covitz-Nadler [1].

The following is our version:

Theorem 7.1. (Covitz-Nadler) *Let* (X,δ) *be a complete quasi-metric space and* $T : X \to \mathrm{Cl}(X)$. *Assume there is an* $h \in [0, 1)$ *such that*

$$H(T(x), T(y)) \leq h\ \delta(x, y) \text{ for all } \quad x, y \in X.$$

If X *is* T*-orbitally complete, then* T *has a fixed point.*

Note that Rus-Hicks-Rhoades theorem (Theorem 3.2) follows from the following:

Theorem 7.2. (H(γ1)) *If a maps* $f: X \to X$ *satisfies* $\delta(f(x),f^2(x)) \leq h,\delta(x,f(x))$ *for all* $x \in X\backslash\{f(x)\}$, *then it has a fixed element* $v \in X$, *that is,* $v = f(v)$.

Remark 7.3. Our Corollary 6.6, that is, Chandra-Joshi-Joshi ([21], Theorem 4), follows from (γ1). In fact, the beginning few lines of its proof in [21] shows

$$d(Tx,T^2x) \leq r\ d(x, Tx) \ \ \forall\, x \in X.$$

Therefore (γ1) works.

8. Other Results without Symmetry

Recall that, in the proofs for Lemma 3.1, Theorems 3.2, 3.5, 4.1 and 5.1, the requirement (a) (self-distance) in Definition 2.1 (of quasi-metric spaces) is not used.

In the present section, we list some known results which does not need the symmetry of metric spaces.

1. Kada-Suzuki-Takahashi [23] in 1996 introduced the concept of W-distances for a metric space. It is known that, for a quasi-metric space, the concepts of w-distances, Cauchy sequences, completeness, and Banach contractions can be defined.

2. Park [4], [24] in 1980 obtained some fixed point theorems for many contractive definitions. One of them was the following forerunner of many known theorems including Theorem 2.5, where $O(u) := \{u, f(u), f^2(u), \ldots\}$.

Theorem 8.1. ([24]) *Let f be a selfmap of a metric space* (X, d). *If there exists a point* $u \in X$ *and a* $\lambda \in [0,1)$ *such that* $\bar{O}(u)$ *is complete and*

$$(*) \quad d(f(x),f(y)) \le \lambda\, d(x,y))$$

holds for any x, $y = f(x) \in O(u)$, *then* $\{f^n(u)\}$ *converges to some* $\xi \in X$, and

$$d(f^i(u),\xi) \le \frac{\lambda^i}{1-\lambda} d(u,f(u)) \ \text{ for } i \ge 1.$$

Further, if f is orbitally continuous at ξ *or if* (*) *holds for any* $x, y \in \bar{O}(u)$, *then* ξ *is a fixed point of f.*

Checking the proof of this theorem in [24], the metric can be replaced by quasi-metric (that is, the symmetry is not needed). Therefore Theorem 8.1 implies Theorem 3.2. Moreover, we gave many consequences or examples of Theorem 8.1 in [24] due to Pal-Maiti, Rhoades, Wong, Ćirić, Fisher, Jaggi, and Taskovitz.

3. Park [22] in 2000: The aim of this paper is to unify the results of Blum-Oettli [25] (1994), Kada-Suzuki-Takahashi [23] (1996), and Oettli-Théra [26] (1993) along the lines of Park-Kang [27] (1993) and to improve the equivalent formulations of Ekeland's principle in various aspects. In fact, we obtain far-reaching generalized forms of Ekeland's principle and its six equivalents (Theorems 1, 10, and 2 in [28]). We also show that one of our formulations readily implies the principle (Theorem 3 in [22]). Finally, we add historical remarks.

4. Suzuki [29] introduced the concept of τ-distance on a metric space, which is a generalized concept of both w-distance and Tataru's distance. He also improved the generalizations of the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, and the nonconvex minimization theorem according to Takahashi.

5. Based on our new 2023 Metatheorem in [9],[12],[22], we can extend the main result in our previous work [4] in 2000.

6. Since the Banach contraction is an RHR map, thousands of known related works can be extended.

9. Epilogue

Traditional monographs or text-books on fixed point theory or general topology stated the Banach contraction principle for metric spaces, for example, Dugundji [30], Willard [16], Smart [31], Istratescu [32], Dugundji-Granas [33], Goebel-Kirk [34], and many others. All of them stated the principle for metric spaces only, but their proofs do not use the symmetry of a metric. Even more, they do not mention the concept of quasi-metric spaces.

Beginning from our [5] in 2022, we have published nearly two dozen articles on our Metatheorem and related topics. These establishes a new foundations of Ordered Fixed Point Theory in [5]. Moreover, our Metatheorem has numerous applications; for recent examples, see[8],[9],[12],[22], [35], [36].

In the present article, we corrected only some of inaccurate statements in Metric Fixed Point Theory and introduced many results related the Rus-Hicks-Rhoades theorem. Since the theorem and its extensions properly include the corresponding ones of the Banach contraction, the future study on Metric Fixed Point Theory would be concentrated to extend the theorem and new applications.

There are thousands of artificial generalizations of metric spaces. Most of them assume the symmetry which can be eliminated as in the present article.

References

[1] S. B. Nadler, Jr. *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969) 475–488.

[2] H. Covitz, S.B. Nadler, Jr. *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. 8 (1970) 5–11.

[3] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226 (1977) 257–290.

[4] S. Park, *On general contractive-type conditions*, J. Korean Math. Soc. 17 (1980) 131–140.

[5] S. Park, Foundations of ordered fixed point theory, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 61(2) (2022) 1–51.

[6] I. A. Rus, *Teoria punctului fix, II*, Univ. Babes-Bolyai, Cluj, 1973.

[7] T. L. Hicks, B. E. Rhoades, *A Banach type fixed point theorem*, Math. Japon. 24 (1979) 327–330.

[8] S. Park, *Relatives of a theorem of Rus-Hicks-Rhoades*, Letters Nonlinear Anal. Appl. 1 (2023) 57–63. ISSN 2958-874X

[9] S. Park, *Almost all about Rus-Hicks-Rhoades maps in quasi-metric spaces*, Adv. Th. Nonlinear Anal. Appl. 7(2) (2023) 455–471. doi: 10.31197/atnaa.1185449

[10] H. Aydi, M. Jellali, E. Karapinar, *On fixed point resultsfor a-implicit contractions in quasi-metric spaces and consequences}*, Nonlinear Anal. Model. Control, 21(1) (2016) 40–56.

[11] M. Jleli, B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appl. 2012:210, 2012.

[12] S. Park, *History of the metatheorem in ordered fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **62**(2)

[13] W. A. Kirk, *Contraction mappings and extensions*, Chapter 1, Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims, eds.), Kluwer Academic Publ. (2001) 1–34.

[14] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 10 (1968), 71–76.

[15] S. K. Chatterjea, *Fixed-point theorems*, C.R. Acad. Bulgare Sci. 25 (1972), 727–730.

[16] S. Willard, *General Topology*, Addison-Wesley, 1970.

[17] L. B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd) (N.S.) 12–26 (1971) 19–26.

[18] V. Berinde, *Approximating fixed points of weak contractions using the picard iteration*, Nonlinear Anal. Forum 9(1) (2004) 43–53.

[19] R. P. Pant, V. Rakočević, D. Gopal, A. Pant, M. Ram, *A general fixed point theorem*, Filomat 35:12 (2021) 4061–4072. doi: 10.2298/FIL2112061P

[20] T. Suzuki, *A generalized Banach contraction principle that generalizes metric completeness*, Proc. Amer. Math. Soc. 136(5) (2008) 1861–1869.

[21] N. Chandra, B. Joshi, M. C. Joshi, *Generalized fixed point theorems on metric spaces*, Mathematica Moravica 26(2) (2022) 85–101. doi: 10.5937/MatMor2202085C

[22] S. Park, *Remarks on the metatheorem in ordered fixed point theory*, Advanced Mathematical Analysis and Its Applications, Chapter 2 (Edited by P. Debnath, D.F.M. Torres, Y.J. Cho), CRC Press (2023) 11–27. doi: 10.1201/9781003388678-2

[23] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica 44 (1996) 381–391.

[24] S. Park, *A unified approach to fixed points of contractive maps*, J. Korean Math. Soc. 16 (1980) 95–105.

[25] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Students 63 (1994) 123–145.

[26] W. Oettli, M. Théra, *Equivalents of Ekeland's principle*, Bull. Austral. Math.Soc. 48 (1993) 385–392.

[27] S. Park, B.G. Kang, *Generalizations of the Ekeland type variational principles*, Chinese J. Math. 21 (1993) 313–325.

[28] S. Park, *On generalizations of the Ekeland type variational principles*, Nonlinear Anal. TMA 39 (2000) 881–889.

[29] T. Suzuki, *The strong Ekeland variational principle*, J. Math. Anal. Appl. 320 (2006) 787–794.

[30] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[31] D. R. Smart, *Fixed point theorems*, Cambridge Univ. Press, 1974.

[32] V. I. Istratescu, *Fixed Point Theory*, D. Reidel Pub. Co. 1981

[33] J. Dugundji, A. Granas, *Fixed Point Theory I*, Polish Scientific Publishers, Warszawa, 1982.

[34] K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Univ. Press 1990.

[35] S. Park, *Equivalents of ordered fixed point theorems of Kirk, Caristi, Nadler, Banach, and others*, Adv. Th. Nonlinear Anal. Appl. 6(4) (2022) 420–439.

[36] S. Park, *Equivalents of some ordered fixed point theorems*, J. Advances Math. Comp, Sci. 38(1) (2023) 52–67.

[37] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (basel) 23 (1972) 292–298.