Results in Nonlinear Analysis 7 (2024) No. 2, 79–88
https://doi.org/10.31838/rna/2024.07.02.008
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

On rings with involution and inner rings

Azzh Saad Alshehry[a], AL Hassan AL-Jarrah[b], Rashid Abu-Dawwas[c], Ala'a AL-Kateeb[d]

[a]Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; [b–d]Department of Mathematics, Yarmouk University, Irbid 21163, Jordan.

Abstract

In this article, we extend several known results from the ring of all $n \times n$ matrices with complex entries to any ring R with nonzero unity 1 and involution*. We introduce various results concerning Hermitian, skew-Hermitian, Unitary and Normal elements of R. Also, we propose two versions of the norm of an element and the orthogonality of two elements of R. Furthermore, we define an order on the elements of R and examine some properties. Finally, we establish the concept of inner rings and study some of its properties.

Key words and phrases. Rings with involution, Hermitian elements, Unitary elements, Normal elements.

Mathematics Subject Classification (2010). Primary 16W10, Secondary 16W99

1. Introduction

Let R be a ring with unity 1. A maping $* : R \to R$ is said to be an involution if the following hold for all $a, b \in R$:

(1) $a^{**} = a$.
(2) $(a + b)^* = a^* + b^*$.
(3) $(ab)^* = b^*a^*$.

Email addresses: asalshihry@pnu.edu.sa (Azzh Saad Alshehry); 2020105018@ses.yu.edu.jo (AL Hassan AL-Jarrah); rrashid@yu.edu.jo (Rashid Abu-Dawwas); alaa.kateeb@yu.edu.jo (Ala'a AL-Kateeb)

Received November 1, 2023; Accepted December 26, 2023; Online March 28, 2024

Example 1.1. [1] *Let* $R = M_n(\mathbb{R})$ *(the ring of all* $n \times n$ *matrices with real entries), and for* $A \in M_n(\mathbb{R})$, $A^* = A^t$ *(the transpose of A) is an involution on R.*

Example 1.2. [2] *Let* $R = \mathbb{C}$ *(the ring of all complex numbers), and for* $a+bi \in \mathbb{C}$, $(a + bi)^* = a + bi = a - bi$ *is an involution on R.*

Example 1.3. [3] *Let* $R = M_n(\mathbb{C})$ *(the ring of all* $n \times n$ *matrices with complex entries), and for* $A \in M_n(\mathbb{C})$, $A^* = \bar{A}^t$ *(the transpose of A, and take the conjugate to the elements of A) is an involution on R.*

Example 1.4. [4] *Let* $R = \mathbb{C}^n$, *and for* $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n,\ x^* = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}$ is an *involution on R.*

An element $a \in R$ is said to be Hermitian if $a^* = a$, skew-Hermitian if $a^* = -a$, unitary if $a^*a = aa^* = 1$ and normal if $a^*a = aa^*$. Clearly, Hermitian, skew-Hermitian and unitary elements are normal. However, the next example shows that the converse is not necessarily true:

Example 1.5. *Consider* $R = M_2(\mathbb{R})$ *with involution* $A^* = A^t$. *Choose* $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, *then A is normal, but A is not Hermitian, not skew-Hermitian and not unitary.*

For more details on rings with involution, see [1, 2, 3, 4, 5, 6, 7]. In this article, we extend several known results from the ring of all $n \times n$ matrices with complex entries to any ring R with nonzero unity 1 and involution $*$. We introduce various results concerning Hermitian, skew-Hermitian, Unitary and Normal elements of R. Also, we propose two versions of the norm of an element and the orthogonality of two elements of R. Furthermore, we define an order on the elements of R and examine some properties. Finally, we establish the concept of inner rings and study some of its properties.

2. Hermitian, skew-Hermitian, Unitary and Normal Elements

In this section, we introduce several results concerning Hermitian, skew-Hermitian, Unitary, and Normal elements of R. Let R be a ring with nonzero unity 1 and involution $*$. Define $d : R \times R \to R$ by $d(a, b) = b^*a$. We begin our results with introducing some basic properties of d.

Lemma 2.1. *Let R be a ring with involution* $*$ *and* $a, b, c \in R$. *Then*

(1) $d(a, a)$ *is Hermitian, and if R is a domain, then* $d(a, a) = 0$ *if andronly if* $a = 0$.
(2) $d(a + b, c) = d(a, c) + d(b, c)$.
(3) $d(a, b + c) = d(a, b) + d(a, c)$.
(4) $d(ac, b) = d(c, a^*b)$.
(5) $d(c, ab) = d(a^*c, b)$.
(6) $d(a, b) = d(b, a)^*$.
(7) $d(-a, b) = -d(a, b)$.
(8) $d(a, -b) = -d(a, b)$.
(9) $d(na, b) = n.d(a, b)$ *for any integer n.*
(10) $d(a, nb) = n.d(a, b)$ *for any integer n.*

Proof.

(1) $d(a, a)^* = (a^*a)^* = a^*a = d(a, a)$ that is $d(a, a)$ is Hermitian. Suppose $d(a, a) = 0$. Then $0 = a^*a$ and since R is a domain, either $a = 0$ or $a^* = 0$, but if $a^* = 0$, then $a = 0^* = 0$. The converse is clear.
(2) $d(a + b, c) = c^*(a + b) = c^*a + c^*b = d(a, c) + d(b, c)$.
(3) Similar to 2.
(4) $d(ac, b) = b^*(ac) = (b^*a)c = (a^*b)^*c = d(c, a^*b)$.
(5) Similar to 4.

(6) $d(a, b) = b^*a = (a^*b)^* = d(b, a)^*$.
(7) $d(-a, b) = b^*(-a) = -(b^*a) = -d(a, b)$.
(8) Similar to 6.
(9) Let n be an integer. If n is positive, then
$d(na, b) = d(a + ... + a, b) = d(a, b) + ... + d(a, b) = n.d(a, b)$. If n is negative, then $n = -m$ for some positive integer m, and then $d(na, b) = d(-ma, b) = -d(ma, b) = -m.d(a, b) = n.d(a, b)$. Also, if $n = 0$, then $d(na, b) = 0 = n.d(a, b)$.
(10) Similar to 8.

Theorem 2.2. *Let R be a ring with involution * and $a \in R$ be a Hermitian element. If $d(ab, b) = 0$ for all $b \in R$, then $a = 0$.*

Proof. Let $x, y \in R$. Then $d(a(x + y), x + y) - d(a(x - y), x - y) = 2d(ax, y) + 2d(ay, x)$, and then by assumption $2d(ax, y)+2d(ay, x) = 0$ it follows that $d(ax, y) = -d(ay, x)$ for all $x, y \in R$. Choose $y = ax$, then $d(ax, ax) = -d(a^2x, x) = -d(ax, ax)$ as a is Hermitian. So, $2d(ax, ax) = 0$ for all $x \in R$, and then $ax = 0$ for all $x \in R$ which implies that $a = 0$.

Theorem 2.3. *Let R be a ring with involution *. Then $a \in R$ is Unitary if and only if $d(ab, ac) = d(b, c)$ for all $b, c \in R$.*

Proof. Suppose that a is an Unitary element. Let $b, c \in R$. Then $d(ab, ac) = d(a^*ab, c) = d(1.b, c) = d(b, c)$. Conversely, choose $c = b$, then $0 = d(ab, ab) - d(b, b) = d(a^*ab, b) - d(b, b) = d(a^*ab - b, b) = d((a^*a - 1)b, b)$ for all $b \in R$. Since $(a^*a - 1)$ is Hermitian, by Theorem 2.2, $a^*a - 1 = 0$, i.e., a is Unitary.

Theorem 2.4. *Let R be a ring with involution ∗ and $a \in R$. Then a is a Normal element if and only if $d(ab, ab) = d(a^*b, a^*b)$ for all $b \in R$.*

Proof. Suppose that a is a Normal element. Then $d(ab, ab) = d(b, a^*ab) = d(b, aa^*b) = d(a^*b, a^*b)$ for all $b \in R$. Conversely, let $x = a^*a - aa^*$. Then x is Hermitian with $d(xb, b) = d((a^*a - aa^*)b, b) = d(a^*ab, b) - d(aa^*b, b) = d(ab, ab)\ d(a^*b, a^*b) = 0$ for all $b \in R$, and then by Theorem 2.2, $x = 0$, i.e., a is a Normal element.

Theorem 2.5. *Let R be a ring with involution ∗ and $a, b \in R$ are Unitary. Then ab is Unitary.*

Proof. Since a and b are Unitary, then $aa^* = 1 = a^*a$ and $bb^* = 1 = b^*b$, and then $(ab)^*(ab) = b^*a^*ab = b^*1b = b^*b = 1$, and $(ab)(ab)^* = abb^*a^* = a1a^* = aa^* = 1$. Thus, ab is Unitary.

Theorem 2.6. *Let R be a ring with involution ∗. Suppose that $2 \in R$ is not a zero divisor. Let $a \in R$ such that $a = b + c$ for some Hermitian element $b \in R$ and Skew-Hermitian element $c \in R$. Then a is a Normal element if and only if $bc = cb$.*

Proof. $a^*a = (b + c)^*(b + c) = (b^* + c^*)(b + c) = (b - c)(b + c) = b^2 + bc - cb - c^2$. Similarly, $aa^* = b^2 - bc + cb - c^2$. So, $a^*a - aa^* = 2(bc - cb)$. If a is Normal, then since 2 is not a zero divisor, $bc = cb$. The converse is clear.

Theorem 2.7. *Let R be a ring with involution ∗ and $a \in R$ be a skew-Hermitian element such that $a^2 - 1$ is a unit. Then $(a + 1)(a - 1)^{-1}$ is an Unitary element.*

Proof. Let $b = (a + 1)(a - 1)^{-1}$. Then $b^*b - 1 = (a^* - 1)^{-1}(a^* + 1)(a + 1)(a - 1)^{-1} - 1 = (a + 1)^{-1}(a - 1)(a + 1)(a - 1)^{-1} - 1 = (a + 1)^{-1}[(a - 1)(a + 1) -(a + 1)(a - 1)](a - 1)^{-1} = (a + 1)^{-1}[a^2 - 1 - (a^2 - 1)](a - 1)^{-1} = (a + 1)^{-1}\ [0]\ (a - 1)^{-1} = 0$. Hence, $b = (a + 1)(a - 1)^{-1}$ is Unitary.

Theorem 2.8. *Let R be a ring with involution ∗ and $a \in R$ such that $a = b^{-1}b^*$ for some unit $b \in R$. Then a is Unitary if andronly if b is Normal.*

Proof. Suppose that a is Unitary. Then $1 = a^*a = b(b^*)^{-1}(b^{-1}b^*) = b(bb^*)^{-1}b^*$, and then $b^{-1} = (bb^*)^{-1}b^*$ it follows that $(bb^*)b^{-1} = b^*$, and then $bb^* = b^*b$. Hence, b is Normal. Conversely, $a^*a = b(b^*)^{-1}b^{-1}b^* = b(bb^*)^{-1}b^* = b(b^*b)^{-1}b^* = bb^{-1}(b^*)^{-1}b^* = 1$. Hence, a is Unitary.

Theorem 2.9. *Let R be a domain with involution $*$ and $a \in R$ be an Unitary element. If $ab = nb$ for some integer n and nonzero $b \in R$, then $a = \pm 1$.*

Proof. Since a is Unitary, $d(b, b) = d(ab, ab) = d(nb, nb) = n^2d(b, b)$. Since $b \neq 0$ and R is a domain, $d(b, b) \neq 0$, and then $n^2 = 1$, i.e., $n = \pm 1$. Hence, $ab = \pm b$, and then $(a \mp 1)b = 0$ and as $b \neq 0$, $a = \pm 1$.

Theorem 2.10. *Let R be a domain with involution $*$ and $a \in R$ be a skew-Hermitian element. If $ab = nb$ for some integer n and nonzero $b \in R$, then $a = 0$.*

Proof. $n.d(b, b) = d(nb, b) = d(ab, b) = d(b, a^*b) = d(b, -ab) = d(b, -nb) = n.d(b, b)$. Since $b \neq 0$ and R is a domain, $d(b, b) \neq 0$ and then $n = -n$. It follows that $n = 0$, and hence $a = 0$.

Theorem 2.11. *Let R be a ring with involution $*$ and $a \in R$.*

(1) *If a is a Hermitian element, then $d(ab, b)$ is a Hermitian element for all $b \in R$.*
(2) *If $d(ab, b)$ is a Hermitian element for some unit $b \in R$, then a is a Hermitian element.*

Proof.

(1) Let $b \in R$. Then $d(ab, b) = d(b, a^*b) = d(b, ab) = d(ab, b)^*$, and hence $d(ab, b)$ is a Hermitian element.
(2) Since b is unit, b^* is unit with $(b^*)^{-1} = (b^{-1})^*$. Now, $b^*ab = d(ab, b) = d(ab, b)^* = d(b, ab) = (ab)^*b = b^*a^*b$, and then $a = a^*$, and hence a is a Hermitian element.

3. Norm and Orthogonality

In this section, we define the concept of the norm of an element and the concept of orthogonality between two elements in R, and finally, we define an order on the elements of R and study some properties.

Definition 3.1. *Let R be a ring with involution $*$. Let $a \in R$. Then the norm of a is denoted by $\|a\|_d$ and is defined by $\|a\|_d = d(a, a)$.*

Theorem 3.2. *Let R be a ring with involution $*$ and $a \in R$. Then the following hold:*

(1) *Suppose that $\|a\|_d$ is Hermitian and R is a domain. Then $\|a\|_d = 0$ if and only if $a = 0$.*
(2) *$\|na\|_d = n^2\|a\|_d$ for any integer n.*
(3) *$\|a\|_d = \|a^*\|_d$ if and only if a is Normal.*
(4) *If R is a domain, then $\|ab\|_d = \|a^*b\|_d$ for all $b \in R$ if and only if a is normal.*

Proof.

(1) The result holds from Lemma 2.1 (1).
(2) The result holds from Lemma 2.1 (9) and (10).
(3) $\|a\|_d = \|a^*\|_d$ if and only if $d(a, a) = d(a^*, a^*)$ if and only if $a^*a = aa^*$ if and only if a is Normal.
(4) The result holds from Theorem 2.4.

Theorem 3.3. *Let R be a ring with involution $*$ and $a, b \in R$. Then*

(1) $\|a + b\|_d + \|a - b\|_d = 2\|a\|_d + 2\|b\|_d$.
(2) $\|a + b\|_d - \|a - b\|_d = 2d(a, b) + 2d(b, a)$.

Proof. $\|a + b\|_d = d(a + b, a + b) = d(a, a) + d(a, b) + d(b, a) + d(b, b) = \|a\|_d + d(a, b) + d(b, a) + \|b\|_d$ and similarly, $\|a - b\|_d = \|a\|_d - d(a, b) - d(b, a) + \|b\|_d$. Then $\|a + b\|_d + \|a - b\|_d = 2\|a\|_d + 2\|b\|_d$ and $\|a + b\|_d - \|a - b\|_d = 2d(a, b) + 2d(b, a)$.

Theorem 3.4. *Let R be a ring with involution and $a \in R$. Then $\|a\|_d = 1$ if and only if a is Unitary.*

Proof. Suppose that $\|a\|_d = 1$. Then $d(a, a) = a^*a = 1$. Now $(a^*a)^* = 1^*$. Then $(a^*)^*a^* = aa^* = 1^* = 1$. Thus a is Unitary. Conversely, $a^*a = 1$. and $d(a, a) = a^*a = 1$. Thus $\|a\|_d = 1$.

Definition 3.5. *Let R be a ring with nonzero unity 1 and involution $*$. Let $a, b \in R$. Then a is said to be orthogonal to b if $d(a, b) = 0$. The set of all elements in R that are orthogonal to a is denoted by $a^\perp$ that is $a^\perp = \{b \in R : d(a, b) = 0\}$.*

Remark 3.6. *Clearly, 0 $a^\perp$, i.e., $a^\perp$ is a non-empty set. Also, if b $a^\perp$, then $d(a, b) = 0$ and then $d(b, a) = d(a, b)^* = 0^* = 0$.*

Theorem 3.7. *Let R be a ring with involution $*$, $a \in R$ and $b, c \in a^\perp$. Then $b + c \in a^\perp$ and $nb \in a^\perp$ for any integer n. In particular, $a^\perp$ is an additive subgroup of R.*

Proof. $d(a, b + c) = d(a, b) + d(a, c) = 0 + 0 = 0$ and then $b + c \in a^\perp$. Let n be an integer. Then $d(a, nb) = n.d(a, b) = n.0 = 0$, and then $nb \in a^\perp$.

Theorem 3.8. *Let R be a ring with involution $*$ and $a \in R$. Then $\|a + b\|_d = \|a - b\|_d$ for all $b \in a^\perp$.*

Proof. The result holds from Theorem 3.3 (2) and Remark 3.6.

Definition 3.9. *Let R be a ring with involution $*$ and $a, b \in R$. Then we write $a \le b$ if $a^*a = a^*b$.*

Theorem 3.10. *Let R be a domain with involution $*$ and $a, b \in R$ such that $a \le b$ and $b \le a$. Then $a = b$.*

Proof. Since $a \le b$, $a^*a = a^*b$ and since $b \le a$, $b^*b = b^*a$. Then $\|a - b\|_d = \|a\|_d - d(a, b) - d(b, a) + \|b\|_d = a\ a - b\ a - a\ b + b\ b = a\ b - b\ a - a\ b + b\ a = 0$ and then by Theorem 3.2 (1), $a - b = 0$, i.e., $a = b$.

Theorem 3.11. *Let R be a ring with involution $*$ and $a \in R$. Then*

(1) *$0 \le a$ for all $a \in R$.*
(2) *$a \le 1$ if and only if a is Hermitian and idempotent.*

Proof.

(1) Since $0^*0 = 0 = 0^*a$, $0 \le a$.
(2) Suppose that $a \le 1$. Then $a^*a = a^*.1 = a^*$, and then $a = (a^*)^* = (a^*a)^* = a^*a = a^*$. Hence, a is Hermitian, and then $a = a^* = a^*a = aa = a^2$. Hence, a is idempotent. Conversely, $a^*a = aa = a^2 = a = a^* = a^*.1$ that means $a \le 1$.

Theorem 3.12. *Let R be a ring with involution $*$ and $a, b \in R$ such that $a \le b$. Then $c^*ac \le c^*bc$ for all Unitary $c \in R$.*

Proof. Since $a \le b$, $a^*a = a^*b$. Let $c \in R$ be an Unitary element. Then $(c^*ac)^*(c^*ac) = c^*a^*c.c^*ac = c^*a^*ac = c^*a^*bc = c^*a^*cc^*bc = (c^*ac)^*(c^*bc)$ that means $c^*ac \le c^*bc$.

The next example shows that if $a \le b$, then it is not necessary to have $a^2 \le b^2$.

Example 3.13. *Consider $R = M_2(\mathbb{R})$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Since $A^*A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A^*B$, $A \le B$. However, $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and then $(A^2)^*A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.* On the other hand, $B^2 = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}$ and then $(A^2)^*B^2 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$. *Note that $(A^2)^*A^2 \neq (A^2)^*B^2$ that means $A^2 \not\le B^2$.*

Theorem 3.14. *Let R be a ring with involution * and $a, b \in R$ such that $a \le b$. If a is Hermitian, then $a^2 \le b^2$.*

Proof. Since $a \le b$, $a^*a = a^*b$ and since a is Hermitian, $a^2 = ab$. So, $(a^2)^*b^2 = a^2b^2 = a(ab)b = a(a^2)b = a^3b = a^2(ab) = a^2(a^2) = (a^2)^*a^2$ that means $a^2 \le b^2$.

4. Inner Rings

In this section, we establish the concept of inner rings and examine several properties.

Definition 4.1. *A ring R is said to be an inner ring if there exists a function* $\langle -, - \rangle : R \times R \to \mathbb{C}$ *satisfies the following:*

(1) $\langle a, a\rangle \ge 0$ *for all* $a \in R$, *and* $\langle a, a\rangle = 0$ *if and only if* $a = 0$.
(2) $\langle a + c, b\rangle = \langle a, c\rangle + \langle b, c\rangle$ *for all* $a, b, c \in R$.
(3) $\langle \alpha a, b\rangle = \underline{\alpha\langle a,} b\rangle$ *for all* $a, b \in R$, *and* $\alpha \in \mathrm{Z}$.
(4) $\langle a, b\rangle = \overline{\langle b, a\rangle}$ *for all* $a, b \in R$.

This function is called inner product on R. Moreover, R is said to be a real inner ring if $\langle -, - \rangle : R \times R \to \mathbb{R}$.

Example 4.2. *Consider* $R = M_n(\mathbb{C})$. *Then* $\langle A, B\rangle = tr(A^*B)$ *is an inner product on R, and so R is an inner ring.*

Example 4.3. *Consider* $R = \mathbb{C}^n$. *Then* $\langle a,b\rangle = \sum_{n=1}^{n} a_i\overline{b_i}$ *is an inner product on R, and so R is an inner ring.*

Example 4.4. *Let Rbe the ring of all continuous real-valued function on* $[a, b]$. *Then* $\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)$ *is an inner product on R, and so R is an inner ring.*

Example 4.5. *Let R be the ring of all continuous complex-valued functions on* $[a, b]$. *Then* $\langle f(x), g(x)\rangle = \int_a^b f(x)\overline{g(x)}$ *is an inner product on R, and so R is an inner ring.*

Remark 4.6. *Since* $\langle a,a\rangle = \overline{\langle a,a\rangle}$, *we have* $\langle a, a\rangle \in \mathrm{R}$ *and* $\langle a, a\rangle \ge 0$. *Thus, we can talk about* $\sqrt{\langle a,a\rangle}$.

Definition 4.7. *Let R be an inner ring and* $a \in R$. *Then the norm of a is defined as* $\|a\| = \sqrt{\langle a,a\rangle}$.

Theorem 4.8. *Let R be an inner ring,* $a, b \in R$ *and* $\alpha \in \mathrm{Z}$. *Then*

(1) $\|a\| \ge 0$, *and* $\|a\| = 0$ *if and only if* $a = 0$.
(2) $\|\alpha a\| = |\alpha|\,\|a\|$.

Proof.

(1) $\|a\| = \sqrt{\langle a,a\rangle} \ge 0$, and $\|a\| = 0$ if and only if $\sqrt{\langle a,a\rangle} = 0$ if and only if $\langle a, a\rangle = 0$ if and only if $a = 0$.
(2) $\|\alpha a\| = \sqrt{\langle \alpha a, \alpha a\rangle} = \sqrt{\alpha^2\langle a,a\rangle} = \sqrt{\alpha^2}\sqrt{\langle a,a\rangle} = |\alpha|\|a\|$.

Definition 4.9. *Let R be an inner ring. Then the distance between* $a, b \in R$ *is defined as* $D(a, b) = \|a - b\|$.

Theorem 4.10. *Let R be an inner ring and* $a, b \in R$. *Then*

(1) $D(a, b) \ge 0$, *and* $D(a, b) = 0$ *if and only if* $a = b$.
(2) $D(a, b) = D(b, a)$.
(3) $D(\alpha a, \alpha b) = |\alpha| D(a, b)$, *for all* $\alpha \in Z$.

Proof.

(1) $D(a, b) = \|a - b\| = \sqrt{\langle a-b, a-b\rangle} \geq 0$, and $D(a, b) = \|a - b\| = 0$. Then $\sqrt{\langle a-b, a-b\rangle} = 0$ if and only if $\langle a - b, a - b\rangle = 0$. So $a - b = 0$. Thus, $a = b$.
(2) $D(a, b) = \|a - b\| = \|-1(b - a)\| = |-1|\,\|(b - a)\| = \|(b - a)\| = D(b, a)$.
(3) $D(\alpha a, \alpha b) = \|\alpha a - \alpha b\| = \|\alpha(a - b)\| = |\alpha|\,\|a - b\| = |\alpha|\,D(a, b)$.

The next example shows that it is not necessarily $\|a + b\| = \|a\| + \|b\|$:

Example 4.11. *Let $R = \mathbb{R}^2$ with inner product as in Example 4.3, $a = (1, 2)$, $b = (3, 4) \in R$. Then $\|a\| = \sqrt{\langle a,a\rangle} = \sqrt{5}$, $\|b\| = \sqrt{\langle b,b\rangle} = \sqrt{5}$, and $\|a + b\| = \sqrt{\langle a+b, a+b\rangle} = \sqrt{52}$, and so $\sqrt{52} \neq 2\sqrt{5}$.*

In the next result, we state Cauchy-Schwartz inequality in inner rings:

Theorem 4.12. *Let R be an inner ring and $a, b \in R$. Then $|\langle a, b\rangle| \leq \|a\|\|b\|$.*

Proof. Let $t \in \mathbb{Z}$. Then $0 \leq \|ta + b\|^2 = \langle ta + b, ta + b\rangle = \langle ta, ta\rangle + \langle ta, b\rangle + \langle b, ta\rangle + \langle b, b\rangle = |t^2|\langle a, a\rangle + t\langle a, b\rangle + t\overline{\langle a,b\rangle} + \langle b, b\rangle = t^2\|a\|^2 + 2tRe(\langle a, b\rangle) + \|b\|^2 \leq t^2\|a\|^2 + 2t|\langle a, b\rangle| + \|b\|^2$. Then $0 \leq \|ta + b\|^2 \leq t^2\|a\|^2 + 2t|\langle a, b\rangle| + \|b\|^2$, and so $0 \leq t^2\|a\|^2 + 2t|\langle a, b\rangle| + \|b\|^2$. Then $(2|\langle a, b\rangle|)^2 - 4\|a\|2\|b\|^2 \leq 0$. So, $|\langle a, b\rangle|^2 - \|a\|^2\|b\|^2 \leq 0$. Thus, $|\langle a, b\rangle| \leq \|a\|\|b\|$.

Remark 4.13. *Clearly, the equality in Cauchy-Schwartz inequality in inner rings occurs if and only if $b = \alpha a$, for some $\alpha \in$ Z.*

Now, we are ready to state the triangle inequality in inner rings:

Theorem 4.14. *let R be an inner ring and $a, b \in R$. Then $\|a + b\| \leq \|a\| + \|b\|$.*

Proof. let $a, b \in R$. Then $\|a + b\|^2 = \langle a + b, a + b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle = \langle a, a\rangle + \langle a, b\rangle + \overline{\langle a, b\rangle} + \langle b, b\rangle = \|a\|^2 + 2Re(\langle a, b\rangle) + \|b\|^2 \leq \|a\|^2 + 2|\langle a, b\rangle| + \|b\|^2$. By Cauchy-Schwartz Inequality for Inner Ring, we have $\|a\|^2 + 2|\langle a, b\rangle| + \|b\|^2 \leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 = (\|a\| + \|b\|)^2$. Thus $\|a + b\| \leq \|a\| + \|b\|$.

Theorem 4.15. *let R be an inner ring and $a, b, z \in R$. Then $D(a, b) \leq D(a, z) + D(z, b)$, for all $a, b, z \in R$.*

Proof. Let $a, b, z \in R$. Then $\|a - b\| = \|(a - z) + (z - b)\| \leq \|(a - z)\| + \|(z - b)\| = D(a, z) + D(z, b)$.

Definition 4.16. *Let R be a real inner ring and $a \neq 0$, $b \neq 0 \in R$. Then the angle between a and b is $\theta \in [0, \pi]$ such that $\theta = cos^{-1}\left(\frac{\langle a,b\rangle}{\|a\|\|b\|}\right)$.*

Example 4.17. *Let R be the ring of all continuous real-valued functions on $[a, b]$ with inner product as in Example 4.4 and let $f(x) = x^2$, $g(x) = x^4 \in R$. Then $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x) = \frac{1}{7}$, $\|f(x)\| = \sqrt{\langle f(x), f(x)\rangle} = \int_0^1 f(x)^2 = \frac{1}{\sqrt{5}}$ and $\|g(x)\| = \sqrt{\langle g(x), g(x)\rangle} = \int_0^1 g(x)^2 = \frac{1}{3}$. Then $\theta = cos^{-1}\left(\frac{3\sqrt{5}}{7}\right)$ is the angle between $f(x)$ and $g(x)$.*

Definition 4.18. *let R be an inner ring. Then $a, b \in R$ are said to be orthogonal if $\langle a, b\rangle = 0$.*

Theorem 4.19. *Let R be an inner ring. Then the zero element is the only element that is orthogonal to every element in R.*

Proof. Let $a \in R$. Then $\langle 0, a\rangle = \langle 0 + 0, a\rangle = \langle 0, a\rangle + \langle 0, a\rangle$. So, $\langle 0, a\rangle = \langle 0, a\rangle - \langle 0, a\rangle = 0$. Thus, 0 is orthogonal to every element in R. Let $a \in R$ such that a is orthogonal to every element in R. Then $\langle a, b\rangle = 0$ for all $b \in R$. Now choose $a = b$, then $\langle a, a\rangle = 0$. Thus, $\|a\|^2 = 0$. and So, $a = 0$.

Now, we investigate the concept of inner rings with the involution:

Definition 4.20. *Let R be an inner ring with involution* $*$. *Then* $a \in R$ *is said to be* $*$*-inner element if* $\langle ax, y\rangle = \langle x, a^*y\rangle$, *for all* $x, y \in R$.

Example 4.21. *Consider* $R = \mathbb{C}^2$ *with involution* $*$ *as in Example 1.4. And let* $a = \begin{pmatrix}1\\1\end{pmatrix}$. *Then* $a^* = \begin{pmatrix}1\\1\end{pmatrix}$, *and so let* $x, y \in \mathbb{C}^2$. *Then* $\left\langle \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\right\rangle$. *Thus a is* $*$*-inner element.*

Example 4.22. *Consider* $R = \mathbb{C}^2$ *with involution* $*$ *as in Example 1.4. And let* $a = \begin{pmatrix}1\\-1\end{pmatrix}$. *Then* $a^* = \begin{pmatrix}1\\-1\end{pmatrix}$, *and so let* $x, y \in \mathbb{C}^2$. *Then* $\left\langle \begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right\rangle = x_1\overline{y_1} - x_2\overline{y_2}$ *and* $\left\langle \begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\right\rangle = x_1\overline{y_1} - x_2\overline{y_2}$. *Thus a is* $*$*-inner element.*

Theorem 4.23. *Let R be an inner ring with involution* $*$ *and* $a \in R$ *be Hermitian* $*$*-inner element. If* $\langle ax, x\rangle = 0$, *for all* $x \in R$, *then* $a = 0$.

Proof. Let $a, x, y \in R$. Then $\langle a(x + y), x + y\rangle - \langle a(x - y), x - y\rangle = 2\langle ax, y\rangle + 2\langle ay, x\rangle$, and then by assumption $2\langle ax, y\rangle + 2\langle ay, x\rangle = 0$, it follows that $2\langle ax, y\rangle = 2\langle ay, x\rangle$ for all $x, y \in R$. Choose $y = ax$, then $\langle ax, ax\rangle = -\langle a^2x, x\rangle$ as a is Hermitian. So, $2\langle ax, ax\rangle = 0$, for all $x \in R$, and then $ax = 0$, for all $x \in R$. Choose $x = 1$. Thus, $a = 0$.

Theorem 4.24. *Let R be an inner ring with involution* $*$ *and* $a \in R$ *be* $*$*-inner element. Then if* $\langle ax, x\rangle$ *is real number for some* $x \in R$, *then* $\langle (a - a^*)x, x\rangle = 0$.

Proof. Let $\langle ax, x\rangle$ be real number for some $x \in R$. Then $\langle ax, x\rangle = \overline{\langle ax, x\rangle} = \langle x, ax\rangle = \langle a^*x, x\rangle$, then $0 = \langle ax, x\rangle - \langle a^*x, x\rangle = \langle ax - a^*x, x\rangle = \langle (a - a^*)x, x\rangle$. Thus, $\langle (a - a^*)x, x\rangle = 0$.

Theorem 4.25. *Let R be an inner ring with involution* $*$ *and* $a \in R$ *be* $*$*-inner element. Then if a is Hermitian, then* $\langle ax, x\rangle$ *is real number, for all* $x \in R$.

Proof. Let $a \in R$ be Hermitian $*$-inner element. Then $\langle ax, x\rangle = \langle x, ax\rangle = \overline{\langle ax, x\rangle}$. Thus $\langle ax, x\rangle$ is real number, for all $x \in R$.

Definition 4.26. *Let R be an inner ring, let* $a \in R$ *be Hermitian* $*$*-inner element. Then*

(1) *a is called positive semi definite element if* $\langle ax, x\rangle \geq 0$, *for all* $x \in R$.
(2) *a is called positive definite element if* $\langle ax, x\rangle > 0$, *for all* $x \in R - \{0\}$.

Clearly, every positive definite element is positive semi definite element. The next example shows that a positive semi definite element is not necessarily positive definite element:

Example 4.27. *Let* $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ *with involution* $*$ *as in Example 1.4. Now,* $a = \begin{pmatrix}1\\1\end{pmatrix}$ *is positive semi definite element since for all* $x = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{Z}_4 \times \mathbb{Z}_4$, $\langle ax, x\rangle = \left\langle \begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}x_1\\x_2\end{pmatrix}\right\rangle = (x_1)\overline{x_1} + (x_2)\overline{x_2} = |x_1|^2 + |x_2|^2 \geq 0$. *But a is not positive definite element since for* $x = \begin{pmatrix}2\\2\end{pmatrix} \in \mathbb{Z}_4 \times \mathbb{Z}_4$, $\langle ax, x\rangle = 0$.

Clearly, every positive semi definite element is Hermitian element. The next example shows that a Hermitian element is not necessarily positive semi definite element:

Example 4.28. *Let $R = \mathbb{C}^2$ with involution $*$ as in Example 1.4. Now* $a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *then* $a^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$*. Thus a is Hermitian. But, a is not positive element, Since for* $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$, $\langle ax, x \rangle = \langle \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle = -1$.

Theorem 4.29. *Let R be an inner ring. If a and $b \in R$ are positive semi definite element (positive definite element), then $a + b$ is positive semi definite element (positive definite element).*

Proof. Since a and b are positive semi definite elements, then a and b are Hermitian, and then $a + b$ is Hermitian. Now let $x \in R$. Then $\langle (a + b)x, x \rangle = \langle ax, x \rangle + \langle bx, x \rangle$. Now we have $\langle ax, x \rangle \geq 0$ and $\langle bx, x \rangle \geq 0$. Thus, $a + b$ is positive semi definite element. Now for positive definite element since a and b are positive definite elements, then a and b are Hermitian, and then $a + b$ is Hermitian. Now let $x \in R$. Then $\langle (a + b)x, x \rangle = \langle ax, x \rangle + \langle bx, x \rangle$. Now we have $\langle ax, x \rangle > 0$ and $\langle bx, x \rangle > 0$. Thus, $a + b$ is positive definite element.

Theorem 4.30. *Let R be an inner ring with involution $*$ and $a \in R$ be $*$-inner element. Then the following are equivalent:*

(1) *a is Unitary.*
(2) $\langle ax, ay \rangle = \langle x, y \rangle$, *any all $x, y \in R$.*
(3) $\|ax\| = \|x\|$, *for any $x \in R$.*

Proof. $(1 \Rightarrow 2)$ Let $x, y \in R$. Then $\langle ax, ay \rangle = \langle x, a^*ay \rangle = \langle x, 1y \rangle = \langle x, y \rangle$.
$(2 \Rightarrow 3)$ Let $x \in R$. Then $\|ax\|^2 = \langle ax, ax \rangle = \langle x, x \rangle = \|x\|^2$ by (2).
$(3 \Rightarrow 1)$ For $x \in R$, $\langle ax, ax \rangle = \|ax\|^2 = \|x\|^2 = \langle x, x \rangle$, and then $0 = \langle ax, ax \rangle - \langle x, x \rangle = \langle a^*ax, x \rangle - \langle x, x \rangle = \langle (a^*a - 1)x, x \rangle = 0$. Then $(a^*a - 1)$ is Hermitian, and then by Theorem 4.23, $a^*a - 1 = 0$, then $a^*a = 1$, i.e., a is Unitary.

Theorem 4.31. *Let R be an inner ring with involution $*$ and $a \in R$ be $*$-inner element. Then a is Normal if and only if $\|ax\| = \|a^*x\|$, for all $x \in R$.*

Proof. Suppose that a is Normal. Let $x \in R$. Then $\|ax\|^2 = \langle ax, ax \rangle = \langle a^*ax, x \rangle = \langle a^*x, a^*x \rangle = \|a^*x\|^2$. Thus, $\|ax\| = \|a^*x\|$. Conversely, for all $x \in R$, $\langle ax, ax \rangle = \|ax\|^2 = \|a^*x\|^2 = \langle a^*x, a^*x \rangle$, then $0 = \langle ax, ax \rangle - \langle a^*x, a^*x \rangle = \langle a^*ax, x \rangle - \langle aa^*x, x \rangle = \langle a^*ax - aa^*x, x \rangle = \langle (a^*a - aa^*)x, x \rangle$. Then $(a^*a - aa^*)$ is Hermitian, and then by Theorem 4.23 $a^*a - aa^* = 0$, then $a^*a = aa^*$, i.e., a is Normal.

Conclusion

In this article, we extended several known results from the ring of all $n \times n$ matrices with complex entries to any ring R with nonzero unity 1 and involution $*$. We introduced various results concerning Hermitian, skew-Hermitian, Unitary and Normal elements of R. Also, we proposed two versions of the norm of an element and the orthogonality of two elements of R. Furthermore, we defined an order on the elements of R and examine some properties. Finally, we established the concept of inner rings and study some of its properties. As a proposal for future work, we are going to establish various inequalities on the norm of elements of R under specific conditions. In fact, we will investigate the two proposed versions of the norm, and we will examine which one will be more suitable.

Acknowledgment

This work was supported by Yarmouk University. The authors would like to thank the referees for the helpful comments and suggestions.

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