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Motion stability study of double and ball pendulums

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Abstract

This research examines the stability of periodic motion for a physics application, which results in a second order differential equation for systems, such as the double and spherical pendulums. The stability of the equilibrium modes is analysed using the Libanov and Getayer methods, along with the principle of energy conservation. Moreover, this study describes the periodic motion and explains the phase-level solution paths and the stability conditions for the double and spherical pendulums by using the MATLAB program.

Key words: Pendulum, stability, motion, spherical and Lyapunov's theory.

1. Introduction

In real-world scenarios, numerous physical phenomena are controlled by second order differential equations and periodic motion, such as in the motion of a simple pendulum. The simple pendulum is a special case of the spherical pendulum. The pendulum has been a subject for the discovery of differential, and its invention is credited to Galileo. The study of pendulums was later continued by Christian Huygens. Huygens invented the clock with a pendulum, based on Galileo's research. However, scientist Ibn Yuns-Al Mary, who passed away in 1009 AD, had also invented the pendulum and used it to measure time with precision and calculate time periods during monitoring. The dynamism of the pendulum has broadened its scope to encompass modem technology, leading to the discovery that certain chemical systems exhibit behaviours similar to a pendulum. This expansion is progressively growing to include organic psychological forecasting and economic medicine.

$$\theta'' + \frac{g}{\ell}\sin\theta = 0 \;\;\&\; \theta(o) = X_o : \theta'(o) = V_o, \tag{1}$$

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where θ is the angle that the pendulum makes with the plumb.

Equation (1) has additional conditions that are not expressed within the equation. In mechanics, these conditions pertain to the initial position (X_o) at time $t = o$ and the initial velocity (V_o) at time $t = o$ for a moving body. Certain characteristics of the solution can be established by incorporating these initial conditions to the differential equation. This process is referred to as the initial value issue and serves as a model for physical problems, with time t being the independent variable [1].

The significance of the field of mechanics is widely acknowledged, not only in our current era but also throughout history. Mechanics, referred to as the 'science of tricks' by the Arabs, is a fundamental component of our civilization's advancement. This branch of natural science centers around the study of body movement, with a focus on the transition from stillness to motion. Amongst these contributions to mechanics are the renowned three laws of motion and Newton's laws, which continue to be one of the most significant scientific achievements to this day.

Numerous types of movements exist, one of which is referred to as periodic movement. Examples of periodic movement include the swinging of a pendulum, the earth's rotation on its axis, and the vibration of an object at the end of a spring [2].

Stability is an increasingly significant concept in modern engineering mathematics. The concept of stability originated from physics. Accordingly, numerous studies emerged concerning periodic motion and the various types of pendulum motion, along with the exploration of their stability.

Chinnery and Hall [3] studied the stability of the periodic motion of a solid body attached to the tip of a spring in its waning state using the Laypunov–Schmidt method.

Corinaldesi [4] discovered the movement of the spherical pendulum and demonstrated that the equation that describes this movement is in the following form:

$$\ddot{\theta} = \frac{-g}{\ell}\sin\theta + \dot{\Psi}^2\sin\theta\cos\theta,$$

$$\ddot{\Psi} = -2\dot{\theta}\dot{\Psi}\cot\theta.$$

Winter [5] illustrated the periodic motion of a simple pendulum by using the MATHLAB program and obtained a numerical solution by using the fourth-order Range–Kutta method.

In 2005, Caiado and Sarycher demonstrated the instability of the equilibrium position of the double inverted pendula by applying the principle of linear approximation. Simmons and Krantz [2] presented an equation that describes the periodic movement of a pendulum connected by a spring, which takes the following form:

$$m\ddot{X}_1 = \frac{-mg}{\ell}X_1 - K\left(X_1 - X_2\right),$$

$$m\ddot{X}_2 = \frac{-mg}{\ell}X_2 - K\left(X_1 - X_2\right).$$

Thus, research in this field continued to advance.

2. Double Pendulum

Double pendulum is a system consisting of two pendulums, with one suspended from the other. The lengths of the two pendulums, denoted as ℓ_1 and ℓ_2, are connected by two masses: m_1 for the first pendulum and m_2 for the second pendulum. The pendulum makes two perpendicular angles, namely, θ_1 and θ_2 [6]. This concept is illustrated in Figure (1).

Where

$$\left.\begin{array}{l} x_1 = \ell_1\sin\theta_1 ; \dot{x}_1 = \ell_1\dot{\theta}_1\cos\theta_1 \\ x_2 = \ell_1\sin\theta_1 + \ell_2\sin\theta_2 ; \dot{x}_2 = \ell_1\dot{\theta}_1\cos\theta_1 + \ell_2\dot{\theta}_2\cos\theta_2 \\ y_1 = -\ell_1\cos\theta_1 ; \dot{y}_1 = \ell_1\dot{\theta}_1\sin\theta_1 \\ y_2 = -\ell_2\cos\theta_1 - \ell_2\cos\theta_2 ; \dot{y}_2 = \ell_1\dot{\theta}_1\sin\theta_1 + \ell_2\dot{\theta}_2\sin\theta_2 \end{array}\right\}.$$

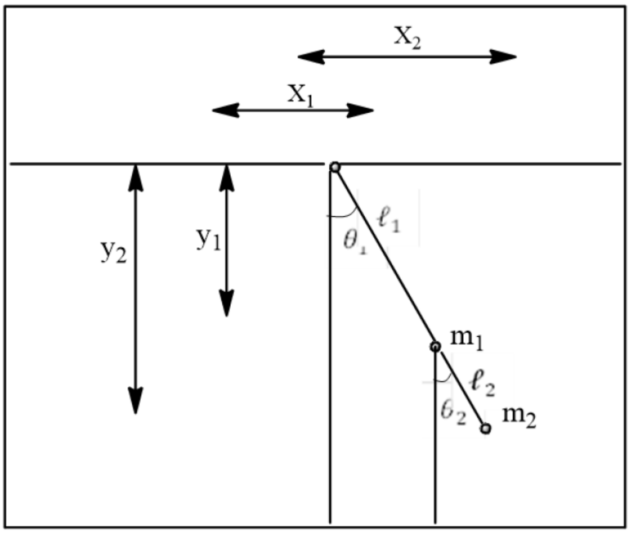

Figure 1. Double pendulum

The kinetic energy resulting from the motion of mass m_1 is expressed as follows:

$$T_1 = \frac{1}{2} m\ell_1^2\dot{\theta}_1^2. \tag{2}$$

The kinetic energy resulting from the motion of mass m_2 is expressed as follows:

$$T_2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right). \tag{3}$$

The potential energy of the system is determined according to Equations (5) and (6):

$$T = T_1 + T_2,$$

such that:

$$T = \frac{1}{2} m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2} m_2 \left(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right. \tag{4}$$

The potential energy stored in mass m_1 is expressed as follows:

$$E_1 = m_1 g y_1. \tag{5}$$

The potential energy stored in mass m_2 is expressed as follows:

$$E_2 = m_2 g y_2. \tag{6}$$

The potential energy of the system is determined by using Equations (5) and (6):

$$E = E_1 + E_2,$$

such that:

$$E = -\left(m_1 + m_2\right) g\ell_1\cos\theta_1 - m_2 g\ell_2\cos\theta_2.$$

Lagrangian equations are used to obtain the system of motion of the double pendulum:

$$\begin{aligned} &L = T - E \\ &L = \frac{1}{2}\left(m_1 + m_2\right)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2} m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) \\ &\quad + \left(m_1 + m_2\right) g\ell_1\cos\theta_1 + m_2\ell_2\cos\theta_2. \end{aligned}$$

Using Euler–Lagrange equation for θ_1 yields:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1}-\frac{\partial L}{\partial \theta_1}=0. \tag{8}$$

Given that:

$$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}_1}&=m_1\ell_1^2\dot{\theta}_1+m_2\ell_1^2\dot{\theta}_1+m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1-\theta_2),\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1}&=(m_1+m_2)\ell_1^2\ddot{\theta}_1+m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2\ell_1\ell_2\dot{\theta}_2\Big)\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2),\\ \frac{\partial L}{\partial \theta_1}&=-\ell g(m_1+m_2)\sin\theta_1-m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2),\end{aligned} \tag{9}$$

substituting Equations (9) and (10) into Equation (8) yields:

$$\begin{aligned}(m_1+m_2)\ell_1^2\ddot{\theta}_1+m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2\ell_1\ell_2\dot{\theta}_2\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\\ +\ell_1 g(m_1+m_2)\sin\theta_1+m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)=0.\end{aligned} \tag{11}$$

Euler–Lagrange Equation is used to determine θ_2:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2}-\frac{\partial L}{\partial \theta_2}=0. \tag{12}$$

Considering that:

$$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}_2}&=m_2\ell_2^2\dot{\theta}_2+m_2\ell_1\ell_2\dot{\theta}_1\cos(\theta_1-\theta_2),\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2}&=m_2\ell_2^2\ddot{\theta}_2+m_2\ell_1\ell_2\ddot{\theta}\cos(\theta_1-\theta_2)-m_2\ell_1\ell_2\dot{\theta}_1\sin(\theta_1-\dot{\theta}_2)(\dot{\theta}_1-\dot{\theta}_2)\ldots,\end{aligned} \tag{13}$$

$$\frac{\partial L}{\partial \theta_2}=m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)-\ell_2 m_2 g\sin\theta_2, \tag{14}$$

substituting Equations (13) and (14) into Equation (12) yields:

$$\begin{aligned}m_2\ell_2^2\ddot{\theta}_2+m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2\ell_1\ell_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2),\\ -m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)+\ell_2 m_2 g\sin\theta_2=0.\end{aligned}$$

After dividing both sides by ℓ_2 and making simplifications, the following expression is obtained:

$$m_2\ell_2\ddot{\theta}_2+m_2\ell_1\cos(\theta_1-\theta_2)\ddot{\theta}_1-m_2\ell_1\sin(\theta_1-\theta_2)\dot{\theta}_1^2+m_2 g\sin\theta_2=0. \tag{15}$$

The equation of motion $\ddot{\theta}_1$ is determined by simultaneously solving Equations (11) and (15). To achieve this task, Equation (15) is multiplied by $\cos(\theta_1-\theta_2)$, resulting in Equations (11) and (15) being expressed in the following form:

$$\begin{aligned}&(m_1+m_2)\ell_1\ddot{\theta}_1+m_2\ell_2\cos(\theta_1-\theta_2)\ddot{\theta}_2+m_2\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2+g(m_1+m_2)\sin\theta_1=0\\ &-m_2\ell_1\cos^2(\theta_1-\theta_2)\ddot{\theta}_1-m_2\ell_2\cos(\theta_1-\theta_2)\ddot{\theta}_2+m_2\ell_1\cos(\theta_1-\theta_2)\sin(\theta_1-\theta_2)\dot{\theta}_1^2\\ &-m_2 g\sin\theta_2\cos(\theta_1-\theta_2)=0\end{aligned}$$

The equation of motion force $\ddot{\theta}_1$ is obtained in the following form by simplifying the two equations:

$$\ddot{\theta}_1 = \frac{\left[-m_2\ell_2\sin\left(\theta_1-\theta_2\right)\dot{\theta}_2^2 - g\left(m_1+m_2\right)\sin\theta_1 + m_2 g\sin\theta_2\cos(\theta_1-\theta_2\right]}{\ell_1\left(m_1+m_2\left(\sin^2\left(\theta_1-\theta_2\right)\right)\right)}. \tag{16}$$

To obtain the equation of motion, Equations (11) $\left(m_2\cos\left(\theta_1-\theta_2\right)\right)$ and (15) are multiplied by $\left(-\left(m_1+m_2\right)\right)$. Accordingly, the two equations become:

$$\left(m_1+m_2\right)\ell_1\ddot{\theta}_1\cos\Big(\theta_1-\theta_2) + m_2\ell_2\cos^2\left(\theta_1-\theta_2\right)\ddot{\theta}_2 + m_2\ell_2\cos(\theta_1-\theta_2)\sin(\theta_1-\theta_2)\dot{\theta}_2^2$$
$$+\left(m_1+m_2\right)g\sin\theta_1\cos\left(\theta_1-\theta_2\right)=0$$
$$-\left(m_1+m_2\right)\ell_1\cos\left(\theta_1-\theta_2\right)\ddot{\theta}_1 - \left(m_1+m_2\right)\ell_2\ddot{\theta}_2 + \left(m_1+m_2\right)\ell_1\sin\left(\theta_1-\theta_2\right)\dot{\theta}_1^2$$
$$-(m_1+m_2)g\sin\theta_2=0.$$

The equation of motion $\ddot{\theta}_2$ is obtained in the following form by simplifying the two equations:

$$\ddot{\theta}_2 = \frac{\left(m_1+m_2\right)\left[\frac{m_2\ell_2\cos\left(\theta_1-\theta_2\right)\sin\left(\theta_1-\theta_2\right)\dot{\theta}_2^2}{m_1+m_2} + g\sin\theta_1\cos\left(\theta_1-\theta_2\right) - g\sin\theta_2 + \ell_2\sin\left(\theta_1-\theta_2\right)\dot{\theta}_2^2\right]}{\ell_2\left(m_1+m_2\sin^2\left(\theta_1-\theta_2\right)\right)}. \tag{17}$$

Equations (16) and (17) represent the double pendulum motion system.

3-Studying the stability of the motion of the double pendulum using Libanov's theory

3. Theoretical Aspect

In the case of the double pendulum, a method called the direct Lipanov method or the second Lipanov method was developed by Laypunov to study the stability of the solutions to nonlinear differential equations without directly solving the problem. Libanov's first stability theorem states that [7]:

Consider the following differential system:

$$\frac{dx_i}{dt} = f_i\left(t, x_1, x_2, \ldots, x_n\right);\ i = 1,2,\ldots,n.$$

Let $V\left(x_1, x_2, \ldots, x_n\right)$ be a function, referred to as the Libanov function, and $x_i \equiv 0$ a fixed equilibrium point that satisfies the following conditions near the origin:

The first condition is:

$$V\left(x_1, x_2, \ldots, x_n\right) \geq 0 \text{ and } V = 0.$$

Only when $x = 0$, function V has a strictly defined minimum limit at the origin.

The second condition is:

$$\frac{dv}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_1} f_i\left(t, x_1, x_2, \ldots, x_n\right) \leq 0; t \geq t_0.$$

Then, equilibrium point $x_i \equiv 0$ is stable:

This theory represents a generalization of the physical principle that states that:In energy-conserving physical systems, a particle is considered to be in a stable position when its potential energy is reaches its minimum value. The research results were obtained on the basis of this theory.

In the case of the double pendulum, our initial goal is to convert the system of motion from a second-order system to a differential system by using the following hypotheses:

$$y_1=\theta_1;\; y_2=\theta_2; y_3=\dot{\theta}_1;\; y_4=\dot{\theta}_2.$$

The equivalent system is expressed in the following form:

$$\dot{y}_1=y_3,$$
$$\dot{y}_2=y_4,$$
$$\dot{y}_3=\frac{\begin{bmatrix}-m_2\ell_2\sin\left(y_1-y_2\right)y_4^2-g\left(m_1+m_2\right)\sin y_1+m_2 g\sin y_2\cos(y_1-y_2)\\ -m_2\ell_1\cos\left(y_1-y_2\right)\sin(y_1-y_2)y_3^2\end{bmatrix}}{\ell_1(m_1+m_2\sin^2\left(y_1-y_2\right))},$$
$$\dot{y}_4=\frac{\left(m_1+m_2\right)\begin{bmatrix}\dfrac{m_2\ell_2\cos\left(y_1-y_2\right)\sin\left(y_1-y_2\right)y_4^2}{m_1+m_2}+g\sin y_1\cos\left(y_1+y_2\right)\\ -g\sin y_2+\ell_1\cos\left(y_1-y_2\right)y_3^2\end{bmatrix}}{\ell_2(m_1+m_2\sin^2\left(y_1-y_2\right))}.$$

The total energy $V\left(y_1,y_2 y_3,y_4\right)$ is determined by utilising Equations (4) and (7) for the kinetic and potential energy, respectively. The definite Libanov function is also considered to be positive:

$$V\left(y_1,y_2,y_3,y_4\right)=\frac{1}{2}\left(m_1+m_2\right)\ell_1^2 y_3^2+\frac{1}{2}m_2\ell_2^2 y_4^2+m_2\ell_1\ell_2 y_3 y_4\cos\left(y_1-y_2\right)$$
$$-\left(m_1+m_2\right)g\ell_1\cos y_1-m_2 g\ell_1\cos y_2.$$

Applying the chain rule, we find:

$$\dot{V}=\frac{\partial v}{\partial y_1}\dot{y}_1+\frac{\partial v}{\partial y_2}\dot{y}_2+\frac{\partial v}{\partial y_3}\dot{y}_3+\frac{\partial v}{\partial y_2}\dot{y}_4,$$

such that:

$$\dot{V}=-\mathrm{m}_2\ell_1\ell_2 y_3^2 y_4\sin\left(y_1-y_2\right)+\left(m_1+m_2\right)g\ell_1 y_3 sin y_1+m_2\ell_1\ell_2 y_3 y_4^2\sin\left(y_1-y_2\right)$$
$$+m_2 g\ell_2 y_4\sin y_2$$
$$-\frac{m_2\left(m_1+m_2\right)\ell_1^2\ell_2 y_3 y_4^2\sin\left(y_1-y_2\right)}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}-\frac{g\left(m_1+m_2\right)^2\ell_1^2 y_3\sin y_1}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}$$
$$+\frac{m_2\left(m_1+m_2\right)g\ell_1^2 y_2\sin y_2\cos\left(y_1-y_2\right)}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}-\frac{m_2\left(m_1+m_2\right)\ell_1^3 y_3^3\cos\left(y_1-y_2\right)\sin\left(y_1-y_2\right)}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}$$
$$-\frac{m_2^2\ell_1\ell_2^2 y_4\cos\left(y_1-y_2\right)\sin\left(y_1-y_2\right)}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}-\frac{m_2\left(m_1+m_2\right)\ell_1\ell_2 g y_4\sin y_1\cos\left(y_1-y_2\right)}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}$$
$$+\frac{m_2^2\ell_1\ell_2 g\sin y_2\cos^2\left(y_1-y_2\right)y_4}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}-\frac{m_2^2\ell_1^2\ell_2 y_3^2 y_1\cos^2\left(y_1-y_2\right)\sin\left(y_1-y_2\right)}{\ell_1\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}$$
$$+\frac{m_2^2\ell_2^2 y_4^3\cos\left(y_1-y_2\right)\sin\left(y_1-y_2\right)}{\ell_2\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}+\frac{m_2\left(m_1+m_2\right)\ell_2^2 g y_4\sin y_1\cos(y_1-y_2)}{\ell_2\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}$$
$$-\frac{m_2\left(m_1+m_2\right)\ell_2^2 g y_4\sin y_2}{\ell_2\left(m_1+m_2\sin\left(y_1-y_2\right)\right)}+\frac{m_2\ell_1\ell_2 y_3^2 y_2\sin\left(y_1-y_2\right)\left(m_1+m_2\right)}{\ell_2\left(m_1+m_2\sin^2\left(y_1-y_2\right)\right)}$$

$$+\frac{m_2^2\ell_1\ell_2^2y_3y_4\cos^2(y_1-y_2)\sin(y_1-y_2)}{\ell_2\left(m_1+m_2\sin^2(y_1-y_2)\right)}+\frac{m_2(m_1+m_2)\ell_1\ell_2gy_3\sin y_1\cos^2(y_1-y_2)}{\ell\left(m_1+m_2\sin^2(y_1-y_2)\right)}$$
$$-\frac{m_2(m_1+m_2)\ell_1\ell_2gy_3\sin y_2\cos(y_1-y_2)}{\ell_2\left(m_1+m_2\sin^2(y_1-y_2)\right)}+\frac{m_2(m_1+m_2)\ell_1^2\ell_2y_3^3\cos(y_1-y_2)\sin(y_1-y_2)}{\ell_2\left(m_1+m_2\sin^2(y_1-y_2)\right)}.$$

Specifically:

$$\begin{aligned}\dot{V} = &-m_2\ell_1\ell_2y_3^2y_4\sin(y_1-y_2)+(m_1+m_2)g\ell_1y_3\sin y_1+m_2\ell_1\ell_2y_3y_4^2\sin(y_1-y_2)\\ &+m_2g\ell_2y_4\sin y_4\\ &-\frac{m_2(m_1+m_2)\ell_1\ell_2y_3y_4^2\sin(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}-\frac{g(m_1+m_2)^2\ell_1y_3\sin y_1}{m_1+m_2\sin^2(y_1-y_2)}\\ &+\frac{m_2^2\ell_2gy_4\sin y_2\cos^2(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}-\frac{m_2^2\ell_1\ell_2gy_3^2y_4\cos^2(y_1-y_2)\sin(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}\\ &-\frac{m_2(m_1+m_2)\ell_2gy_4\sin y_2}{m_1+m_2\sin^2(y_1-y_2)}+\frac{m_2(m_1+m_2)\ell_1\ell_2y_3^2y_4\sin(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}\\ &+\frac{m_2^2\ell_1\ell_2y_3y_4^2\cos^2(y_1-y_2)\sin(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}+\frac{m_2(m_1+m_2)\ell_1gy_3\sin y_1\cos^2(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}.\end{aligned}$$

The following conditions must be satisfied for the determinant of a sign to be negative:

$$1-(m_1+m_2)g\ell_1y_3\sin y_1\le 0;$$
$$2-m_2\ell_1\ell_2y_3y_4^2\sin(y_1-y_2)\le 0;$$
$$3-m_2g\ell_2y_4\sin y_2\le 0;$$
$$4-\frac{m_2^2\ell_2gy_4\sin y_2\cos^2(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}\le 0;$$
$$5-\frac{m_2(m_1+m_2)\ell_1\ell_2y_3^2y_4\sin(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}\le 0;$$
$$6-\frac{m_2^2\ell_1\ell_2y_3y_4^2\cos^2(y_1-y_2)\sin(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}\le 0;$$
$$7-\frac{m_2(m_1+m_2)\ell_1gy_3\sin_1 y_1\cos^2(y_1-y_2)}{m_1+m_2\sin^2(y_1-y_2)}\le 0.$$

Thus, the above-mentioned conditions indicate the requirements for ensuring the stability of the movement of the double pendulum.

4. Practical Aspect

Figures (2) and (6) were generated using the MATLAB program, which help in illustrating the periodicity motion, phase levels, and the stability state of a double pendulum:

5. Spherical Pendulum

The movement of the spherical pendulum follows a periodic trajectory, which was discovered by the Dutch physicist Christian Huygens. The spherical pendulum is a an example of a classic bound

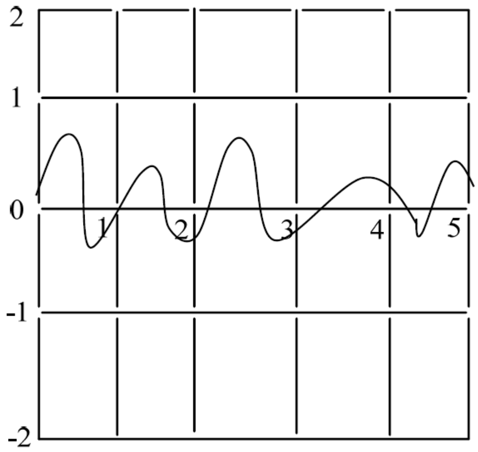

Figure 2: Solution path $\theta_1(t)$ for the double pendulum motion at $(\theta_1(o) = 0, \theta_2(0) = 1, \theta_1^{'}(0) = \sqrt{2}$ and $\theta_2^{'}(0) = 0)$ and $(0 \leq t \leq 5)$

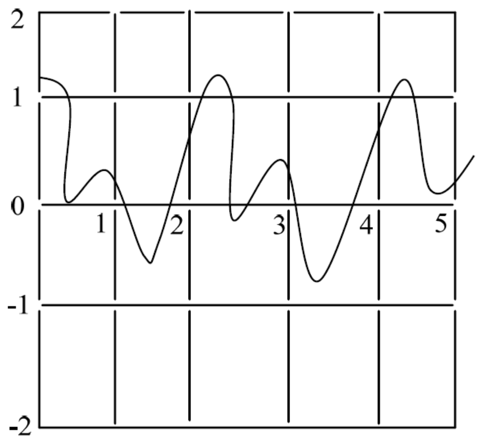

Figure 3: Solution path $\theta_2(t)$ for the double pendulum motion at $(\theta_1(o) = 0, \theta_2(0) = 1, \theta_1^{'}(0) = \sqrt{2}$ and $\theta_2^{'}(0) = 0)$ and $(0 \leq t \leq 5)$

motion, where a sphere of mass (m) is attached to the an inelastic string of length (ℓ) and can freely move in any direction around a fixed point.

The spherical pendulum moves in three dimensions (x, y, z), represented by the Cartesian coordinates, and (r, θ, ϕ), the spherical coordinates, which provide two degrees of freedom for the spherical pendulum, namely, θ and ϕ [8]. The expressions for the kinetic energy equations and latent are derived by using spherical coordinates while keeping $r = \ell =$ constant. The following expressions are also considered:

$$x = \ell \sin\theta \cos\phi$$
$$y = \ell \sin\theta \sin\phi$$
$$z = -\ell \cos\theta$$

Accordingly, the potential energy is:

$$E = mgz,$$

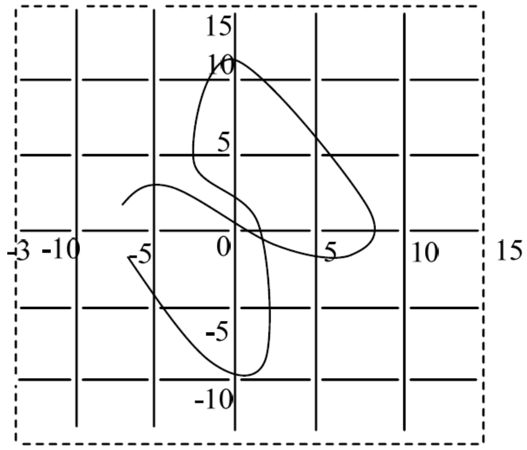

Figure 4: Solution paths for the motion of the double pendulum at $(-1 \le t \le 1)$

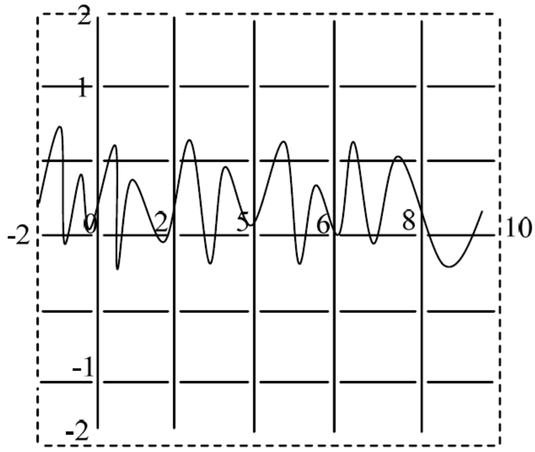

Figure 5: Double pendulum motion at $(m_1 = 2, m_2 = 1, \ell_1 = 5, \ell_2 = 6, g = 32)$ and $(-2 \le t \le 10)$

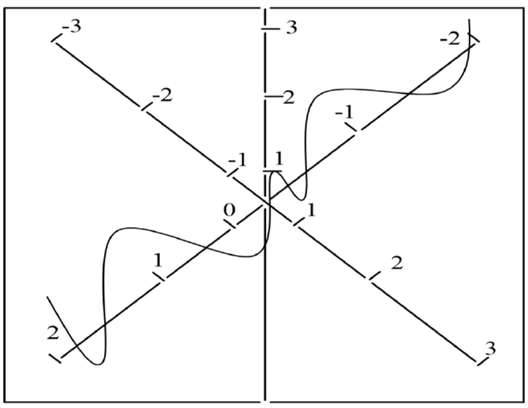

Figure 6: Phase place of double pendulum motion at $(-2 \le t \le 2)$

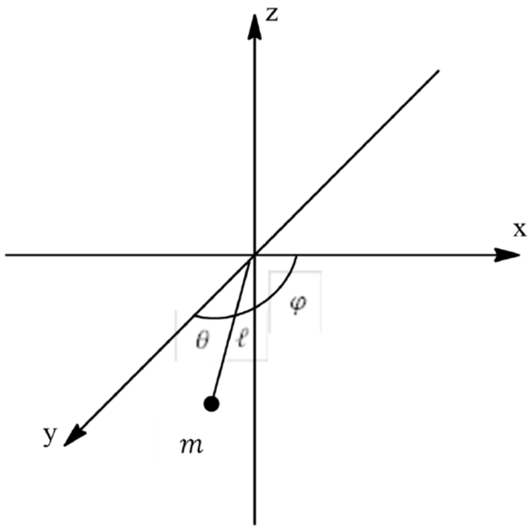

Figure 7: Spherical pendulum

such that:

$$E = -mg\ell\cos\theta$$

and the kinetic energy is expressed as follows:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right).$$

After simplication, the following expression is obtained:

$$T = \frac{1}{2}m\ell^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \quad . \tag{19}$$

The Lagrangian equations are used to obtain the system of motion of the spherical pendulum:

$$L = T - E,$$

such that:

$$L = \frac{1}{2}m\ell^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mg\ell\cos\theta.$$

Applying the Euler–Lagrange equation yields:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0,$$
$$\frac{d}{dt}\frac{\partial L}{\partial\phi'} - \frac{\partial L}{\partial\phi} = 0,$$

such that:

$$\left.\begin{aligned} \ddot{\theta} &= \frac{-g}{\ell}\sin\theta + \dot{\phi}^2\sin\theta\cos\theta \\ \ddot{\phi} &= -2\dot{\phi}\dot{\theta}\cot\theta \end{aligned}\right\} \tag{20}$$

The differential system (20) represents the spherical pendulum motion system.

Studying the stability of the motion of the spherical pendulum using the Getayer method

6. Theoretical Aspect

In the case of a spherical pendulum, mathematician Chetayer proposed the construction of the Libanov function with the help of form integrals:

$$V = \lambda_1\left[F_1 - F_1(0)\right] + \cdots + \lambda_m\left[F_m - F_m(0)\right] + B_1\left[F_1^2 - F_1^2(0)\right] + \cdots + B_{\mathrm{m}}\left[F_{\mathrm{m}}^2 - F_m^2(0)\right],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\beta_1, \ldots, \beta_m$ are the optional constants.

Accordingly, if β_j, λ_{j} (where $j = 1, \ldots, m$) is chosen such that V is a positive determinant function, then all the conditions of Libanov's theory of stability would be satisfied. Chetayer's method for constructing the Libanov function using integrals is a highly effective method. In the case of a spherical pendulum, its motion in influenced by inherent gravity forces, and it has an axis ϕ periodic. Furthermore, kinetic energy [9-16] depends on the speed ϕ and is independent on the axis ϕ. Meanwhile, the forces acting on the pendulum along these exes are equal to zero, indicating that:

$$\frac{\partial E}{\partial \phi} = 0.$$

Accordingly, the differential system (20) has two integrals, which can be expressed using Equations (18) and (19):

$$\left.\begin{aligned} & T + E = \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - \frac{2g}{\ell}\cos\theta = h \\ & \frac{\partial T}{\partial \phi'} = \phi^2 \sin^2\theta = n \end{aligned}\right\}, \tag{21}$$

where h and n are constants.

Assuming that:

$$\theta = \propto + x_1;\ \dot{\theta} = x_2;\ \dot{\phi} = \eta + x_3,$$

the two integrals can be expressed by substituting the assumptions into Equations (21):

$$\left.\begin{aligned} & F\left(x_1, x_2, x_3\right) = \left(x_2^2 + \sin^2\left(\propto + x_1\right)\left(\eta + x_3\right)^2\right) - \frac{2g}{\ell}\cos\left(\propto + x_1\right) = h \\ & F_2\left(x_1, x_2, x_3\right) = \sin^2\left(\propto + x_1\right)\left(\eta + x_3\right) = \eta \end{aligned}\right\}. \tag{22}$$

Both integrals in Equation (22) are functions of indefinite sign. Accordingly, Chetayer's method is applied to construct the Libanov function by $\lambda_1 = 1$ and $\lambda = \lambda_2$. Thus, the function is:

$$V = \left[F_1 - F_1(0)\right] + \lambda\left[F_2 - F_2(0)\right],$$

such that:

$$\begin{aligned} V = & \left[x_2^2 + \sin^2\left(\propto + x_1\right)\left(\eta + x_3\right)^2 - \frac{2g}{\ell}\cos\left(\propto + x_1\right) - \eta^2\sin^2\propto + \frac{2g}{\ell}\cos\propto\right] \\ & + \left[\lambda\sin^2\left(\propto + x_1\right)\left(\eta + x_3\right) - \lambda\eta\sin^2\propto\right]. \end{aligned}$$

Specifically,

$$\begin{aligned} V = & \eta\left[(\lambda + \eta)\cos 2\propto + \eta\cos^2\propto\right]x_1^2 + x_2^2 + x_3^2\sin^2\propto + \eta(\lambda + 2\eta)x_1\sin\propto \\ & + (\lambda + 2\eta)x_3\sin^2\propto + (\lambda + 2\eta)x_1x_3\sin 2\propto. \end{aligned}$$

For V to be apositive definite quadratic function, the terms that contain the variables x_1, x_2, and x_3 of the first order must be eliminated by setting ($\lambda = -2\eta$) in V, resulting in the following form:

$$V = \eta^2 \sin^2 \propto \cdot x_1^2 + x_2^2 \sin^2 \propto - x_3^2 + \cdots,$$

such that:

$$\frac{dv}{dt} = 0.$$

According to Gitay's theory and based on integrals (22), the motion of the spherical pendulum is stable. Moreover, the spherical pendulum movement system is an energy conservation system based on the result $\frac{dv}{dt} = 0$.

7. Practical Aspect

Figures (8)–(12) were generated by using the MATLAB program, which help in demonstrating the periodic motion, phase levels, and stability of the spherical pendulum.

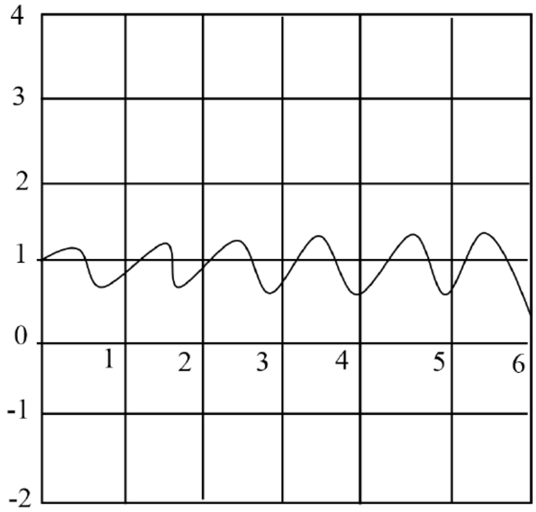

Figure 8: The solution path $\theta(t)$ for the movement of the spherical pendulum at ($\theta(o) = 1$, $Ø(o) = 0$, $\theta'(o) = \sqrt{2}Ø'(o) = 1$) and ($0 \leq t \leq 6$)

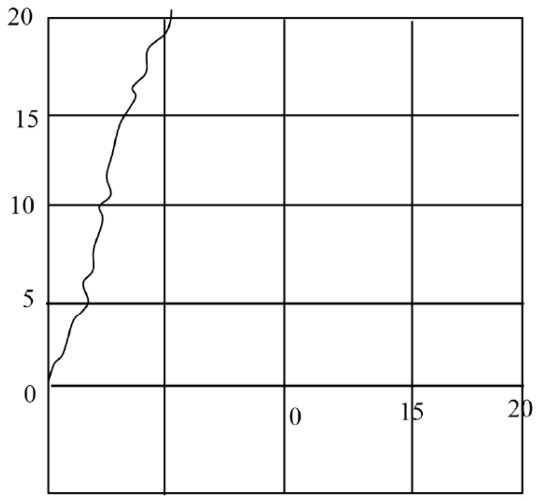

Figure 9: The solution path $\theta(t)$ for the movement of the spherical pendulum at ($\theta(o) = 1$, $Ø(o) = 0$, $\theta'(o) = \sqrt{2}Ø'(o) = 1$) and ($0 \leq t \leq 20$)

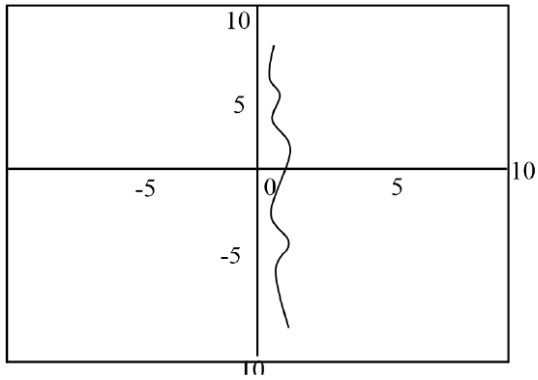

Figure 10: The path of solution for the movement of the spherical pendulum at (–1.5 ≤ t ≤ 2)

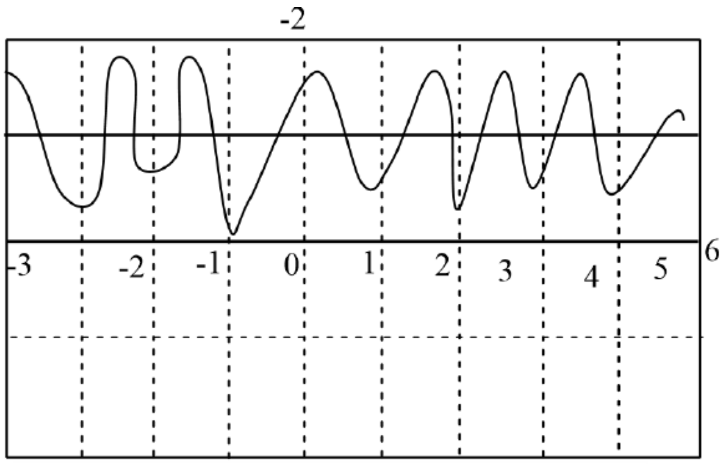

Figure 11: The movement of the spherical pendulum at (ℓ = 3, g = 32) and (–3 ≤ t ≤ 6)

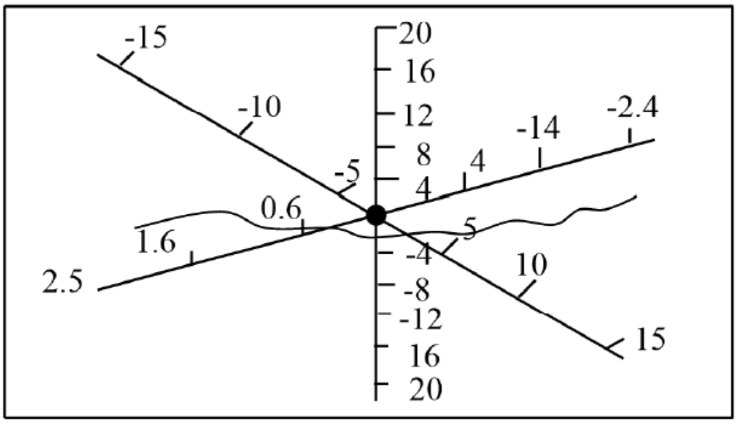

Figure 12: The phase level of the spherical pendulum motion at (–2.5 ≤ t ≤ 2.5)

8. Conclusions

Libanov's theories of stability can be applied to the motion of the double pendulum (Double Pendulum). And finding conditions that make its movement stable.

- The pendulum spherical system is considered one of the energy conservation systems according to the principle of conservation energy, and according to the Getayev method of constructing the Libanov function with the help of integrals the motion of the spherical pendulum is considered stable.

9. Recommendations

- It is possible to apply this study to many electrical and mechanical devices in which the pendulum is considered basis for its work, such as the seismograph, which is used to monitor earthquakes.

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