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On recurrence in dendrite flows

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Consider a finitely generated group generated G that acts on a dendrite X by transformations. (G, X) is called a flow. In this note, it was proven that if the flow (G, X) is pointwise recurrent, then (G, X) is almost periodic. Furthermore, we give a transitive flow having only two recurrent points.

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1. Introduction

A continuous action of a topological group G on a compact topological space X is called a *flow* (G, X). Particular attention was paid to the study of groups acting on dendrites [14, 10, 18, 15, 16, 12, 9, 6, 1]. The interest of studying groups of transformations on these one dimensional spaces is motivated first by the appearance of dendrites as Julia subsets into complex analysis [4] and secondly by the study of hyperbolic geometry in dimension three [15]. Recently, in [7], the authors studied the rigidity in the sense of Zimmer for higher rank lattice actions on dendrites.

In the context of a finitely generated group acting on a compact metric space, several researchers studied the correspondences between the following dynamical properties:

(1) the flow (G, X) is pointwise recurrent;
(2) the flow (G, X) is almost periodic;
(3) the orbit closure relation of the flow (G, X) is closed;
(4) the flow (G, X) is equicontinuous.

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In the papers [3] and [8] it is proven that the four properties are equivalent in the context of a finitely generated group G acting on a compact space of dimension zero, a topological graph or a dendrite with a countable subset of endpoints.

In this article we prove that if the flow (G, X) is pointwise recurrent, then (G, X) is almost periodic in the context of a finitely generated group G which acts on a general dendrite (Theorem 4.5).

2. Dendrites

A compact connected metric space is *a continuum*. A topological space is *path connected* if two of its points can be connected by an arc. A locally connected continuum containing no simple closed curves (equivalent to the unit circle $\mathbb{S}^1$) is called *a dendrite*. Remember that two elements x and y in a dendrite D can be connected by a unique arc where the ends are x and y, which is denoted by $[x, y]$. We set $(x,y)=[x,y]\setminus\{x,y\}$, $[x,y)=[x,y]\setminus\{y\}$ and $(x,y]=[x,y]\setminus\{x\}$. According to [13, Corollary 10.6], any subcontinuum of a dendrite is a dendrite. Furthermore, each dendrite is hereditarily locally connected [13].

In a dendrite D, the cardinality of edges issuing from x is named *the order of an element* x [5]. In the setting of dendrites, this cardinal is equal to the number of connected components of $D\setminus\{x\}$ [19]. We designate it by ord(x, D). If this cardinal is not finite, then it will be countable, and the connected components of $D\setminus\{x\}$ have diameters which converge to zero [5]. In this setting, we put ord(x, D) is equal to ω. In this paper ω denotes the first ordinal limit. The elements of order one are named *end-points*. The family of all endpoints of D is represented by $E(D)$. The elements of order greater than three are named *branching points* and the class of all branching points are represented by $R(D)$. For all n in $\{1,2,\dots,\omega\}$, we note $R_n(D)$ the subset of all elements of D of order n. It is obvious that $\overline{R_2(D)}=D$, and that $R(D)$ is at most countable [13].

3. Flows

In this article, by flow we indicate a pair (G, X), where G is a topological group acting, by transformations, on a compact metric space (X, d). If $g \in G$ and $x \in X$ we will identify g and the related transformation and we will write gx to denote the action of g on x. The subset $Gx=\{gx : g\in G\}$ is named the *orbit of* x. The set of return times from $x\in X$ to $A\subset X$ is $T_A(x)=\{g\in G : gx\in A\}$. A point x of X is said to be *periodic under* G if its orbit Gx has finitely many elements.

A subspace Y in X is an *invariant subest* when Gy is a subset of Y for all y in Y. The complement, the interior and the closure of an invariant subspace are invariant subspaces.

A subspace $W\subset X$ is named a *minimal set* of the flow (G, X) if it is non-empty, closed, invariant and no proper subspace of W verifies the above three notions, equivalently $\overline{Gx}=W$ for all $x\in W$. Note that the closure of each orbit includes a minimal subset. A flow (G, X) is named *minimal* if the space X itself is a minimal subset. x is an *almost periodic* point if and only if the closure, $\overline{Gx}$, of Gx is a minimal subset. A flow (G, X) is *pointwise almost periodic* if each element $x\in X$ is almost periodic. We denote by $AP(G)$ the subset of all almost periodic points.

(G, X) is a *transitive* flow if $\overline{Gx}=X$ for some $x\in X$.

The orbit closure relation is $R(G)=\{(x,y): y\in\overline{Gx}\}$.

The flow (G, X) is equicontinuous (with respect to a metric d) if for all $\varepsilon>0$, there is a $0<\delta<\varepsilon$ satisfying $d(gx,gy)<\varepsilon$ for all $x, y\in X$ where $d(x,y)<\delta$ and each $g\in G$.

The notion of recurrence for group action is defined in [3]. Consider G a finitely generated group $\Gamma=\{f_1,\dots,f_p\}$. Let B_r be the class of points of G having a length less than or equal to r. Put $K(g)=B_{|g|-1}.g$, for $g\in G$, such that $|g|$ denotes the length of g. A class $C\subset G$ is called a *cone* if there is a subsequence $g_n\in G$ such that $|g_n|\to+\infty$ and C is equal to $\lim\limits_{n\to\infty}K(g_n)^1$. According to [3, Proposition 1.5], note that for each cone C one can find a sequence c_n such that B_n . C_n is a subset of C and for every $g \in G$, gc_n belongs to C for some integer n.

Definition 3.1. [3] Let (G, X) denote a flow such that X is a compact metric space and a finitely generated group acting by transformations on X. Put C a subset of G not containing the identity element e. A point x is called *recurrent*, when it is C-recurrent for any cone C. A point $x \in X$ is *C-recurrent*, whenever for any open neighbor U of x, the intersection $Cx \cap U$ is not empty.

$R(G)$ denote the subset of all recurrent points. (G, X) is called *pointwise recurrent* flow when $R(G) = X$.

If $G = \mathbb{Z}$ (discrete flow) is a transitive flow, then $\overline{R(G)} = X$. In the following example, we show that this result is not true for a general flow.

Example 3.2. Let $X = [0,1]$ and G be the group generated by 2 homeomorphisms $f = h^{-1} \circ T_1 \circ h$ and $g = h^{-1} \circ T_{\sqrt{(2)}} \circ h$ where h is a the homeomorphism of (0,1) to the real line $\mathbb{R}$ and T_1 and $T_{\sqrt{2}}$ are 2 translations.

Theorem 3.3. *The flow* (G, [0,1]) *is transitive and the only recurrent points are the endpoints 0 and 1.*

Proof. If $x \in (0,1)$, then $\overline{Gx} = [0,1]$. So (G, [0,1]) is transitive.

We show that only the fixed endpoints 0 and 1 are recurrent. The group G is algebraically isomorphic to $\mathbb{Z} \times \mathbb{Z}$, so to construct a cone in G we can do it in $\mathbb{Z} \times \mathbb{Z}$ first. Let $g_n = (0,n)$ for all $n = 1,2,3,...$, and let $C = lim_n K(g_n)$. It's quite easy to check that $C = \{(m,n) : m = 1,2,3,..., |n| < m\}$.

Moreover, in that every cone of $\mathbb{Z} \times \mathbb{Z}$ contains a certain type of translation of an orthant (in this case, a quadrant). This cone is a rotated quadrant.

Now, to bring this cone to G, we can define $C' = \{g^m o f^n : (m,n) \in C\}$.

It is easier to do the analysis with the T_1 and $T_{\sqrt{2}}$ maps on the real line. If we take x in $\mathbb{R}$ and (m,n) in C, then

$$T_{\sqrt{2}}^m o T_1^n(x) \geq m\sqrt{2} + n + x = m\sqrt{2} - (m-1) + x \geq (\sqrt{2}-1)m + 1 + x > 1 + x.$$

Thus we cannot approximate x arbitrarily using the homeomorphisms $T_{\sqrt{2}}^m o T_1^n$, where $(m,n) \in C$. Therefore, the points in the interval (0,1) are not C'-recurrent in the flow (G, [0, 1]).

4. Main Theorem

In this paragraph we show the following theorem.

Theorem 4.1. *Let G be a group acting by transformations on a dendrite D where G is a finitely generated group. Then the following statements are equivalent:*

(1) *the flow* (G, D) *is pointwise recurrent;*
(2) *the flow* (G, D) *is almost periodic;*
(3) *the orbit closure relation of the flow* (G, D) *is closed;*
(4) *the flow* (G, D) *is equicontinuous.*

Let us recall the definition of some important sets given in [17].

For each $x \in X$, the limit ω defined under G is:

$$\omega(x,G) = \{y \in X : \exists\ g_n \in G\ with\ |g_n| \to +\infty;\ g_n x \to y\}.$$

The weak limit ω defined under the action of the group G is:

$$w\omega(x,G) = \{y \in X : \exists\ g_n \in G\ with\ |g_n| \to +\infty\ and\ \exists x_n \in X\ with\ x_n \to x;\ g_n x_n \to y\}.$$

Lemma 4.2. [17] *Let G denote a group acting by transformations on a compact metric space X. Then, wω(x, G) and ω(x, G) are non-empty, closed, invariant subsets of X under G and we always have* $\omega(x,G) \subset w\omega(x,G)$.

Lemma 4.3. *Let (G, X) denote a flow where X is a compact metric space and G is a finitely generated group.* $w\omega(x,G) \subset \overline{Gx}$ *if and only if each open invariant set containing a point* $y \in w\omega(x,G)$ *contains x.*

Proof. If $w\omega(x,G) \subset \overline{Gx}$ and U is an open invariant containing $y \in w\omega(x,G)$ then $\overline{Gx} \cap U \neq \emptyset$ hence $Gx \cap U \neq \emptyset$ which implies that $x \in U$.

Conversely if $y \notin \overline{Gx}$ then $y \in X \setminus \overline{Gx}$ which is invariant and open then $x \in X \setminus \overline{Gx}$ which is impossible.

Lemma 4.4. [17] *If* $w\omega(x,G) \subset \overline{Gx}$ *for all* $x \in X$ *then AP(G) = X.*

Theorem 4.5. *Let (G, X) be a dendritic flow, where G is a finitely generated group. Then R(G) = X if and only if AP(G) = X.*

Proof. By [3, Proposition 1.7], we obtain the "if " part of the theorem.

Conversely, let $x \in X$ and suppose that $R(G) = X$, we distinguish two cases here:

Case 1. The dimension of $\overline{Gx}$ is equal to zero. By [3, Theorem 1.8] we obtain $x \in AP(G)$.

Case 2. $\overline{Gx}$ *is one-dimensional*. In this case $\text{int}(\overline{Gx}) \neq \emptyset$. Let $y \in w\omega(x,G)$ then there exists a sequence (X_n) which converges to x and a sequence (g_n) in G with $|g_n|$ tends to infinity such as the sequence $(g_n x_n)$ converges to y. Let U be an invariant open set of X containing y then there is N such that for $n \geq N$, $g_n x_n \in U$. From U is invariant, $x_n \in U$ for $n \geq N$. Therefore, $x \in \bar{U}$. If $x \notin U$, then $\overline{Gx} \subset \bar{U} \setminus U$. The fact that $int(\bar{U} \setminus U) = \emptyset$ implies that $int(\overline{Gx}) = \emptyset$, which is impossible. Therefore, $w\omega(x,G) \subset \overline{Gx}$. By Lemma 4.4, $x \in AP(G)$.

We can now prove the Theorem 4.1.

Proof. By [3][Proposition 1.7], (2) implies (1) and by Theorem 4.5, (1) implies (2). By [3][Proposition 1.1], (3) implies (2) and by [11][Theorem 5.5], (2) implies (3). By [2][Lemma 3. p. 37], (4) implies (2). It follows from [1, Exercise 6. p. 46] that (3) and (4) are equivalent. Thus Theorem 4.1 is proved.

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