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An indirect spectral shifted Gegenbauer collocation method for discretizing fractional optimal control problems

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Many best properties of the shifted Gegenbauer functions were used to obtain a new closed formula of the left and right Riemann-Liouville fractional derivative. The new formulas have been used to approximate the solution of the fractional optimal control problems (FOCPs). The indirect spectral-shifted Gegenbauer collocation method is applied to discretizing FOCPs with a dynamic fractional differential equation. The FOCPs were reduced to the system of algebraic equations. Special attention is given to studying the convergence analysis and estimating an error upper bound of the presented formulas. Illustrative numerical examples are integrated to show the truth and applicability of this new technique.

Key words and phrases: Riemann-Liouville fractional derivative, Gegenbauer polynomials, Optimal control problem

Mathematics Subject Classification (2010): 49M05.

1. Introduction

Fractional optimal control problems could be applied in different applications, such as in scientific life and engineering fields. Recently, they have received wide attentions [10]. Some time cannot find the analytic solution for FOCPs, so the researcher has to use numerical methods to calculate the solution

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for these problems. Majorly, we have indirect and direct methods for numirical solution. The first step of indirect methods, is to obtain the optimality conditions of FOCPs. This leads to boundary-value problems (BVPs), then the BVPs can be solved numerically to obtain the extremals, and finally, the optimal solution can be obtained, for examples: the indirect shooting method, indirect collocation method, and indirect multiple-shooting method. While, the direct method is included the transcription of FOCPs to a nonlinear programming problem (NLP), then the NLP will be solved by using well-known optimization techniques [13, 14].

In general, the numerical methods for FOCPs can be classified into local and global categories. The finite difference and finite element methods are based on local arguments, whereas the spectral method is usually global [20, 21]. Mostly, used the advantage and properties of the global spectral methods in current work. The best advantage of spectral methods is that used the discretization to approximate optimal control problems and partial differential equations, almost all solutions of problems may be infinitely smooth, and converge to real solutions [20, 21]. Beyond, classical spectral methods also meet some limits, such as loss of global accuracy when facing problems with non-smooth/singular solutions, More detail is found in Jie et al. [22].

There are many globally smooth functions such as trial/test functions like the Jacobi spectral method, Gegenbauer spectral method, Chebyshev spectral method, and ace. The good features of these functions give the spectral methods more properties to deal with the field of science. Mostly, when using the spectral methods to approximate solutions to the problems, the problems are reduced to a system of linear or nonlinear algebraic equations, thus, it's easy to represent this system of equations by using the operational matrices.

Therefore, the strategy of substituting nodes of the collocation method with operational matrices will give results that have more accuracy, fast convergent with few numbers of collocation points, and easy to implement [23].

The shifted Gegenbauer polynomials are just another basis set that offers considerable advantages over basis sets, allowing us to attack problems and use them to get suitable numerical methods [14, 12]. The shifted Gegenbauer polynomials considerable of the best methods is the approximation by the orthogonal family of functions [19]. In the current work, a collocation strategy is applied to approximate solution of linear/nonlinear FOCPs indirectly. A good trait of some properties of the global approximate of shifted Gegenbauer polynomials has been used to get new approximate formulas of the Caputo and Riemann-Liouville fractional derivative. These approximate formulas gave us a good chance to discretize FOCPs by the spectral collocation method.

The mane topics of the paper includs some features of the shifted Gegenbauer polynomials, fractional derivative, and integrals formulas have been written, also the problem statements were introduced. moreover, contains a newly evaluated fractional derivative for the shifted Gegenbauer polynomials, and convergence analysis. Also, it contains approximately examples to show the correctness of the numerical method. And gives a brief conclusion and some remarks.

2. Preliminaries and notations

In this section: some important features of the shifted Gegenbauer polynomials, fractional derivative formula, and the problem statements are inserted.

2.1. Riemann-Liouville and Caputo fractional derivative

Let $0 \le \alpha < 1$ a real number and $g:[a,b] \to R$ is continuous function then the left and right Riemann-Liouvill fractional integrals of order α, respectively are defined as:

$$ {}_aI_u^{\alpha} g(u) = \frac{1}{\Gamma(\alpha)} \int_a^u (u-t)^{\alpha-1} g(t)dt, \qquad u \in [a,b], \tag{1} $$

$$ {}_{u}I_{b}^{\alpha} g(u)=\frac{1}{\Gamma(\alpha)}\int_{u}^{b}(t-u)^{\alpha-1}g(t)dt, \quad u\in[a,b]. \tag{2}$$

The left and right Riemann-Liouvill fractional derivatives of order α can be defined respectively as:

$$ {}_{a}^{R}D_{u}^{\alpha} g(u)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{du}\int_{a}^{u}(u-t)^{-\alpha}g(t)dt, \tag{3}$$

$$ {}_{u}^{R}D_{b}^{\alpha} g(u)=\frac{1}{\Gamma(1-\alpha)}\frac{(-1)d}{du}\int_{u}^{b}(t-u)^{-\alpha}g(t)dt. \tag{4}$$

The left and right Caputo fractional derivatives of order α can be defined respectively as:

$$ {}_{a}^{C}D_{u}^{\alpha} g(u)=\frac{1}{\Gamma(1-\alpha)}\int_{a}^{u}(u-t)^{-\alpha}\dot{g}(t)dt, \tag{5}$$

$$ {}_{u}^{C}D_{b}^{\alpha} g(u)=\frac{(-1)}{\Gamma(1-\alpha)}\int_{u}^{b}(t-u)^{-\alpha}\dot{g}(t)dt, \tag{6}$$

Such that $\dot{g}(t)$ is the first derivative of the function $g(t)$.

The next remark summarizes some properties of the Riemann-Liouville and Caputo fractional derivatives that may need it in the next sections.

Remark 2.1. *There are many properties of the Riemann-Liouville and Caputo fractional derivative:*

1- *Linearity*: If ξ and μ are constant then:

$$ {}_{a}^{C}D_{u}^{\alpha}(\mu g(u)+\zeta f(u))=\mu\, {}_{a}^{C}D_{u}^{\alpha} g(u)+\zeta\, {}_{a}^{C}D_{u}^{\alpha} f(u).$$

2- ${}_{u}^{C}D_{b}^{\alpha}\zeta={}_{a}^{C}D_{u}^{\alpha}\zeta=0$, ${}_{a}^{R}D_{u}^{\alpha}\zeta=\frac{\zeta.(u-a)^{-\alpha}}{\Gamma(1-\alpha)}$, and ${}_{u}^{R}D_{b}^{\alpha}\zeta=\frac{\zeta.(b-u)^{-\alpha}}{\Gamma(1-\alpha)}$, such that ξ is constant.

3- Let $m > 1$, and $0<\alpha<1$, $m\in R$, $n\in Z$, then:

$$ {}_{a}^{R}D_{u}^{\alpha}(u-a)^{m}=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}(u-a)^{m-\alpha}, \text{ and } {}_{u}^{R}D_{b}^{\alpha}(b-u)^{m}=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}(b-u)^{m-\alpha}.$$

4- ${}_{u}^{C}D_{b}^{\alpha} g(u)={}_{u}^{R}D_{b}^{\alpha} g(u)-\sum_{k=0}^{r-1}\frac{g^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-u)^{k-\alpha}$ and $r-1<\alpha<r\in Z$.

5- ${}_{a}^{C}D_{u}^{\alpha} g(u)={}_{a}^{R}D_{u}^{\alpha} g(u)-\sum_{k=0}^{r-1}\frac{g^{(k)}(a)}{\Gamma(k-\alpha-1)}(u-a)^{k-\alpha}$ and $r-1<\alpha<r\in Z$.

For more information see [3].

2.2. Problem statement and optimality conditions

No doubt that with the indirect method the optimal control problem is transformed into a system of differential equations with a set of optimality conditions. The first step is to find the Euler– Lagrange equations and Pontryagin's maximum principle and find the solution of this differential equation in the second step, see[15].

Our goal is to solve the FOCP by the indirect method and find the control and the state functions $u(t)$ and $x(t)$. The performance index $J(u)$ of FOCP can write it as follows:

$$ mimimze\ \ J(u)=\int_{a}^{t_f}F(t,x(t),u(t))dt, \tag{7}$$

subject to the system's dynamic constraints:

$$ A\dot{x}(t)+B\,{}_{a}^{c}D_{t}^{\alpha}x(t)=G(t,x(t),u(t)), \quad and \quad x(a)=x_0, \quad 0<\alpha<1, \tag{8}$$

where $t \in [a, t_f]$ represents the time, *A, B* are real numbers nonequal to zero, *F, G* are two arbitrary real functions and t_f is the final time. The necessary and optimality conditions in terms of a Hamiltonian for the problem (7) and (8) can be find it in [16, 17, 4]. Where Hamiltonian function is defined by:

$$H(t,x,u,\gamma) = F(t,x,u) + \gamma G(t,x,u),$$
$$\frac{\partial H}{\partial \gamma}(t,x(t),u(t),\gamma(t)) = M(t,x(t),u(t),\gamma(t)),\ and$$
$$-\frac{\partial H}{\partial x}(t,x(t),u(t),\gamma(t)) = N(t,x(t),u(t),\gamma(t)),$$

such that $\gamma(t)$ is the Lagrange multiplier vector or the adjoin function. Then the above FOCP (7) and (8) can be written as the following system of fractional differential equations:

$$A\dot{x}(t) + B\,{}_a^cD_t^\alpha x(t) = M(t,x(t),u(t),\gamma(t)), \tag{9}$$
$$A\dot{\gamma}(t) - B\,{}_t^RD_{t_f}^\alpha \gamma(t) = N(t,x(t),u(t),\gamma(t)), \tag{10}$$

where $x(a) = x_0$, $\gamma(t_f) = 0$, $M(t,x(t),\gamma(t))$ and $N(t,x(t),\gamma(t))$ are known functions [16, 17].

If *G* and *F* be two convex functions in terms of *u* and *x*, then the system of equations (9) and (10) contains necessary and sufficient conditions for optimal solutions u^* and x^*.

2.3. Shifted Gegenbauer polynomials

Let's talk in general about shifted Gegenbauer orthogonal polynomials $G_{m,\lambda}(t)$, which are defined in the interval [0,1] with respect to the weight function $(t-t^2)^{\lambda-1/2}$. The sequence of generalized shifted Gegenbauer polynomials $\{G_{m,\lambda}(t)\}_{m=0}^{\infty}$, $\lambda > \frac{-1}{2}$, see [18, 7]. The significant properties of the shifted Gegenbauer polynomials $G_{m,\lambda}(t)$ can be summarized as follows:

1. The derivative formula of shifted Gegenbauer functions can be written as follows:

$$\frac{d^r}{dt^r} G_{m,\lambda}(t) = \frac{2^{2r}(\lambda + r - 1)!}{(\lambda - 1)!} G_{m-r,\lambda+r}(t), r \in N. \tag{11}$$

2. The symmetry of the shifted Gegenbauer polynomials is emphasized by the relation:

$$G_{m,\lambda}(t) = (-1)^m . G_{m,\lambda}(-t). \tag{12}$$

3. The closed-form equation for shifted Gegenbauer polynomials of degree *m* [8] can be written as:

$$G_{m,\lambda}(t) = \sum_{k=0}^{m} (-1)^{m-k} \frac{\Gamma(\lambda + \frac{1}{2})\Gamma(m + k + 2\lambda)}{\Gamma(k + \lambda + \frac{1}{2})\Gamma(2\lambda)(m-k)!k!} t^k,$$
$$G_{j,\lambda}(0) = (-1)^j \frac{\Gamma(j + 2\lambda)}{\Gamma(2\lambda)j!}, \quad and \quad G_{j,\lambda}(1) = \frac{\Gamma(j + 2\alpha)}{j!\Gamma(2\alpha)}. \tag{13}$$

4. The shifted Gegenbauer polynomials can be generated directly by using the following three- term recurrence equation:

$$G_{0,\lambda}(t) = 1, \tag{14}$$
$$G_{1,\lambda}(t) = 2t - 1, \tag{15}$$
$$G_{j+1,\lambda}(t) = \frac{2(j+\lambda)}{j+2\lambda}(2t-1)G_{j,\lambda}(t) - \frac{j}{j+2\lambda}G_{j-1,\lambda}(t), \qquad j \geq 1, \tag{16}$$
$$G_{j,\lambda}(t) = \frac{1}{2(j+1)}G'_{j+1,\lambda}(t) - \frac{j}{2(j+2\lambda)(j+2\lambda-1)}G'_{j-1,\lambda}(t). \tag{17}$$

From the basis of family of shifted Gegenbauer polynomials of degree m: $\{G_{i,\lambda}(t)\}_{i=0}^{m}$, can be approximate any function $x(t) \in C^{\infty}[0,1]$ see [18,7]. The expansion series of the function $x(t)$ can be written as follows:

$$x(t) \approx x_m(t) = \sum_{j=0}^{m} c_j . G_{j,\lambda}(t), \qquad t \in [0,1], \tag{18}$$

the coefficients c_j of equation (18) are given by:

$$c_j = h_{j,\lambda}^{-1} \int_0^1 (t - t^2)^{\left(\lambda - \frac{1}{2}\right)} G_{j,\lambda}(t) . x(t) dt, \quad j=0,1,2,3...m, \tag{19}$$

and the normalization factor can be calculated by the following relation:

$$h_{j,\lambda}^{-1} = \frac{2^{(\lambda-1)} \quad j! \quad \Gamma^2(\lambda)(j+\lambda)}{\pi \Gamma(j+2\lambda)}, \tag{20}$$

The shifted Gegenbauer Gauss nodes (GG) can be defined as $S_k = \{t_k | , k=0,1,...,m\}$. To calculate the set of GG points S_k and quadrature weights see [18].

$$c_j = h_{j,\lambda}^{-1} . \sum_{k=0}^{m} w_{k,\lambda} . G_{j,\lambda}(t_k) . x(t_k), \quad and \quad (w_{k,\lambda})^{-1} = \sum_{j=0}^{m} h_{j,\lambda}^{-1} . \left(G_{j,\lambda}(t_k)\right)^2 . \tag{21}$$

In general, expansion series to the function $x(t)$ can be written as follows:

$$x(t) \approx \sum_{j=0}^{m} \sum_{k=0}^{m} h_{j,\lambda}^{-1} . w_{k,\lambda} . G_{j,\lambda}(t_k) . G_{j,\lambda}(t) . x(t_k), \tag{22}$$

such that $\{t_k\}$ are GG nodes. For more information see [2,11, 18, 1, 9, 7].

3. Numerical Approximation

This section establishes to get a new formula to approximate the left and right fractional integrals and derivatives by using shifted Gegenbauer functions and developing an algorithm to approximate the solution of the system of fractional differential equations (9) and (10). Also, the new two theorems have been inserted into convergence analysis to estimate the errors.

3.1. Evaluate the new formulas of the fractional derivative

Let us start to find the left and right Riemann-Liouville fractional integrals of order α for the shifted Gegenbauer polynomials. For any $0<\alpha<1$, the fractional integral ${}_0^R I_t^{\alpha} G_{j,\lambda}(t)$ can be find it as follows:

$${}_0^R I_t^{\alpha} G_{j,\lambda}(t) = \widehat{G}_{j,\lambda}^{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} G_{j,\lambda}(\tau) d\tau, \quad j=0,1,2,...,m. \tag{23}$$

The first and second terms of shifted Gegenbauer polynomials can be find the fractional integral of it easily as:

$$\widehat{G}_{0,\lambda}^{\alpha}(t) = \frac{t^{\alpha}}{\alpha \Gamma(\alpha)}, \tag{24}$$

$$\widehat{G}_{1,\lambda}^{\alpha}(t) = \frac{4}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{t^{\alpha}}{\alpha \Gamma(\alpha)} . \tag{25}$$

To find $\widehat{G}_{j+1,\lambda}^{\alpha}(t)$ where $j=1,2,3,4,...,m$ can be using the equations (17) with (23) and itegral by part, finally by using second property of the symmetry of the shifted Gegenbauer polynomials obtained:

$$\widehat{G}_{j+1,\lambda}^{\alpha}(t) = \theta_j^{\lambda,\alpha} . \left\{ \frac{2(j+\lambda)t}{(j+2\lambda)} \widehat{G}_{j,\lambda}^{\alpha}(t) - \upsilon_j^{\lambda,\alpha} \widehat{G}_{j-1,\lambda}^{\alpha}(t) - \beta_j^{\lambda,\alpha} . (t+1)^{\alpha} \right\}, \tag{26}$$

where

$$\theta_j^{\lambda,\alpha} = \frac{2(j+2\lambda)(j+1)}{2(j+2\lambda)(j+1)-(2j+\lambda)\alpha}, \tag{27}$$

$$\upsilon_j^{\lambda,\alpha} = \frac{2j(j+2\lambda)(j+2\lambda-1)+(2j+\lambda)j\alpha}{2(j+2\lambda)^2(j+2\lambda-1)}, \tag{28}$$

and

$$\beta_j^{\lambda,\alpha} = \frac{(-1)^j(2j+\lambda)}{2\Gamma(\alpha)(j+2\lambda)}\left\{\frac{G_{j+1,\lambda}(1)}{(j+1)} - \frac{jG_{j-1,\lambda}(1)}{(j+2\lambda)(j+2\lambda-1)}\right\}, \ \ for\ j \geq 1. \tag{29}$$

It's clear that Equations (24),(25), and (26) represent the three-recurrence formula generating the left Riemann-Liouvill fractional integrals of order α. Also can use these three- recurrenc formula to generating the left Riemann-Liouvill fractional derivative of order α as follows:

$$ {}_0^R D_t^\alpha G_{j,\lambda}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\widehat{G}_{j,\lambda}^{\alpha}(t). \tag{30}$$

From properties of the Riemann-Liouville and Caputo fractional derivative the left Caputo fractional derivative of the shifted Gegenbauer polynomials can be written as follows:

$$ {}_0^C D_t^\alpha G_{j,\lambda}(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{dt}\widehat{G}_{j,\lambda}^{\alpha}(t) - \frac{G_{j,\lambda}^{\alpha}(0)}{\Gamma(1-\alpha)}t^{-\alpha}, \tag{31}$$

$$ {}_0^C D_t^\alpha G_{j,\lambda}(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{dt}\widehat{G}_{j,\lambda}^{\alpha}(t) - (-1)^j\frac{\Gamma(j+2\lambda)}{\Gamma(2\lambda)j!}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{32}$$

Further more, the formula represents the Caputo fractional derivative and it can be obtained easily by three-recurrence formula (24)–(26) for the $0<\alpha<1$ as follows:

$$ {}_0^C D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{x}(\tau)\,d\tau,$$

$$x(t) \approx x_m(t) = \sum_{j=0}^{m} c_j . G_{j,\lambda}(t), \qquad t \in [0,1],$$

$$ {}_0^C D_t^\alpha x_m(t) = \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{m} c_j \int_0^t (t-\tau)^{-\alpha}\dot{G}_{j,\lambda}(\tau)d\tau,$$

by using the derivative property of shifted Gegenbauer polynomials

$$ {}_0^C D_t^\alpha x_m(t) = \frac{.2^2(\lambda)}{\Gamma(1-\alpha)(\lambda-1)!}\sum_{j=0}^{m} c_j \int_0^t (t-\tau)^{-\alpha} G_{j-1,\lambda+1}(\tau)\,d\tau,$$

$$ {}_0^C D_t^\alpha x_m(t) = \frac{.2^2(\lambda)}{\Gamma(1-\alpha)(\lambda-1)!}\sum_{j=0}^{m} c_j \widehat{G}_{j-1,\lambda+1}^{1-\alpha}(t). \tag{33}$$

Now, we obtain a closed-form of the right side fractional integral of shifted Gegenbauer functions where $0<\alpha<1$,

$$ {}_t^R I_1^\alpha G_{j,\lambda}(t) = \frac{1}{\Gamma(\alpha)}\int_{u=t}^{u=1}(u-t)^{\alpha-1}.G_{j,\lambda}(u)du,$$

by using the change of variable $u=1-s$ and $t,s,u \in [0,1]$, together with the third property of the Gegenbauer polynomials (12), one will get:

$$ {}_t^R I_1^\alpha G_{j,\lambda}(t) = \frac{1}{\Gamma(\alpha)}\int_{s=1-t}^{s=0}(1-s-t)^{\alpha-1}.G_{j,\lambda}(1-s)ds,$$

$$ {}_{t}^{R}I_{1}^{\alpha}G_{j,\lambda}(t)=\frac{(-1)}{\Gamma(\alpha)}\int_{s=0}^{s=1-t}((1-t)-s)^{\alpha-1}.G_{j,\lambda}(1-s)ds, $$

$$ {}_{t}^{R}I_{1}^{\alpha}G_{j,\lambda}(t)=\frac{(-1)^{j+1}}{\Gamma(\alpha)}\int_{s=0}^{s=1-t}((1-t)-s)^{\alpha-1}.G_{j,\lambda}(s-1)ds, $$

$$ {}_{t}^{R}I_{1}^{\alpha}G_{j,\lambda}(t)=(-1)^{j+1}[{}_{0}^{R}I_{t}^{\alpha}G_{j,\lambda}(t)]\,|_{t\to(1-t)}=(-1)^{j+1}\widehat{G}_{j,\lambda}^{\alpha}\left(1-t\right). \tag{34} $$

Based on the equation (34), the right Riemann-Liouville fractional derivative of the shifted Gegenbauer polynomials can be found it as follows:

$$ {}_{t}^{R}D_{1}^{\alpha}G_{j,\lambda}(t)=\frac{(-1)^{j+2}}{\Gamma(1-\alpha)}\frac{d}{dt}\widehat{G}_{j,\lambda}^{\alpha}(1-t). \tag{35} $$

Thus, the right Riemann-Liouville fractional derivatives of the shifted Gegenbauer functions can be generated by applying three recurrence formulas (24) and (25).

3.2. Discretization of the FOCP

Our goal is to approximate the unknown functions $x(t)$, $u(t)$, $\gamma(t)$ in problem (9)–(10) via spectral collocation method based on the shifted Gegenbauer Gauss nodes. The first step is to transform the interval $[a,t_f]$ to the $[0,1]$. Where $x(\tau)$ is a real-valued function defined on the interval $[a,t_f]$, and $\{t_k,k=0,1,...,m\}$ is the set of collocation shifted Gegenbauer Gauss nodes on the reference interval [0,1], then:

$$ t=\frac{4\tau-a-3t_f}{t_f-a},\ t\in[0,1]. $$

And let us define the shifted Gegenbauer Gauss nodes with wight function $(t^2-t)^{\lambda-1/2}$ on the interval the [0,1] as: $\{t_k,k=0,\dots,m\}$ where t_k's are the zeros of $\widehat{G}_{m+1,\lambda}(t)$.

For a positive integer m, suppose that the state function $x(t)$ can be approximated it by $x_m(t)$ as:

$$ x\left(t\right)\approx x_m\left(t\right)=\sum_{j=0}^{m}c_j.G_{j,\lambda}\left(t\right), \qquad t\in[0,1], \tag{36} $$

then the state function $x(t)$ can be written as Equation (22):

$$ x_m(t)=\sum_{j=0}^{m}\sum_{k=0}^{m}h_{j,\lambda}^{-1}.w_{k,\lambda}.G_{j,\lambda}(t_k).G_{j,\lambda}(t).x(t_k)=G(t)c^T. \tag{37} $$

So, we conclude

$$ \dot{x}_m(t)=\sum_{j=0}^{m}\sum_{k=0}^{m}h_{j,\lambda}^{-1}.w_{k,\lambda}.G_{j,\lambda}(t_k).\dot{G}_{j,\lambda}(t).x(t_k)=\dot{G}(t)c^T, \tag{38} $$

where $\dot{G}_{j,\lambda}(t)$ can be found from the Formula (11).

Also, the approximation of the Lagrange multiplier function $\gamma(t)$ can be written as follows:

$$ \gamma(t)\approx\gamma_m(t)=\sum_{j=0}^{m}\sum_{k=0}^{m}h_{j,\lambda}^{-1}.w_{k,\lambda}.G_{j,\lambda}(t_k).G_{j,\lambda}(t).\gamma(t_k)=G(t)d^T. \tag{39} $$

So, we conclude

$$ \dot{\gamma}_m(t)=\sum_{j=0}^{m}\sum_{k=0}^{m}h_{j,\lambda}^{-1}.w_{k,\lambda}.G_{j,\lambda}(t_k).\dot{G}_{j,\lambda}(t).\gamma(t_k)=\dot{G}(t)d^T. \tag{40} $$

With the help of the Equations (30) and (35), the fractional derivative for any $0<\alpha<1$, of state and Lagrange multiplier functions, at $t\in[0,1]$ can be approximated as follows:

$$ {}_{0}^{C}D_t^{\alpha}x_m(t)=\sum_{j=0}^{m}\sum_{k=0}^{m}h_{j,\lambda}^{-1}.w_{k,\lambda}.G_{j,\lambda}(t_k).x(t_k).{}_{0}^{C}D_t^{\alpha}G_{j,\lambda}(t), \tag{41} $$

$$ {}_{0}^{C}D_t^{\alpha}x_m(t)=\widehat{G}_{left}(t)c^T, $$

$$ {}_{t}^{R}D_1^{\alpha}\gamma_m(t)=\sum_{j=0}^{m}\sum_{k=0}^{m}h_{j,\lambda}^{-1}.w_{k,\lambda}.G_{j,\lambda}(t_k).\gamma(t_k)\frac{(-1)^{n+j+1}}{\Gamma(n-\alpha)}\frac{d}{dt}\widehat{G}_{j,\lambda}^{\alpha}(1-t'), \tag{42} $$

$$ {}_{t}^{R}D_1^{\alpha}\gamma_m(t)=\widehat{G}_{right}(1-t)d^T, $$

such that unknown vectors c^T and d^T expended on the shifted Gegenbauer Gauss nodes t_k which can be wrttin as:

$$ \mathbf{c^T}=\begin{bmatrix} \sum_{k=0}^{m}h_{0,\lambda}^{-1}.w_{k,\lambda}.G_{0,\lambda}(t_k).x(t_k) \\ \sum_{k=0}^{m}h_{1,\lambda}^{-1}.w_{k,\lambda}.G_{1,\lambda}(t_k).x(t_k) \\ \vdots \\ \sum_{k=0}^{m}h_{m,\lambda}^{-1}.w_{k,\lambda}.G_{m,\lambda}(t_k).x(t_k) \end{bmatrix}, \quad \mathbf{d^T}=\begin{bmatrix} \sum_{k=0}^{m}h_{0,\lambda}^{-1}.w_{k,\lambda}.G_{0,\lambda}(t_k).\gamma(t_k) \\ \sum_{k=0}^{m}h_{1,\lambda}^{-1}.w_{k,\lambda}.G_{1,\lambda}(t_k).\gamma(t_k) \\ \vdots \\ \sum_{k=0}^{m}h_{m,\lambda}^{-1}.w_{k,\lambda}.G_{m,\lambda}(t_k).\gamma(t_k) \end{bmatrix}, $$

where

$$ \mathbf{c^T}=\begin{bmatrix} h_{0,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{0,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{0,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ h_{1,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_o) & h_{1,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{1,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{m,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{m,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \end{bmatrix}.\begin{bmatrix} x(t_0) \\ x(t_1) \\ \vdots \\ x(t_m) \end{bmatrix}, $$

$$ \mathbf{d^T}=\begin{bmatrix} h_{0,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{0,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{0,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ h_{1,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_o) & h_{1,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{1,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{m,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{m,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \end{bmatrix}.\begin{bmatrix} \gamma(t_0) \\ \gamma(t_1) \\ \vdots \\ \gamma(t_m) \end{bmatrix}. $$

From above formulas for vectors c and d, it's clear that the state and adjoin functions $x(t_k)$, $\gamma(t_k)$, are unknowns values for all $k=0,1,\ldots,m$.

$$ G(t)=[G_0(t),G_1(t),\ldots,G_m(t)], $$

$$ \dot{G}(t)=[\dot{G}_0(t),\dot{G}_1(t),\ldots,\dot{G}_m(t)], $$

$$ \widehat{G}_{left}(t)=[{}_{0}^{C}D_t^{\alpha}G_{0,\lambda}^{\alpha}(t),{}_{0}^{C}D_t^{\alpha}G_{1,\lambda}^{\alpha}(t),\ldots,{}_{0}^{C}D_t^{\alpha}G_{m,\lambda}^{\alpha}(t)], $$

$$ \widehat{G}_{right}(1-t)=[\frac{(1)}{\Gamma(1-\alpha)}\frac{d}{dt}\widehat{G}_{0,\lambda}^{\alpha}(1-t),\frac{(-1)}{\Gamma(1-\alpha)}\frac{d}{dt}\widehat{G}_{1,\lambda}^{\alpha}(1-t),\ldots,\frac{(-1)^{m+2}}{\Gamma(1-\alpha)}\frac{d}{dt}\widehat{G}_{m,\lambda}^{\alpha}(1-t)]. $$

Now, to apply the spectral collocation strategy let the vector of time t represents the same nodes of the shifted Gegenbauer Gauss nodes. The best advantage of this strategy is that we don't need to derive operational matrices of differentiation and this method can be implemented in any mathematical software.

Let us define the shifted Gegenbauer Gauss nodes with the wight function $(t^2-t)^{\lambda-1/2}$ on the interval [0,1] as: $\mathbf{t}=\{\tau_i,\ i=0,\ldots,m\}$ where τ_i's are the zeros of $\widehat{G}_{m+1,\lambda}(t)$. By collocating strategy at nodes $\tau_i,\ i=0,\ldots,m$, the system of fractional problem (10)-(10) in matrix form will be:

$$ \mathrm{A}.\dot{G}(t)c^T+\mathrm{B}.\widehat{G}_{left}(t)c^T=M(t), \tag{43} $$

$$\text{A}.\dot{G}(t)\, d^T - \text{B}.\widehat{G}_{right}(1-t)\, d^T = -N(t), \tag{44}$$

where

$$G(t)=\begin{bmatrix} G_{0,\lambda}(\tau_0) & G_{0,\lambda}(\tau_1) & & G_{0,\lambda}(\tau_m) \\ G_{1,\lambda}(\tau_0) & G_{1,\lambda}(\tau_1) & & G_{1,\lambda}(\tau_m) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m,\lambda}(\tau_0) & G_{m,\lambda}(\tau_1) & & G_{m,\lambda}(\tau_m) \end{bmatrix},$$

$$\dot{G}(t)=\begin{bmatrix} \dot{G}_{0,\lambda}(\tau_0) & \dot{G}_{0,\lambda}(\tau_1) & & \dot{G}_{0,\lambda}(\tau_m) \\ \dot{G}_{1,\lambda}(\tau_0) & \dot{G}_{1,\lambda}(\tau_1) & & \dot{G}_{1,\lambda}(\tau_m) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{G}_{m,\lambda}(\tau_0) & \dot{G}_{m,\lambda}(\tau_1) & & \dot{G}_{m,\lambda}(\tau_m) \end{bmatrix},$$

$$\widehat{G}_{left}(t)=\begin{bmatrix} b^+_{00} & b^+_{01} & \dots & b^+_{0m} \\ b^+_{10} & b^+_{11} & \dots & b^+_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ b^+_{n0} & b^+_{n1} & \dots & b^+_{mm} \end{bmatrix}, \quad \widehat{G}_{right}(1-t)=\begin{bmatrix} b^-_{00} & b^-_{01} & \dots & b^-_{0m} \\ b^-_{10} & b^-_{11} & \dots & b^-_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ b^-_{m0} & b^-_{m1} & \dots & b^-_{mm} \end{bmatrix},$$

where

$$b^+_{jk}=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\widehat{G}_{j,\lambda}(\tau_k)-(-1)^j\frac{\Gamma(j+2\lambda)}{\Gamma(2\lambda)\,j!}\frac{\tau_k^{-\alpha}}{\Gamma(1-\alpha)}, \quad b^-_{jk}=\frac{(1)}{\Gamma(1-\alpha)}\frac{d}{dt}\widehat{G}_{j,\lambda}(1-\tau_k),$$

such that $\widehat{G}_{j,\lambda}(\tau_k)$ can be generated by three- recurrenc formula (24),(25), and (26) to the left Riemann-Liouvill fractional derivative of order α. and

$$M(t)=\begin{bmatrix} M(\tau_0,x_n(\tau_0),\gamma_n(\tau_0)) \\ M(\tau_1,x_n(\tau_1),\gamma_n(\tau_1)) \\ \vdots \\ M(\tau_m,x_m(\tau_m),\gamma_m(\tau_m)) \end{bmatrix}, \quad N(t)=\begin{bmatrix} N(\tau_0,x_n(\tau_0),\gamma_n(\tau_0)) \\ N(\tau_1,x_n(\tau_1),\gamma_n(\tau_1)) \\ \vdots \\ N(\tau_m,x_m(\tau_m),\gamma_m(\tau_m)) \end{bmatrix}.$$

The functions $A(t)$ and $B(t)$ can be written in a matrix form as:

$$A(t)=\operatorname{diag}\big(A(\tau_0),\dots,A(\tau_m)\big) \quad and \quad B(t)=\operatorname{diag}\big(B(\tau_0),\dots,B(\tau_m)\big).$$

In the final, main problem (9)-(9) will reduce to the system of algebraic equations (43)-(44), then can be solved by any simple method for solving the system of algebraic equations.

3.3. Error upper bound of the approximate fractional derivatives

To achieve this objective, we need the following theorem.

Theorem 3.1. *If H is a Hilbert space, and X is the closed subspace defined on H, and let* $\{x_j\}_{j=1}^m$ *be a basis for X of dimension m, if x belongs to H and* x_0 *is the unique best approximation to x out of X, then*

$$\| x-x_0 \|_2^2=\frac{D(x,x_1,x_2,\dots,x_m)}{D(x_1,x_2,\dots,x_m)}$$

where

$$D(x,x_1,x_2,...,x_m)=\begin{vmatrix} \langle x,x\rangle & \langle x,x_1\rangle & \dots & \langle x,x_m\rangle \\ \langle x_1,x\rangle & \langle x_1,x_1\rangle & \dots & \langle x_1,x_m\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle x_m,x\rangle & \langle x_m,x_1\rangle & \dots & \langle x_m,x_m\rangle \end{vmatrix}.$$

The proof of the above theorem can be found in [13].

Now, to evaluate an upper error bound for the approximating fractional derivatives, let $\{G_{k,\lambda}(t), k=0,1,...,m\}$ as the basis of functions and $G_{k,\lambda}(t)$ is shifted Gegenbauer functions. A good feature of these basis functions is that the left and right fractional derivatives can be obtained in a closed form, which the upper error bound for fractional derivatives of $\varphi_k(t)$ is equal to zero. On the other hand, using Theorem 3.1, it is easy to evaluate the error upper bound of $x(t)$ such that $x(t)$ can be approximated by $x_m(t)$ as:

$$x(t)\approx x_m(t)=\sum_{j=0}^{m} G_{j,\lambda}(t)c_j,$$

$$\| x(t)-x_m(t)\|_2 =\left(\frac{D(x(t),G_{0,\lambda}(t),G_{1,\lambda}(t),...,G_{m,\lambda}(t))}{D(G_{0,\lambda}(t),G_{2,\lambda}(t),...,G_{m,\lambda}(t))}\right)^{\frac{1}{2}},$$

where $\mathbf{c}_j\in\mathbb{R}^{n_x}$ are unknown vectors.

On the other hand, to find the truncation error of approximating a smooth function by using the shifted Gegenbauer expansion series can use the next two theorems:

Theorem 3.2. *The left fractional Caputo derivative of order* α *for shifted Gegenbauer functions can be written in terms of the shifted Gegenbauer polynomials themselves.*

Proof: To prove the theorem, the closed-form for shifted Gegenbauer polynomials (2) can be used, then employing the property of linearity of the left fractional Caputo derivative is to obtained as follows:

$${}_0^cD_t^\alpha G_{m,\lambda}(t)=\sum_{k=0}^{m}(-1)^{m-k}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(j+k+2\lambda)}{\Gamma\left(k+\lambda+\frac{1}{2}\right)\Gamma(2\lambda)(m-k)!k!}\cdot {}_0^cD_t^\alpha t^k.$$

If $m-2k<1$, then

$${}_0^cD_t^\alpha G_{m,\lambda}(t)=0,$$

and if $m-2k\geq 1$ then

$${}_0^cD_t^\alpha G_{m,\lambda}(t)=\sum_{k=0}^{m}(-1)^{m-k}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(j+k+2\lambda\right)\Gamma\left(k+1\right)}{\Gamma\left(k+1+\alpha\right)\Gamma\left(k+\lambda+\frac{1}{2}\right)\Gamma\left(2\lambda\right)\left(m-k\right)!k!}t^{k-\alpha}.$$

Also, let

$$(t)^{k-\alpha}=\sum_{j=0}^{m}a_j.G_{j,\lambda}(t),$$

where a_j's are the coefficients of Gegenbauer polynomials, then

$${}_0^cD_t^\alpha G_{m,\lambda}(t)=\sum_{j=0}^{m}\sum_{k=0}^{m}a_j(-1)^{m-k}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(j+k+2\lambda)\Gamma(k+1)}{\Gamma(k+1+\alpha)\Gamma\left(k+\lambda+\frac{1}{2}\right)\Gamma(2\lambda)(m-k)!k!}G_{j,\lambda}(t). \tag{45}$$

Theorem 3.3. (Truncation Error). *Let* $x(t)\in C^\infty[0,1]$ *be approximated by the shifted Gegenbauer expansion series* (18)*, then for each* $t\in[0,1]$*, a number* $\xi(t)\in[0,1]$ *exists such that the truncation error* $R(t,\xi,m,\lambda)$ *is given by*

$$R(t,\xi,m,\lambda)=x(t)-x_m(t)=\frac{x^{(m+1)}(\xi)}{(m+1)!.K_{m+1,\lambda}}.G_{m+1,\lambda}(t)$$

where

$$K_{m+1,\lambda}=\frac{2^m.\Gamma(m+\lambda+1).\Gamma(2\lambda+1)}{\Gamma(m+2\lambda+1).\Gamma(\lambda+1)},$$

and $\| R(t,\xi,m,\lambda)\| \leq Max \frac{1}{(m+1)!.K^{\lambda}_{m+1}} \| x^{(m+1)}(\xi)\|, \quad -1\leq \xi \leq 1$.

The proof of the above theorem is the same as the proof of Theorem (??) can be found it in El-Hawary [1] for the Gegenbauer polynomials. Anyway, the next theorem helps us to get the truncation error of the Caputo fractional derivative for shifted Gegenbauer functions.

Theorem 3.4. *Let* $x(t)\in C^{\infty}[0,1]$ *be approximated by the shifted Gegenbauer expansion series* (18)*, then for each* $t\in[0,1]$*, a number* $\xi(t)\in[0,1]$ *exists such that the truncation error of the Caputo fractional drivetive of shifted Gegenbauer polynomials* $E(t,\xi,m,\lambda,\alpha)$ *is given by*

$$E(t,\xi,m,\lambda,\alpha)=\frac{x^{(m+1)}(\xi)}{(m+1)!.K_{m+1,\lambda}}\sum_{j=0}^{m+1}\sum_{k=0}^{m+1}a_j f(\lambda,\alpha,j,k,m).G_{j,\lambda}(t)$$

where a_j is the coefficient of shifted Gegenbauer polynomials,

$$K_{m+1,\lambda}=\frac{2^m.\Gamma(m+\lambda+1).\Gamma(2\lambda+1)}{\Gamma(m+2\lambda+1).\Gamma(\lambda+1)},$$

$$f(\lambda,\alpha,j,k,m)=(-1)^{m+1-k}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(j+k+2\lambda)\Gamma(k+1)}{\Gamma(k+1+\alpha)\Gamma\left(k+\lambda+\frac{1}{2}\right)\Gamma(2\lambda)(m-k+1)!k!},$$

and

$$\| E(t,\xi,m,\lambda,\alpha)\| \leq Max\frac{1}{(m+1)!.K_{m+1,\lambda}}\sum_{j=0}^{m+1}\sum_{k=0}^{m+1}a_j.f(\lambda,\alpha,j,k,m).\| x^{(m+1)}(\xi)\|.$$

Proof:

$$E\left(t,\xi,m,\lambda,\alpha\right)=._0^C D_t^{\alpha}x\left(t\right)-_0^C D_t^{\alpha}x_m\left(t\right),$$

from the linearty property of Caputo fractional derivative, one obtains:

$$E(t,\xi,m,\lambda,\alpha)=._0^C D_t^{\alpha}[x(t)-x_m(t)].$$

By the help of Theorem (25), one gets:

$$E\left(t,\xi,m,\lambda,\alpha\right)=._0^C D_t^{\alpha}\left[\frac{x^{(m+1)}\left(\xi\right)}{(m+1)!.K_{m+1,\lambda}}.G_{m+1,\lambda}\left(t\right)\right],$$

$$E(t,\xi,m,\lambda,\alpha)=\frac{x^{(m+1)}(\xi)}{(m+1)!.K_{m+1,\lambda}}._0^C D_t^{\alpha}G_{m+1,\lambda}(t),$$

by using Theorem (24) will be get:

$$E(t,\xi,m,\lambda,\alpha)=\frac{x^{(m+1)}(\xi)}{(m+1)!.K_{m+1,\lambda}}\sum_{j=0}^{m+1}\sum_{k=0}^{m+1}a_j f(\lambda,\alpha,j,k,m).G_{j,\lambda}(t),$$

then this error is bounded since

$$\| E(t,\xi,m,\lambda,\alpha) \| = \frac{1}{(m+1)!.K_{m+1,\lambda}} \sum_{j=0}^{m+1}\sum_{k=0}^{m+1} a_j f(\lambda,\alpha,j,k,m). \| x^{(m+1)}(\xi) \| \| G_{j,\lambda}(u) \|$$

$$\| E(t,\xi,m,\lambda,\alpha) \| \le Max \frac{1}{(m+1)!.K_{m+1,\lambda}} \sum_{j=0}^{m+1}\sum_{k=0}^{m+1} a_j . f(\lambda,\alpha,j,k,m). \| x^{(m+1)}(\xi).$$

4. Numerical results

This section develops an algorithm of the collocation strategy method for the approximateing the solution of FOCPs. Two illustrative examples have been solved to show the efficiency and accuracy of the numerical method. The obtained results were compared with the analytical solution of the FOCP. The proposed method was implemented with MATLAB 2018a on a PC.

Algorithm of collocation strategy method to approximate FOCP (9)-(10):

Step one: Compute the Hamiltonian H and drive the necessary optimality conditions.
Step two: Transform interval $[a,t_f]$ to the interval [0,1] and approximate $x(t)$ and $\gamma(t)$ by shifted Gegenbauer polynomials.
Step three: Find shifted Gegenbauer Guss nodes on the interval [0,1].
Step four: Apply shifted Gegenbauer Guss nodes with the system of matrix equations (43)-(44), to get approximate solution of coupled system of matrix equations. Generally, the system of matrix will be:

$$c^T = [A.\dot{G}(t) + B.\widehat{G}_{letf}(t)]^{-1} M(t), \tag{46}$$

$$d^T = \left[A.\dot{G}(t) - (-1)B.\widehat{G}_{right}(1-t) \right]^{-1} N(t), \tag{47}$$

Step five: Find unknown vectors c^T, d^T using Matlab, then obtain $x(t)$, $\gamma(t)$ as follows:

$$\begin{bmatrix} x(t_0) \\ x(t_1) \\ \vdots \\ x(t_m) \end{bmatrix} = \begin{bmatrix} h_{0,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{0,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{0,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ h_{1,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_o) & h_{1,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{1,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{m,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{m,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \end{bmatrix}^{-1} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix},$$

$$\begin{bmatrix} \gamma(t_0) \\ \gamma(t_1) \\ \vdots \\ \gamma(t_m) \end{bmatrix} = \begin{bmatrix} h_{0,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{0,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{0,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ h_{1,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_o) & h_{1,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{1,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m,\lambda}^{-1}.w_{0,\lambda}.G_{0,\lambda}(t_0) & h_{m,\lambda}^{-1}.w_{1,\lambda}.G_{0,\lambda}(t_1) & & h_{m,\lambda}^{-1}.w_{m,\lambda}.G_{0,\lambda}(t_m) \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_m \end{bmatrix}.$$

Step six: Find the optimal control function $u(t)$ from Hamiltonian condition.

Step seven: Define the error function at nodes t_i by $Error(t_i) = | x(t_i) - x_m(t_i) |$, where $x(t)$ is the exact solution, then report the maximum absolute error at the end as $Error(t) = \max_{t_i} |Error(t_i)|$.

Because of the shifted Gegenbauer Guss nodes, it does not include the upper and lower bounds for the interval [0,1], or to get more flexibility and satisfy the boundary conditions let:

$$x(t) \approx x_m(t) := x_0 + (x_\ell - x_0)t + \sum_{j=1}^{m} c_j G_{j,\lambda}(t), \; where \;\; x(0) = x_0, x(1) = x_\ell,$$

$$\gamma(t) \approx \gamma_m(t) := \gamma_0 + (\gamma_\ell - \gamma_0)t + \sum_{j=1}^{m} d_j G_{j,\lambda}(t), \; where \;\; \gamma(0) = \gamma_0, \gamma(1) = \gamma_\ell.$$

Example 1: *Consider FOCPs as follows* [5]*:*

$$min \; J(u) = \frac{1}{2} \int_0^1 \left[\tau u(\tau) - \alpha x(\tau) - 2x(\tau) \right]^2 d\tau,$$

with dynamic system of fractional differential equation

$$\dot{x}(\tau) + {}_{0}^{C}D_{\tau}^{\alpha} x(\tau) = u(\tau) + \tau^{2},$$

with conditions $x(0)=0$, $x(??)=\dfrac{2}{\Gamma(3+\alpha)}$.

The exact solution of control and stat functions are given by $u^{*}(\tau)=\dfrac{2\tau^{\alpha+1}}{\Gamma(2+\alpha)}$ and $x^{*}(\tau)=\dfrac{2\tau^{\alpha+2}}{\Gamma(3+\alpha)}$. Let's the Hamiltonian functions define as:

$$H(\tau, x, u, \gamma) = [\tau u(\tau) - (\alpha+2).x(\tau)]^{2} + \gamma(\tau).[u(\tau) + \tau^{2}].$$

Then the necessary and optimality conditions in terms of a Hamiltonian for the FOCP will be changed to a system of fractional differential equations as follows:

$$\dot{x}(\tau) + {}_{0}^{C}D_{\tau}^{\alpha} x(\tau) + \frac{\gamma(\tau)}{2\tau^{2}} - \frac{(\alpha+2)x(\tau)}{\tau} - \tau^{2} = 0,$$

$$\dot{\gamma}(\tau) - {}_{1}^{R}D_{\tau}^{\alpha} \gamma(\tau) + \frac{(\alpha+2)}{\tau}\gamma(\tau) = 0,$$

$$x(0)=0,\ x(1)=\frac{2}{\Gamma(3+\alpha)},\ \gamma(0)=0,\ \gamma(1)=0, \tau \neq 0.$$

To approximate the state and control functions of above fractional system, apply the presented method in terms of the collocation strategy. Figures 1 shows that the obtained control and state functions with their error functions for $\lambda = 1.5$, $m = 15$, and $\alpha = 0.7$. The Figure 2 shows that the obtained control function $u_{15}(t)$ for different values of $\alpha = 0.6, 0.7, 0.8, 0.9$.

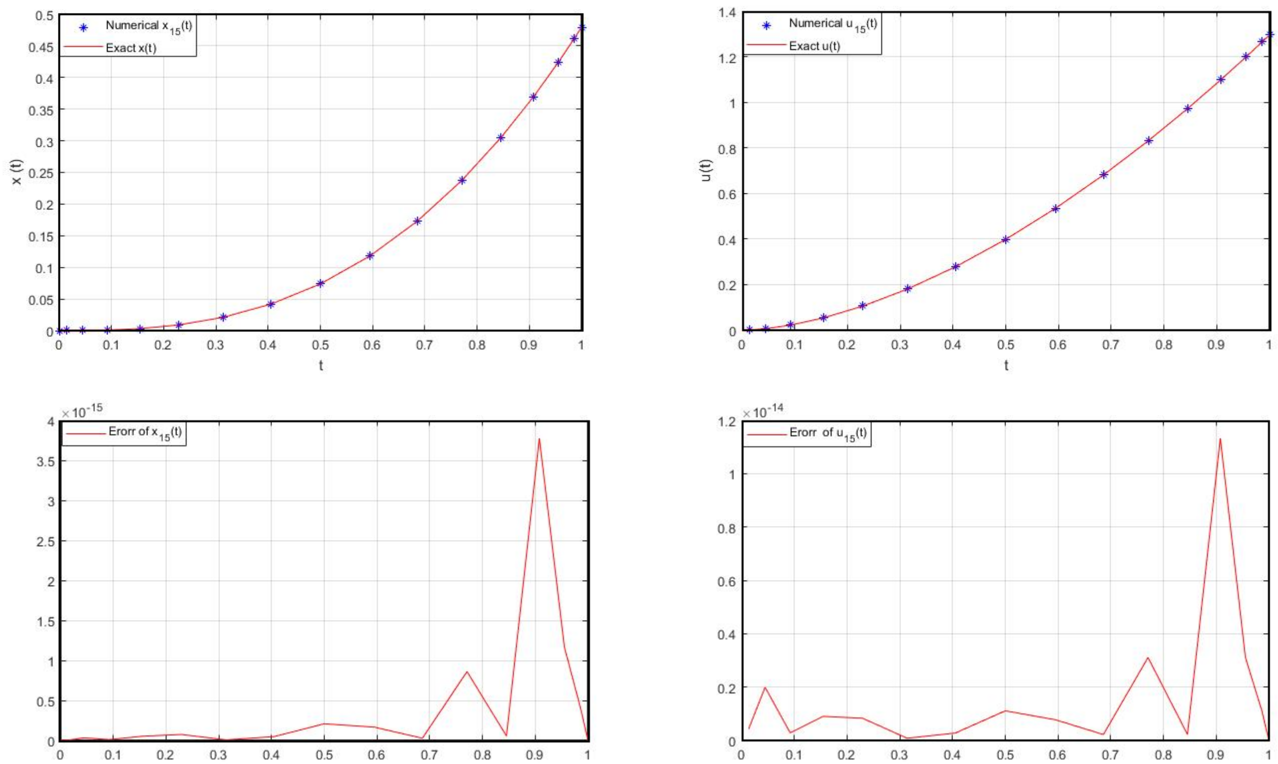

Figure 1: The obtained state and control functions with error functions from the presented method, for $m = 15$ and $\alpha = 0.7$.(Example 1).

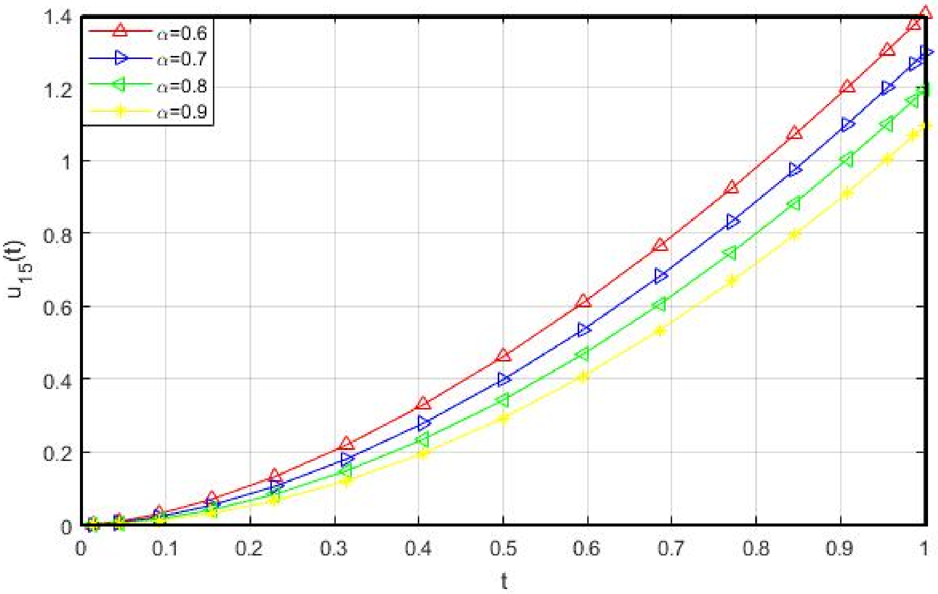

Figure 2: The obtained control function from the presented method for different values of α and m = 15.(Example 1).

Example 2: *Consider FOCPs as follows* [5]*:*

$$min\ \ J(u)=\int_0^1[(x_1(\tau)-\tau^{3.3})^2+(x_2(\tau)-(1+\tau)^2)^2+(u(\tau)-x_1(\tau))^2]d\tau,$$

with the system of fractional equation

$$\dot{x}_1(\tau)+\ {}_0^CD_\tau^\alpha x_1(\tau)=3.3\tau^{-1}x_1(\tau)+\frac{\Gamma(4.3)}{\Gamma(4.3-\alpha)}\tau^{3.3-\alpha},$$

$$\dot{x}_2(\tau)-{}_0^CD_\tau^\alpha x_2(\tau)=2(\tau+1)-\frac{\Gamma(3)}{\Gamma(3-\alpha)}\tau^{2-\alpha}-\frac{4}{\Gamma(2-\alpha)}\tau^{1-\alpha},$$

$$x_1(0)=0,\ \ x_1(1)=1,\ \ x_2(0)=1,\ \ x_2(1)=4,$$

the analytical solutions of the problem are:

$$x_1^*(\tau)=\tau^{3.3},\ \ x_2^*(\tau)=(\tau+1)^2,\ \ u^*(\tau)=\tau^{3.3}.$$

To approximate the solution of FOCP with boundary value conditions, apply the algorithm of collocation strategy method. The Hamiltonian equation can be defined as:

$$H(\tau,x,u,\lambda)=F(\tau,x,u)+\lambda G(\tau,x,u),$$

$$H=(x_1(\tau)-\tau^{3.3})^2+(x_2(\tau)-(\tau+1)^2)^2+\ (u(\tau)-x_1(\tau))^2+\gamma_1(\tau)[3.3\tau^{-1}x_1(\tau)+\frac{\Gamma(4.3)}{\Gamma(4.3-\alpha)}\tau^{3.3-\alpha}]+$$

$$\gamma_2(\tau).\left[2(\tau+1)-\frac{\Gamma(3)}{\Gamma(3-\alpha)}\tau^{2-\alpha}-\frac{4}{\Gamma(2-\alpha)}\tau^{1-\alpha}\right].$$

Then, the necessary and optimality conditions in terms of a Hamiltonian for the FOCP will be changed to a system of fractional differential equations as follows:

$$\dot{x}_1(\tau) + {}_0^C D_\tau^\alpha x_1(\tau) = 3.3\tau^{-1}x_1(\tau) + \frac{\Gamma(4.3)}{\Gamma(4.3-\alpha)}\tau^{3.3-\alpha},$$

$$\dot{x}_2(\tau) - {}_0^C D_\tau^\alpha x_2(\tau) = 2(\tau+1) - \frac{\Gamma(3)}{\Gamma(3-\alpha)}\tau^{2-\alpha} - \frac{4}{\Gamma(2-\alpha)}\tau^{1-\alpha},$$

$$\dot{\lambda}_1(\tau) + {}_\tau^C D_\tau^\alpha \gamma_1(\tau) = -2\left[x_1(\tau) - \tau^{3.3}\right] + 2\left[u(\tau) - x_1(\tau)\right] + 3.3\tau^{-1}\gamma_1(\tau),$$

$$\dot{\gamma}_2(\tau) - {}_\tau^C D_1^\alpha \lambda_2(\tau) = -2[x_2(\tau) - (\tau+1)^2],$$

$$x_1(0) = 0,\ x_1(1) = 1,\ x_2(0) = 1,$$

$$x_2(1) = 4,\ \gamma_1(1) = 0,\ \gamma_2(1) = 0,$$

with the optimal control law $\frac{\partial H}{\partial u}=0$, and then $u(t)=x_1(t)$. By applying the algorithm of collocation method, Figures 3 represents the obtained control and state functions $x_1(t)$, $x_2(t)$, $u(t)$ with error functions by the presented method for $\lambda = 1.5$, $m = 10$, and $\alpha = 0.6$. Figure 4 represents norm-2 of the error $E_m^2(u)$ for the different values of $m = 1,2,\ldots,150$, $\alpha = 0.6$ and

$$E_m^2(u) := \left(\frac{1}{m}\sum_{i=1}^{m}\left(u(\tau_i) - u_m(\tau_i)\right)^2\right)^{1/2}.$$

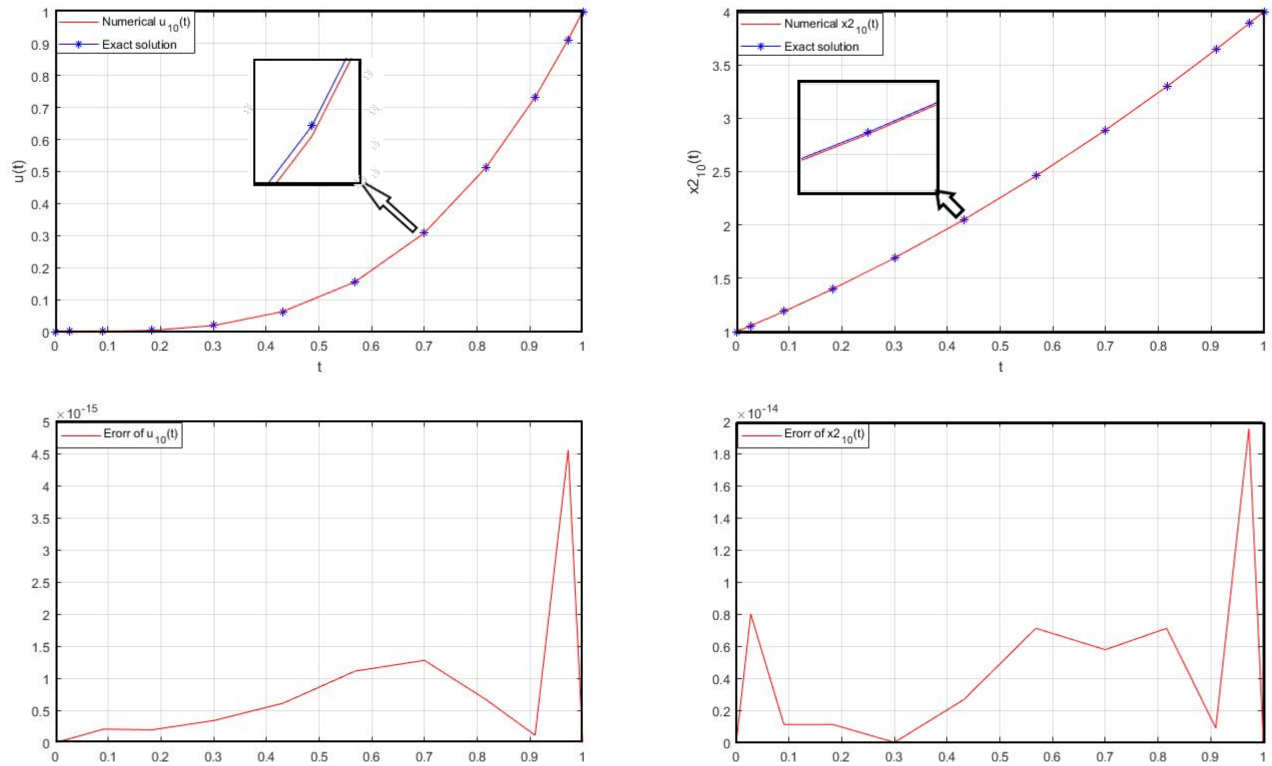

Figure 3: Obtained state, control, and error functions from the presented method for m = 10 and α = 0.6.(Example 2).

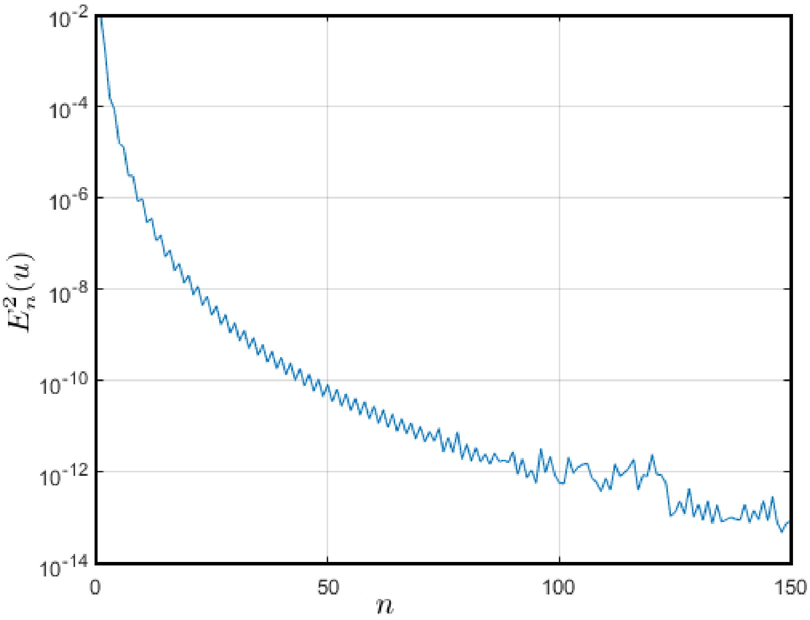

Figure 4: The figure of norm-2 of the error for the different values of n = 1,2,...,150, and α = 0.6 (Example 2).

5. Conclusion

A new approximate formula of the Caputo fractional derivatives of the shifted Gegenbauer functions was derived and the GG collocation method was used to approximate solutions of a category of FOCPs. The properties of shifted Gegenbauer polynomials and three recurrence formulas were used to generate the left and right Riemann-Liouville fractional integrals. Global approximations to functions defined on the interval [0,1] were constructed. The special attention was to study the convergence analysis and estimating an error upper bound of the presented formulas. The discretization of the FOCP were inserted and Illustrative numerical examples were integrated to show the validity and applicability of this new technique with obtained good approximate solutions and the number of operations was less than $O(N^3)$.

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References

[1] H.M. El-Hawary, M.S. Salim, H.S. Hussien, An optimal ultraspherical approximation of integrals, *Int. J. Comput. Math.* **76** (2000) 219–237.
[2] K.T. Elgindy, K.A. Smith-Miles, *Optimal Gegenbauer quadrature over arbitrary integration nodes*, 2012 (submitted for publication).
[3] Li, Changpin, and Fanhai Zeng. *Numerical methods for fractional calculus.* Vol. 24. CRC Press, 2015.
[4] Pooseh S, Almeida R, Torres DFM. A numerical scheme to solve fractional optimal control problems. In: Conference Papers in Mathematics, 2013; 2013. 10p [Article ID:165298].

[5] salh Ali, Mushtaq, and Mohammed K. Almoaeet. "An Indirect Spectral Collocation Method Based on Shifted Jacobi Functions for Solving Some Class of Fractional Optimal Control Problems." *Journal of Physics: Conference Series*. Vol. 1818. No. 1. IOP Publishing, 2021.

[6] Kareem T. Elgindy, Kate A. Smith-Miles Optimal Gegenbauer quadrature over arbitrary integration nodes, *Journal of Computational and Applied Mathematics* **242** (2013) 82–106.

[7] E-Gindy, T., H. Ahmed, and Marina Melad. Shifted Gegenbauer operational matrix and its applications for solving fractional differential equations. *Journal of the Egyptian Mathematical Society* **26.1** (2018): 72–90.

[8] Hafez, R. M., and Y. H. Youssri. Shifted Gegenbauer—Gauss collocation method for solving fractional neutral functional-differential equations with proportional delays. *Kragujevac Journal of Mathematics* **46.6** (2022): 981–996.

[9] K.T. Elgindy, Kate A. Smith-Miles, Solving boundary value problems, integral, and integro-differential equations using Gegenbauer integration matrices, *Journal of Computational and Applied Mathematics* **237** (2013) 307–325.

[10] J.J. Trujillo, On Riemann-Liouvill Generalized Taylor's Formula, *Journal of Mathematical Analysis and Applications*, **231**, 255–265(1999).

[11] Kanti B Datta, M Mohan *Orthogonal functions in systems and control* book, World Scientific 1995.

[12] B. Spain, M.G. Smith *Functions of mathematical physics*, Van Nostrand Reinhold Company, London, 1970. Chapter 10 deals with Laguerre polynomials.

[13] Kreyszig, Erwin. *Introductory functional analysis with applications*. Vol. 17. John Wiley and Sons, 1991.

[14] S. S. Bayin, *Mathematical Methods in Science and Engineering*, Wiley, (2006), Chapter 3.

[15] Shakoor Pooseh, Ricardo Ameida and Delfim F. M. Torres, *Numerical approximations to fractional problems of the calculus of variations and optimal control*, Chapter V, Fractional calculus in analysis, Dynamics and optimal control (Editor: Jacky Cresson), Series: Mathematics Research Developments, Nova Science Publishers, New York, 2014.

[16] O.P. Agrawa, Aquadratic numerical scheme for fractional obtimal control problems, *Trans. ASME, J. Dyn. Syst. Meas. control* **130** (2008), No.1,011010-011016.

[17] A. Lotfi, S.A. Yousefi, M. Dehghan, Numerical solution of a class of fractional optimal control problems via the Legender orthonormal basis combined with the operational matrix and the Gauss quadrathure rule, *J. comput. Appl. Math.*, **250**, pp.143–160, 2013

[18] Elgindy, Kareem T. High–order, stable, and efficient pseudospectral method using barycentric Gegenbauer quadratures, *Applied Numerical Mathematics*, **113** (2017): 1–25 .

[19] Yaghoubi, S., H. Aminikhah, and K. Sadri, A spectral shifted gegenbauer collocation method for fractional pantograph partial differential equations and its error analysis, *Sādhanā* **48.4** (2023): 213

[20] Sadri, Khadijeh, et al., A robust scheme for Caputo variable-order time-fractional diffusion-type equations, *Journal of Thermal Analysis and Calorimetry* **148.12** (2023): 5747–5764.

[21] Sadri, Khadijeh, et al., A pseudo-operational collocation method for variable-order time-space fractional KdV–Burgers–Kuramoto equation, *Mathematical Methods in the Applied Sciences* (2023).

[22] Jie Shen Tao Tang, Li-Lian Wang, Spectral Methods Algorithms, Analysis and Applications, *Springer,* (2011).

[23] Yaghoubia, Sara, Hossein Aminikhaha, and Khadijeh Sadric, An effective operational matrix method based on shifted sixth-kind Chebyshev polynomials for solving fractional integro-differential equations with a weakly singular kernel, *Filomat* **38.7** (2024): 2457–2486.