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On a new version of grierer-meinhardt model using fractional discrete calculus

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Abstract

As mathematical models of biological pattern generation, this study investigates the dynamics of the fractional discrete Gierer-Meinhardt reaction-diffusion system. After deriving the discrete non-integer fractional variant of the Gierer-Meinhardt system and establishing that the system has a unique equilibrium, we analyze the system's local asymptotic behavior in both the presence and absence of diffusion. The conditions for the global stability of the steady-state solution are determined using relevant approaches and the Lyapunov method. Throughout the study, two comprehensive biological models and simulations are employed to validate the utility of the considered approach.

Key words and phrases. Discrete fractional-order reaction–diffusion Gierer-Meinhardt model; Second order difference operator; Caputo ћ-difference operator; Local-global asymptotic stability; Lyapunov method

Mathematics Subject Classification:

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1. Introduction

Nonlinear reaction-diffusion equations provide a variety of applications in pattern development in the fields of chemistry, physics, and biology [1–3]. Reaction-diffusion equations are also applicable to a wide range of additional issues of interest to the targeted research community [4, 5]. Its theoretical framework may be derived from [6–8].

Fractional reaction-diffusion models have captured a lot of debate during the past several years [9], this fractional reaction-diffusion models display self-organization occurrences on one hand and add an additional component to these systems known as fractional derivative indices., that provides an increased amount of freedom for many different self-organization phenomena on the other hand. At the same time, both the analytical and numerical processes for assessing such fractional reaction diffusion models are substantially more sophisticated.

Discrete fractional calculus has gained momentum as a distinct and engaging realm of mathematical exploration in the past decade [10–13]. Its appeal has attracted mathematicians, scholars, and researchers, as it finds relevance in diverse fields like biology, ecology, and applied sciences [14–17]. What adds to the allure of this domain is its discrete fractional operators, which serve as versatile tools for dissecting real-world complexities. Notably, these operators have been employed to unravel challenges across various sectors, as indicated in references [18, 25]. Meanwhile, partial difference equations have gotten a lot of interest in recent years because of their relevance in applications incorporating population dynamics as well as regional migrations, chemical processes, and even computing and analyzing finite difference equations [26]. However, there has been little research on discrete reaction-diffusion models. In [27], the fractional discrete Glycolysis system was investigated using the second order difference operator. For other works, the reader may refer to [28, 31].

Among the most well-known models in biological pattern generation is the Gierer-Meinhardt model. The scientific observation and analysis of the kinetic processes led them to construct a Gierer-Meinhardt model that includes a saturating factor. Such a saturating term restricts the activator's intensity to a maximum amount, allowing the interaction capacity to be controlled. The activation region might change in relation to the overall structure's size. The Gierer-Meinhardt model with a saturating term is believed to be especially appropriate for modeling biological systems, particularly in terms of controlling features such as size maintenance and structure spacing regulation [32]. Numerous studies have been done on the Grierer-Meinhardt model; for instance, in [33], biological pattern creation in plants was discussed. The model was also investigated in [34, 35], which demonstrated the global presence of solutions to a particular case. A unique nonnegative global solution to the Grierer-Meinhardt system was demonstrated in [36].

The basic objective of this study is to offer the fractional discrete reaction diffusion Gierer-Meinhardt model, and also to thoroughly investigate the dynamical behaviors such as local and global stability of the equilibrium of the systems under consideration. Thus, here is the summary of this paper: In Section 2, the discrete fractional reaction diffusion Gierer-Meinhardt model is described. In Section 3, we explored the local in the absence and presence of the influence of spatial diffusion. In Section 4, we adress global stability of the proposed system. In Section 5, numerical approximations for the investigated system with certain specific parameter values and beginning circumstances are also performed to validate our theories.

2. The fractional discrete Gierer–Meinhardt model

Due to the fact that it has been well reported over the last years that combining fractional discrete calculus with the dynamics of systems may provide incredible results. We investigate the discrete reaction-diffusion fractional Gierer-Meinhardt system. We provide first the following important definition.

Definition 1. *[23] The Caputo ℏ-difference operator is outlined by*

$${}^{C}_{\hbar}\Delta^{\mathfrak{c}}_{a}\chi(t) = {}_{\hbar}\Delta_{a}^{-(n-\mathfrak{c})}\Delta^{n}_{\hbar}\chi(t), \quad t \in (\hbar\mathbb{N})_{a+\mathfrak{c}\hbar}, \quad 0 < \mathfrak{c} < 1, \tag{1}$$

and the c^{th}-order ℏ-sum is expressed by

$${}_{\hbar}\Delta_{\mathfrak{a}}^{-\mathfrak{c}}\chi(t) = \frac{\hbar}{\Gamma(\mathfrak{c})}\sum_{\frac{t}{\hbar}-\mathfrak{c}}^{s=\frac{a}{\hbar}}(t-\sigma(s\hbar))^{(\mathfrak{c}-1)}\chi(s\hbar), \quad \sigma(s\hbar) = (s+1)\hbar, \tag{2}$$

with the set $(\hbar\mathbb{N})_{\mathfrak{a}+\mathfrak{c}\hbar}$ *defined by*

$$(\hbar\mathbb{N})_{\mathfrak{a}+\mathfrak{c}\hbar} = \{\mathfrak{a} + (1-\mathfrak{c})\hbar, \mathfrak{a} + (2-\mathfrak{c})\hbar, \ldots\}.$$

The forward difference operator $\Delta_{\hbar}$ *is then defined as*

$$\Delta_{\hbar}\chi(t) = \frac{\chi(t+1)-\chi(t)}{\hbar}; \quad \ell \in \mathbb{N} \tag{3}$$

The Gierer–Meinhardt reaction-diffusion system, as is well-known, was proposed in [37, 38] as follows:

$$\begin{cases} u_t = a_1\Delta u + \sigma - \mu u + \dfrac{u^p}{v^q}, & \mathfrak{x} \in \Omega, \quad t > 0, \\ v_t = a_2\Delta v - \nu v + \dfrac{u^r}{v^s}, & \mathfrak{x} \in \Omega, \quad t > 0, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^n$ represents a bounded domain with smooth boundary $\partial\Omega$, $a_1, a_2 > 0$, $\mu, \nu, \sigma > 0$, while the indices $p, q, r,$ and s are positive such that $p > 1$, with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \mathfrak{x} \in \partial\Omega, t > 0,$$

and the initial conditions

$$u(\mathfrak{x},0) = \phi_0(\mathfrak{x}) > 0, v(\mathfrak{x},0) = \phi_1(\mathfrak{x}) > 0, \quad \mathfrak{x} \in \Omega.$$

Because real-world data is discrete, the discrete system is better suited to simulating the state of biological process. Therefore, we rely on the model (4) and the method of discretization employed in [28]. Provided that $\mathfrak{x} \in [0, L]$, we can get $\mathfrak{x}_{i+1} = \mathfrak{x}_i + k$ for $i = 0, 1, 2, \ldots, m$, and by applying the central difference formula for $\mathfrak{x}, \frac{\partial^2 u(\mathfrak{x},t)}{\partial \mathfrak{x}}$ and $\mathfrak{x}, \frac{\partial^2 v(\mathfrak{x},t)}{\partial \mathfrak{x}}$, we can approximate it as

$$\begin{cases} \dfrac{\partial^2 u(\mathfrak{x},t)}{\partial \mathfrak{x}^2} \approx \dfrac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{k^2}, \\ \dfrac{\partial^2 v(\mathfrak{x},t)}{\partial \mathfrak{x}^2} \approx \dfrac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{k^2}. \end{cases} \tag{5}$$

Now, we can apply the following definition of the second-order difference operator of u_i and v_i provided in [39]:

$$\Delta 2\chi(\ell) = \chi(\ell+2) - 2\chi(\ell+1) + \chi(\ell), \ \ell \in \mathbb{N} \tag{6}$$

to obtain

$$\begin{cases} \dfrac{\partial^2 u(\mathfrak{x},t)}{\partial \mathfrak{x}^2} \approx \dfrac{\Delta^2 u_{i-1}(t)}{k^2}, \\ \dfrac{\partial^2 v(\mathfrak{x},t)}{\partial \mathfrak{x}^2} \approx \dfrac{\Delta^2 v_{i-1}(t)}{k^2}. \end{cases} \tag{7}$$

Thus, we can present the discrete reaction-diffusion fractional Gierer-Meinhardt system as shown below:

$$\begin{cases} {}_{\hbar}^{C}\Delta_{t_0}^{\mathfrak{c}} u_i(t) = \dfrac{a_1}{k^2}\Delta^2 u_{i-1}(t+\mathfrak{c}\hbar) + \sigma - \mu u_i(t+\mathfrak{c}\hbar) + \dfrac{u_i^p(t+\mathfrak{c}\hbar)}{v_i^q(t+\mathfrak{c}\hbar)}, \\ C\Delta_{t_0}^{\mathfrak{c}} v_i(t) = \dfrac{a_2}{k^2}\Delta^2 v_{i-1}(t+\mathfrak{c}\hbar) - \nu v_i(t) + \dfrac{u_i^r(t+\mathfrak{c}\hbar)}{v_i^s(t+\mathfrak{c}\hbar)}, \end{cases} \tag{8}$$

with the periodic boundary conditions

$$\begin{cases} u_0(t) = u_m(t), \quad u_1(t) = u_{m+1}(t), \\ v_0(t) = v_m(t), \quad v_1(t) = v_{m+1}(t), \end{cases} \tag{9}$$

and the initial condition

$$u_i(t_0) = \phi_1(x_i) \geq 0, \qquad v_i(t_0) = \phi_2(x_i) \geq 0.$$

3. Analytical results

3.1. Local stability

3.1.1. Local stability of the free diffusions system

In this part, we establish appropriate conditions for the local stability of the free diffusion model described below.

$$\begin{cases} {}_{\hbar}^{C}\Delta_{t_0}^{\mathfrak{c}} u(t) = \sigma - \mu u(t+\mathfrak{c}\hbar) + \dfrac{u^p(t+\mathfrak{c}\hbar)}{v^q(t+\mathfrak{c}\hbar)}, \\ {}_{\hbar}^{C}\Delta_{t_0}^{\mathfrak{c}} v(t) = -\nu v(t+\mathfrak{c}\hbar) + \dfrac{u^r(t+\mathfrak{c}\hbar)}{v^s(t+\mathfrak{c}\hbar)}, \end{cases} \tag{10}$$

with the initial conditions

$$u_0 \geq 0, \qquad v_0 \geq 0.$$

Before we work on the investigation of the desired stability, we need the next theorem.

Theorem 1. ([40])**.** *Let (u^*, v^*) be an equilibrium point of (10). If all the eigenvalues of $J^*_{(u^*,v^*)}$ are in $S_{\hbar}^{\mathfrak{c}}$, then (u^*, v^*) is asymptotically stable, where*

$$S_{\hbar}^{\mathfrak{c}} = \left\{ w \in \mathbb{C} : |Arg(w)| > \frac{\mathfrak{c}\pi}{2} \text{ or } |w| > \frac{2^{\mathfrak{c}}}{\hbar^{\mathfrak{c}}} \cos^{\mathfrak{c}}\left(\frac{Arg(w)}{\mathfrak{c}}\right) \right\}. \tag{11}$$

In order to discuss the stability we wish, we must identify the equilibrium point. It is worth noting that the equilibrium points of system (10) have the following property:

$$\begin{cases} \sigma - \mu u^* + \dfrac{u^{*p}}{v^{*q}} = 0, \\ -\nu v^* + \dfrac{u^{*r}}{v^{*s}} = 0. \end{cases} \tag{12}$$

Herein, system (10) has a unique positive equilibrium (u^*, v^*) that verifies $v^* = \left(\frac{1}{\nu} u^{*r}\right)^{\frac{1}{s+1}}$. Now, by considering the following two functions:

$$\begin{cases} \phi_0(u(t),v(t)) = \sigma - \mu u(t) + \dfrac{u^p(t)}{v^q(t)}, \\ \psi_0(u(t),v(t)) = -\nu v(t) + \dfrac{u^r(t)}{v^s(t)}, \end{cases}$$

we can obtain the Jacobin matrix of equilibrium point stated as follows:

$$J(u^*,v^*) = \begin{pmatrix} \dfrac{\partial \phi_0}{\partial u}(u,v) & \dfrac{\partial \phi_0}{\partial v}(u,v) \\ \dfrac{\partial \psi_0}{\partial u}(u,v) & \dfrac{\partial \psi_0}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -\mu + p\nu^{\frac{q}{s+1}} u^{*p-\frac{rq}{s+1}-1} & -q\nu^{\frac{q+1}{s+1}} u^{*p-\frac{rq+r}{s+1}} \\ r\nu^{\frac{s}{s+1}} u^{*\frac{r}{s+1}-1} & -\nu - s\nu \end{pmatrix}. \tag{13}$$

Theorem 2. *System (10) is locally asymptotically stable if the following conditions hold:*

- *If* $\mu\nu^2 + s\mu\nu > (p(\nu+s) - rq)\nu^{1+\frac{q}{s+1}} u^{*p+\frac{s-rq}{s+1}-1}$ *and* $p\nu^{\frac{q}{s+1}} u^{*p-\frac{rq}{s+1}-1} < \nu^2 + s\nu + \mu$
- *If* $\mu\nu^2 + s\mu\nu > (p(\nu+s) - rq)\nu^{1+\frac{q}{s+1}} u^{*p+\frac{s-rq}{s+1}-1}$ *and* $p\nu^{\frac{q}{s+1}} u^{*p-\frac{rq}{s+1}-1} < \nu^2 + s\nu + \mu.$

Proof. We start first by proving that system (10) has a unique positive equilibrium. For this purpose, we note that since (u^*, v^*) is an equilibrium of system (10), then it will satisfy (12). This implies that u^* satisfying

$$\mu u^* - \sigma = \nu^{\frac{q}{s+1}} u^{*^{p-\frac{rq}{s+1}}}. \tag{14}$$

Define

$$I_1(u) = \frac{\nu^{\frac{q}{s+1}} u^{p-\frac{rq}{s+1}}}{\mu u - \sigma}. \tag{15}$$

If (14) is true, it is simply to demonstrate that $I_1(u)$ is a strictly decreasing function for $\mu u > \sigma$. Furthermore, the property

$$\lim_{\mu u \to \sigma^+} I_1(u) = \infty, \lim_{\mu u \to \infty} I_1(u) = 0$$

is satisfied. Now, if (14) has been fulfilled, then $I_2(u) = 1 + \nu u^*$, and it, on the other hand, will fulfill $I_2(0) = 1$ and increase on $(0, \infty)$. As a result, if (14) is met, the system has a unique positive constant equilibrium (u^*, v^*), where u^* is the unique positive root of (15), and $v^* = \left(\frac{1}{\nu} u^{*r}\right)^{\frac{1}{s+1}} - \nu v(t) + \frac{u^r(t)}{v^s(t)}$.

In the same regard, if we move on to the stability of (u^*, v^*), we find the characteristic equation of the Jacobian matrix (13) as follows:

$$\lambda^2 - \mathrm{tr}(J_{(u^*,v^*)})\lambda + \det(J_{(u^*,v^*)}) = 0. \tag{16}$$

To investigate the stability of the eigenvalues problem, we should calculate

$$\mathrm{tr}(J_{(u^*,v^*)}) = p\nu^{\frac{q}{s+1}} u^{*\,p-\frac{rq}{s+1}-1} - \nu - s\nu - \mu, \tag{17}$$

and

$$\det(J_{(u^*,v^*)}) = \mu\nu + s\mu\nu + (p(s+1) - \nu q)\nu^{1+\frac{q}{s+1}} u^{*\,p-\frac{rq}{s+1}-1}, \tag{18}$$

which leads us to the following discriminant:

$$\begin{aligned}\Delta &= (\mathrm{tr}(J_{(u^*,v^*)}))^2 - 4\det(J_{(u^*,v^*)}),\\ &= (p\nu^{\frac{q}{s+1}} u^{*\,p-\frac{rq}{s+1}-1} - \mu)^2 + 2\nu^{\frac{q}{s+1}-1} u^{*\,p-\frac{rq}{s+1}-1}(p(s+1) - 2rq) + \nu(s+1)(\nu(s+1) - 6\mu).\end{aligned}$$

Because the sign of Δ is crucial to investigate the stability of the system (10), we will discuss each case separately, depending on the sign of Δ.

- If $\Delta > 0$ and $\det(J_{(u^*,v^*)}) > 0$. As a result, the sign of $\mathrm{tr}(J_{(u^*,v^*)})$ defines the negativity of λ_1 and λ_2, which are real numbers, and they can be expressed as follows:

$$\lambda_1 = \frac{\mathrm{tr}(J_{(u^*,v^*)}) + \sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{\mathrm{tr}(J_{(u^*,v^*)}) - \sqrt{\Delta}}{2}. \tag{19}$$

  – If $\mathrm{tr}(J_{(u^*,v^*)} < 0$, then we have

$$\lambda_1 = \frac{\mathrm{tr}(J_{(u^*,v^*)}) + \sqrt{\Delta}}{2}, \qquad (\ *,\ *)$$

  Hence, we have $\mathrm{Arg}(\lambda_1) = \pi$. Now, due to λ_1 and λ_2 are both real numbers, it is self-evident that $\mathrm{Arg}(\lambda_1) = \mathrm{Arg}(\lambda_2) = \pi$. As a result and according to Theorem 1, we find that (u^*, v^*) is stable.
  – If $\mathrm{tr}(J_{(u^*,v^*)}) < 0$, then

$$\lambda_2 = \frac{\mathrm{tr}(J_{(u^*,v^*)}) + \sqrt{\Delta}}{2} > 0.$$

  Thus, we have $\mathrm{Arg}(\lambda_1) = \mathrm{Arg}(\lambda_2) = 0$, and based on Theorem 1, we can confirm that system (10) is unstable.
- If $\Delta < 0$ and $\det(J_{(u^*,v^*)}) > 0$, then the eigenvalues of the problem are defined as follows:

$$\lambda_1 = \frac{\mathrm{tr}(J_{(u^*,v^*)})}{2} + \frac{\sqrt{\Delta}}{2}i, \quad \lambda_2 = \frac{\mathrm{tr}(J_{(u^*,v^*)})}{2} - \frac{\sqrt{\Delta}}{2}i. \tag{20}$$

  We can then evaluate the desired solutions depending on the sign of $\mathrm{tr}(J_{(u^*,v^*)})$.
  – If $\mathrm{tr}(J_{(u^*,v^*)}) < 0$ or $\mathrm{tr}(J_{(u^*,v^*)}) > 0$, then the same case addressed previously will be followed, and hence system (10) is asymptotically stable.

– If $\mathrm{tr}(J_{(u^*,v^*)}) = 0$, then we have

$$Arg\left(\frac{-i\sqrt{-\Delta}}{2}\right) = Arg\left(\frac{i\sqrt{-\Delta^*}}{2}\right) = \frac{\pi}{2},$$

and so system (10) is asymptotically stable.

- If $\Delta = 0$ and $\det(J_{(u^*,v^*)}) > 0$, then $\mathrm{tr}(J_{(u^*,v^*)})$) cannot be equal to zero. The signs of λ_1 and λ_2 are identical to the sign of $\mathrm{tr}(J_{(u^*,v^*)})$. As a consequence, (u^*, v^*) is asymptotically stable for any $c \in (0, 1]$). Besides, $\mathrm{tr}(J_{(u^*,v^*)}) < 0$ and unstable if $\mathrm{tr}(J_{(u^*,v^*)}) > 0$.

3.1.2. Local stability of the diffusion system

In this section, we will prove that in the presence of diffusion that asserts (u^*, v^*) can be stable provided that some particular parameter conditions are hold. To do so, we will use the identical method as in [41], and we will start by computing the eigenvalues of the following equation:

$$\begin{cases} \Delta^2 \chi_{i-1}(t) + \theta_i \chi_i(t) = 0, \quad t \in \mathbb{N}, \\ \chi_0(t) = \chi_1(t), \quad t \in \mathbb{N}, \\ \chi_N(t) = \chi_{N-1}(t), \quad t \in \mathbb{N}. \end{cases} \tag{21}$$

Consequently, we can obtain

$$\begin{aligned} {}^{C}_{\hbar}\Delta^{\mathfrak{c}}_{t_0} u(t) &= \frac{a_1}{k^2}\theta_i u(t+\mathfrak{c}\hbar) + \sigma - \mu u(t+\mathfrak{c}\hbar) + \frac{u^p(t+\mathfrak{c}\hbar)}{v^q(t+\mathfrak{c}\hbar)}, \\ {}^{C}_{\hbar}\Delta^{\mathfrak{c}}_{t_0} v(t) &= \frac{a_2}{k^2}\theta_i v(t+\mathfrak{c}\hbar) - \nu v(t+\mathfrak{c}\hbar) + \frac{u^r(t+\mathfrak{c}\hbar)}{v^s(t+\mathfrak{c}\hbar)}. \end{aligned} \tag{22}$$

Linearizing the reaction-diffusion system (22) with respect to the steady-state (u^*, v^*) leads us to the matrix $J_{i(u^*,v^*)}$, which can be given as

$$J_{i(u^*,v^*)} = \begin{pmatrix} -\frac{a_1}{k^2}\theta_i - \mu + p\nu^{\frac{q}{s+1}} u^{*\,p-\frac{rq}{s+1}-1} & -q\nu^{\frac{q+1}{s+1}} u^{*\,p-\frac{rq+r}{s+1}} \\ r\nu^{\frac{s}{s+1}} u^{*\,\frac{r}{s+1}-1} & -\frac{a_2}{k^2}\theta_i - \nu - s\nu \end{pmatrix}. \tag{23}$$

Theorem 3. *System (8) is asymptotically stable at (u^*, v^*) under the following states:*

- *If we suppose that*

$$\left(\frac{1}{4} - \frac{a_1 a_1}{k^4}\right)\left(p\nu^{\frac{q}{s+1}} u^{*\,p-\frac{rq}{s+1}-1} - \mu\right)^2 < 2\nu^{\frac{q}{s+1}-1} u^{*\,p-\frac{rq}{s+1}-1}(p(s+1) - 2rq) + \nu(s+1)(\nu(s+1) - 6\mu),$$

and we have the following two cases:

– *If $a_1 < a_2$ and* $\frac{a_1}{k^2}\theta_i - \mu + p\nu^{\frac{q}{s+1}} u^{*\,p-\frac{rq}{s+1}-1} \geq 0$.

– *If $a_1 > a_2$ and* $\frac{a_1}{k^2}\theta_i - \mu + p\nu^{\frac{q}{s+1}} u^{*\,p-\frac{rq}{s+1}-1} \geq 0$. *In addition, the eigenvalues satisfy $Arg(\lambda_j(\theta_i)) > \frac{\mathfrak{c}\pi}{2}$, where*

$$\lambda_j(\theta_i) = \frac{\mathrm{tr}(J_{i(u^*,v^*)}) \pm \sqrt{\mathrm{tr}(J_{i(u^*,v^*)})^2 - 4\det(J_{i(u^*,v^*)})}}{2}, \quad j = 1,2.$$

- *If we suppose that*

$$\left(\frac{1}{4}-\frac{a_1a_1}{k^4}\right)\left(pv^{\frac{q}{s+1}}u*^{p-\frac{rq}{s+1}-1}-\mu\right)^2 > 2v^{\frac{q}{s+1}-1}u*^{p-\frac{rq}{s+1}-1}(p(s+1)-2rq)+v(s+1)(v(s+1)-6\mu),$$

and the eigenvalues are complex described as follows:

$$\lambda_j(\theta_i)=\frac{\mathrm{tr}(J_{i(u^*,v^*)})\pm i\sqrt{\mathrm{tr}(J_{i(u^*,v^*)})^2-4\det(J_{i(u^*,v^*)})}}{2},\quad j=1,2.$$

Besides, it must also satisfy $Arg(\lambda_j(\theta_i))>\frac{\mathrm{c}\pi}{2}$.

Proof. We have

$$\lambda^2(\theta_i)-\mathrm{tr}(J_{i(u^*,v^*)})\lambda(\theta_i)+\det(J_{i(u^*,v^*)})=0. \tag{24}$$

Then, we can obtain

$$\mathrm{tr}(J_{i(u^*,v^*)})=\left(\frac{a_1}{k^2}+\frac{a_2}{k^2}\right)\theta_i+\mathrm{tr}(J_{(u^*,v^*)}), \tag{25}$$

and

$$\det(J_{i(u^*,v^*)})=\frac{a_1a_2}{k^4}\theta_i^2+\left(-\frac{a_1}{k^2}v(s+1)+\frac{a_2}{k^2}\left(-\mu+pv^{\frac{q}{s+1}}u*^{p-\frac{rq}{s+1}-1}\right)\right)\theta_i+\det(J_{(u^*,v^*)}). \tag{26}$$

Herein, the discriminant can be defined by

$$\begin{aligned}\Delta_i&=(\mathrm{tr}(J_{(u^*,v^*)}))^2-4\det(J_{(u^*,v^*)}),\\&=\left(\frac{a_1}{k^2}+\frac{a_2}{k^2}\right)^2\theta_i^2+\mathrm{tr}(J_{(u^*,v^*)})^2-4\left(-\frac{a_1}{k^2}v(s+1)+\frac{a_2}{k^2}\left(-\mu+pv^{\frac{q}{s+1}}u*^{p-\frac{rq}{s+1}-1}\right)\right)\theta_i-4\det(J_{(u^*,v^*)})\\&=\left(\frac{a_1}{k^2}-\frac{a_2}{k^2}\right)^2\theta_i^2+4\left(\frac{a_1}{k^2}v(s+1)-\frac{a_2}{k^2}(-\mu+pv^{\frac{q}{s+1}}u^{*p-\frac{rq}{s+1}-1})\right)\theta_i+\Delta.\end{aligned}$$

The sign of the discriminant Δ_i is crucial for the investigation of the solution of the characteristic equation (24). Therefore, based on the sign of Δ_{θ_i}, we distinguish between the following cases:

- If $\Delta_{\theta i}>0$. This means

$$\left(\frac{1}{4}-\frac{a_1a_1}{k^4}\right)\left(pv^{\frac{q}{s+1}}u^{*p-\frac{rq}{s+1}-1}-\mu\right)^2 > 2v^{\frac{q}{s+1}-1}u*^{p-\frac{rq}{s+1}-1}(p(s+1)-2rq)+v(s+1)(v(s+1)-6\mu).$$

Now, due to $a_1\neq a_2$, we should then distinguish between the following two cases:

– If $a_1<a_2$, then $\Delta>0$, and hence the two solutions of the equation $\Delta^*_{\theta_i}=0$ are both negative. Thus, we have $\Delta_{\theta_i}>0$, and the roots of the problem will be as described in (28). Note that the solutions in question are all real numbers with $\lambda_1(\theta_i)<0$. In addition, if

$$\frac{a_1}{k^2}\theta_i-\mu+pv^{\frac{q}{s+1}}u*^{p-\frac{rq}{s+1}-1}\geq 0,$$

then $\lambda_2(\theta_i) < 0$. This consequently leads to

$$|Arg(\lambda_1(\theta_i))| = |Arg(\lambda_2(\theta_i))| = \pi, \tag{27}$$

which ensures that (u^*, v^*) is asymptotic stability for which

$$\lambda_1(\theta_i) = \frac{\mathrm{tr}(J_{i(u^*,v^*)}) + \sqrt{\Delta_i}}{2}, \quad \lambda_2(\theta_i) = \frac{\mathrm{tr}(J_{i(u^*,v^*)}) - \sqrt{\Delta_i}}{2}. \tag{28}$$

– If $a_1 > a_2$, then we have $\Delta^* > 0$. This returns us to the previous scenario. Again, we can have

$$\frac{a_1}{k^2}\theta_i - \mu + p\nu^{\frac{q}{s+1}} u^{*\, p - \frac{rq}{s+1} - 1} \geq 0,$$

such that $\det J^*_{i(u^*,v^*)} > 0$. Hence, λ_1 and λ_2 are negative and must meet the conditions of Theorem 1.

- If $\Delta^*_{\theta_i} < 0$. This means

$$\left(\frac{1}{4} - \frac{a_1 a_1}{k^4}\right)\left(p\nu^{\frac{q}{s+1}} u^{*\, p - \frac{rq}{s+1} - 1} - \mu\right)^2 > 2\nu^{\frac{q}{s+1} - 1} u^{*\, p - \frac{rq}{s+1} - 1}(p(s+1) - 2rq) + \nu(s+1)(\nu(s+1) - 6\mu) > 0.$$

Now, due to $a_1 \neq a_2$, then the roots of (**??**) will be complex as defined in (29), and they must satisfy $Arg(\lambda_j(\theta_i)) > \frac{\mathfrak{c}\pi}{2}$ for which

$$\lambda_1(\theta_i) = \frac{\mathrm{tr}(J_{i(u^*,v^*)})}{2} + i\frac{\sqrt{\Delta_i}}{2}, \quad \lambda_2(\theta_i) = \frac{\mathrm{tr}(J_{i(u^*,v^*)})}{2} - i\frac{\sqrt{\Delta_i}}{2}. \tag{29}$$

3.2. Global stability

In this part, we expand the Lyapunov function approach to address the global asymptotic stability of the fractional discrete Gierer-Meinhardt model (8). We provide first important some theorems concerning with the discrete fractional systems.

Lemma 1. *[28] The following inequality holds:*

$${}^{C}_{h}\Delta^{\mathfrak{c}}_{a}\chi^2(t) \leq 2\chi(t + \mathfrak{c}\hbar)^{C}_{h}\Delta^{\mathfrak{c}}_{a}\chi(t), \quad t \in (\hbar\mathbb{N})_{a+\mathfrak{c}\hbar}. \tag{30}$$

Theorem 4. *[29] Let (u^*, v^*) be the system's equilibrium point of model (8). The equilibrium point is asymptotically stable if there exists a positive definite and declining scalar function, where ${}^{C}_{h}\Delta^{c}_{a}V(t) \leq 0$.*

Theorem 5. *The system (8) is globally asymptotically stable if*

$$\min\left\{\mu u^* - \frac{pu^{*p}}{v^{*q}} - \frac{ru^{*r}}{v^{*s}}, \nu v^* - \frac{u^{*r}}{v^{*s}}(1+s) - \frac{qu^{*p}}{v^{*q}}\right\} \geq 0,$$

Proof. To exemplify this result, we use the Lyapunov function (31) and analyze the following functions:

$$V_3(t) = \sum_{i=0}^{N} u^* L\left(\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right), \quad V_4(t) = \sum_{i=0}^{N} v^* L\left(\frac{v_i(t+\mathfrak{c}\hbar)}{v^*}\right), \tag{31}$$

Also, we consider

$$V(t) = V_3(t) + V_4(t).$$

Now, we must first assume that $v^* \neq 0,\ u^* \neq 0$. Then we have

$$\begin{aligned}
{}_{\hbar}^{C}\Delta_{t_0}^{\mathfrak{c}} V_3(t) \leq & \sum_{i=1}^{N} \mu u^* \left(1-\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)\left(1-\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right) \\
& -\frac{u^{*p}}{v^{*q}}\left(1-\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)\left(1-\frac{v^{*q}}{u^{*p}}\frac{u_i^p(t+\mathfrak{c}\hbar)}{v_i^q(t+\mathfrak{c}\hbar)}\right), \\
\leq & \sum_{i=1}^{N} \mu u^* \left[-L\left(\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)-L\left(\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right)\right]-\sum_{i=1}^{N}\frac{u^{*p}}{v^{*q}}\left[-L\left(\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)\right. \\
& \left.-L\left(\frac{v^{*q}u_i^p(t+\mathfrak{c}\hbar)}{u^{*p}v_i^q(t+\mathfrak{c}\hbar)}\right)+L\left(\frac{v^{*q}u_i^{p-1}(t+\mathfrak{c}\hbar)}{u^{*p-1}v_i^q(t+\mathfrak{c}\hbar)}\right)\right], \\
\leq & \sum_{i=1}^{N} \mu u^* \left[-L\left(\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)-L\left(\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right)\right]-\sum_{i=1}^{N}\frac{u^{*p}}{v^{*q}}\left[-L\left(\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)\right. \\
& \left.+L\left(\frac{v^{*q}u_i^{p-1}(t+\mathfrak{c}\hbar)}{u^{*p-1}v_i^q(t+\mathfrak{c}\hbar)}\right)-L\left(\frac{v^{*q}}{v_i^q(t+\mathfrak{c}\hbar)}\frac{u_i^p(t+\mathfrak{c}\hbar)}{u^{*p}}\right)\right], \\
\leq & \sum_{i=1}^{N} \mu u^* \left[-L\left(\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)-L\left(\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right)\right]-\sum_{i=1}^{N}\frac{u^{*p}}{v^{*q}}\left[-L\left(\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)\right. \\
& \left.+L\left(\frac{v^{*q}u_i^{p-1}(t+\mathfrak{c}\hbar)}{u^{*p-1}v_i^q(t+\mathfrak{c}\hbar)}\right)-qL\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)-pL\left(\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right)\right], \\
\leq & \sum_{i=1}^{N} -\left(\mu u^*-\frac{u^{*p}}{v^{*q}}\right)L\left(\frac{u^*}{u_i(t+\mathfrak{c}\hbar)}\right)-\left(\mu u^*-\frac{pu^{*p}}{v^{*q}}\right)L\left(\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right) \\
& -\frac{u^{*p}}{v^{*q}}L\left(\frac{v^{*q}u_i^{p-1}(t+\mathfrak{c}\hbar)}{u^{*p-1}v_i^q(t+\mathfrak{c}\hbar)}\right)+\frac{qu^{*p}}{v^{*q}}L\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right).
\end{aligned}$$

Consequently, ${}_{\hbar}^{C}\Delta_{t_0}^{c} V_2(t)$ can be evaluated as follows:

$$\begin{aligned}
{}_{\hbar}^{C}\Delta_{t_0}^{\mathfrak{c}} V_4(t) \leq & \sum_{i=0}^{N} \nu v^* \left(1-\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)\left(1-\frac{v_i(t+\mathfrak{c}\hbar)}{v^*}\right) \\
& -\frac{u^{*r}}{v^{*s}}\left(1-\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)\left(1-\frac{v^{*s}u_i^r(t+\mathfrak{c}\hbar)}{u^{*r}v_i^s(t+\mathfrak{c}\hbar)}\right), \\
\leq & \sum_{i=0}^{N} \nu v^* \left[-L\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)-L\left(\frac{v_i(t+\mathfrak{c}\hbar)}{v^*}\right)\right]-\sum_{i=0}^{N}\frac{u^{*r}}{v^{*s}}\left[-L\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)\right. \\
& \left.-L\left(\frac{v^{*s}u_i^r(t+\mathfrak{c}\hbar)}{u^{*s}v_i^s(t+\mathfrak{c}\hbar)}\right)+L\left(\frac{v^{*s+1}u_i^r(t+\mathfrak{c}\hbar)}{u^{*s}v_i^{s+1}(t+\mathfrak{c}\hbar)}\right)\right],
\end{aligned}$$

$$\leq \sum_{i=0}^{N} \nu v^*\left[-L\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)-L\left(\frac{v_i(t+\mathfrak{c}\hbar)}{v^*}\right)\right]-\sum_{i=0}^{N}\frac{u^{*r}}{v^{*s}}\left[-L\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)\right.$$
$$\left.-L\left(\frac{v^{*s}}{v_i^s(t+\mathfrak{c}\hbar)}\frac{u_i^r(t+\mathfrak{c}\hbar)}{u^{*s}}\right)+L\left(\frac{v^{*s+1}u_i^r(t+\mathfrak{c}\hbar)}{u^{*s}v_i^{s+1}(t+\mathfrak{c}\hbar)}\right)\right],$$
$$\leq \sum_{i=0}^{N} \nu v^*\left[-L\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)-L\left(\frac{v_i(t+\mathfrak{c}\hbar)}{v^*}\right)\right]-\frac{u^{*r}}{v^{*s}}\left[-L\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)\right.$$
$$\left.-sL\left(\frac{v^*}{v_i(t+\mathfrak{c}\hbar)}\right)-rL\left(\frac{u_i(t+\mathfrak{c}\hbar)}{u^*}\right)+L\left(\frac{v^{*s+1}u_i^r(t+\mathfrak{c}\hbar)}{u^{*s}v_i^{s+1}(t+\mathfrak{c}\hbar)}\right)\right].$$

This implies

$$ {}_{\hbar}^{C}\Delta_{t_0}^{c} V_4(t) \leq \sum_{i=0}^{N} -(\nu v^* - \frac{u^{*r}}{v^{*s}}(1+s))L\left(\frac{v^*}{v_i(t+c\hbar)}\right)-\nu v^* L\left(\frac{v_i(t+c\hbar)}{v^*}\right)$$
$$+\frac{ru^{*r}}{u^{*s}}L\left(\frac{u_1(t+c\hbar)}{u^*}\right)-\frac{u^{*r}}{v^{*s}}L\left(\frac{v^{*s+1}u_i^r(t+c\hbar)}{u^{*s+1}v_i^r(t+c\hbar)}\right).$$

Due to the following assumption

$$\Delta V(t) = \Delta V_1(t) + \Delta V_2(t) \tag{32}$$

is true, we can obtain

$$ {}_{\hbar}^{C}\Delta_{t_0}^{c} V(t) \leq \sum_{i=1}^{N} -\left(\mu u^* - \frac{u^{*p}}{v^{*q}}\right)L\left(\frac{u^*}{u_i(t+c\hbar)}\right)-\left(\mu u^* - \frac{pu^{*p}}{v^{*q}}\right)L\left(\frac{u_i(t+c\hbar)}{u^*}\right)$$
$$-\frac{u^{*p}}{v^{*q}}L\left(\frac{v^{*q}u_i^{p-1}(t+c\hbar)}{u^{*p-1}v_i^q(t+c\hbar)}\right)+\frac{qu^{*p}}{v^{*q}}L\left(\frac{v^*}{v_i(t+c\hbar)}\right)$$
$$-\left(\nu v^* - \frac{u^{*r}}{v^{*s}}(1+s)\right)L\left(\frac{v^*}{v_i(t+c\hbar)}\right)-\nu v^* L\left(\frac{v_i(t+c\hbar)}{v^*}\right)+\frac{ru^{*r}}{v^{*s}}L\left(\frac{u_i(t+c\hbar)}{u^*}\right)$$
$$-\frac{u^{*r}}{v^{*s}}L\left(\frac{v^{*s+1}u_i^r(t+c\hbar)}{u^{*s}v_i^{s+1}(t)}\right),$$
$$\leq \sum_{i=1}^{N} -\left(\mu u^* - \frac{u^{*p}}{v^{*q}}\right)L\left(\frac{u^*}{u_i(t+c\hbar)}\right)-\left(\mu u^* - \frac{pu^{*p}}{v^{*q}}-\frac{ru^{*r}}{v^{*s}}\right)L\left(\frac{u_i(t+c\hbar)}{u^*}\right)$$
$$-\left(\nu v^* - \frac{u^{*r}}{v^{*s}}(1+s)-\frac{qu^{*p}}{v^{*q}}\right)L\left(\frac{v^*}{v_i(t+c\hbar)}\right)-\nu v^* L\left(\frac{v_i(t+c\hbar)}{v^*}\right)$$
$$-\frac{u^{*p}}{v^{*q}}L\left(\frac{v^{*q}u_i^{p-1}(t+c\hbar)}{u^{*p-1}v_i^q(t+c\hbar)}\right)-\frac{u^{*r}}{v^{*s}}L\left(\frac{v^{*s+1}u_i^r(t+c\hbar)}{u^{*s}v_i^{s+1}(t+c\hbar)}\right),$$

where

$$\mathfrak{r} = \min\left\{\mu u^* - \frac{pu^{*p}}{v^{*q}}-\frac{ru^{*r}}{v^{*s}}, \nu v^* - \frac{u^{*r}}{v^{*s}}(1+s)-\frac{qu^{*p}}{v^{*q}}\right\}$$

such that $\mathfrak{r} \geq 0$. So, we can have ${}_{\hbar}^{C}\Delta_{\hbar}^{c} V(t) \leq 0$, and hence system (8) is asymptotically stable.

4. Numerical simulations

In this part, we show the exemplary simulations of the theoretical properties of the stability of the discrete Gierer-Meinhardt reaction-diffusion system with non-integer orders. This will allows us to observe how the system behaves when its parameters and order are modified. It should be mentioned here that all numerical solutions performed next are in relation to system (8), which can be appeared as follows:

$$\begin{cases} u_i(n\hbar) = \phi_1(\mathfrak{x}_i) + \dfrac{\hbar^{\mathfrak{c}}}{\Gamma(\mathfrak{c})}\displaystyle\sum_{j=1}^{n}\dfrac{\Gamma(n-j+\mathfrak{c})}{\Gamma(n-j+1)}\Bigg[\dfrac{a_1}{k^2}(u_{i-1}((j-1)\hbar) - 2u_i((j-1)\hbar) + u_{i+1}((j-1)\hbar)) \\ \qquad +\sigma - \mu u_i((j-1)\hbar) + \dfrac{u_i^p((j-1)\hbar)}{v_i^q((j-1)\hbar)}\Bigg], \quad 1 \le i \le m, \quad n > 0. \\ v_i(n\hbar) = \phi_2(\mathfrak{x}_i) + \dfrac{\hbar^{\mathfrak{c}}}{\Gamma(\mathfrak{c})}\displaystyle\sum_{j=1}^{n}\dfrac{\Gamma(n-j+\mathfrak{c})}{\Gamma(n-j+1)}\Bigg[\dfrac{a_2}{k^2}[v_{i-1}((j-1)\hbar) - 2v_i((j-1)\hbar) + v_{i+1}((j-1)\hbar)) \\ \qquad -\nu v_i((j-1)\hbar) + \dfrac{u_i^r((j-1)\hbar)}{v_i^s((j-1)\hbar)}\Bigg], \quad 1 \le i \le m, \quad n > 0. \end{cases} \tag{33}$$

Example 1. *In this example, we assume the parameter values of the system in question as* $(a_1, a_2, \sigma, \mu, \nu, p, q, r, s) = (0.02, 0.03, 1, 0.1, 0.18, 1, 1, 1, 1)$, $t \in [0, 30]$, $\mathfrak{x} \in [0, 20]$, *the boundary conditions* $(u_0, v_0) = (1, 2)$, *and the initial conditions as follows:*

$$\begin{cases} \phi_1(\mathfrak{x}) = \mathfrak{x} + e^{x} + 5\cos(\pi\mathfrak{x}), \\ \phi_2(\mathfrak{x}) = \mathfrak{x} + 2e^{\mathfrak{x}} + 5\sin(\pi\mathfrak{x}), \end{cases} \tag{34}$$

In light of these assumptions, we can notice that the model has a equilibrium point $(u^*, v^*) = (35.1555, 13.9753)$, *which is asymptotically stable. Figures* 1A *and* 1B *show that the numerical solution we obtained for system (8) is consistent with the hypotheses presented in the preceding sections.*

Example 2. *Herein, we consider the following parameter values:* $(a_1, a_2, \sigma, \mu, \nu, p, q, r, s) = (0.02, 0.03, 0.04, 1.2, 0.16, 0.5, 1, 1, 1, 1)$ *and* $t \in [0, 60]$, $\mathfrak{x} \in [0, 40]$. *Also, we take the boundary conditions as* $(u_0, v_0) = (4, 3)$, *whereas the initial conditions as*

$$\begin{cases} \phi_1(\mathfrak{x}) = \mathfrak{x} - e^{\mathfrak{x}} - \cos(\pi\mathfrak{x}), \\ \phi_2(\mathfrak{x}) = \mathfrak{x} - 2e^{\mathfrak{x}} - \sin(\pi\mathfrak{x}). \end{cases} \tag{35}$$

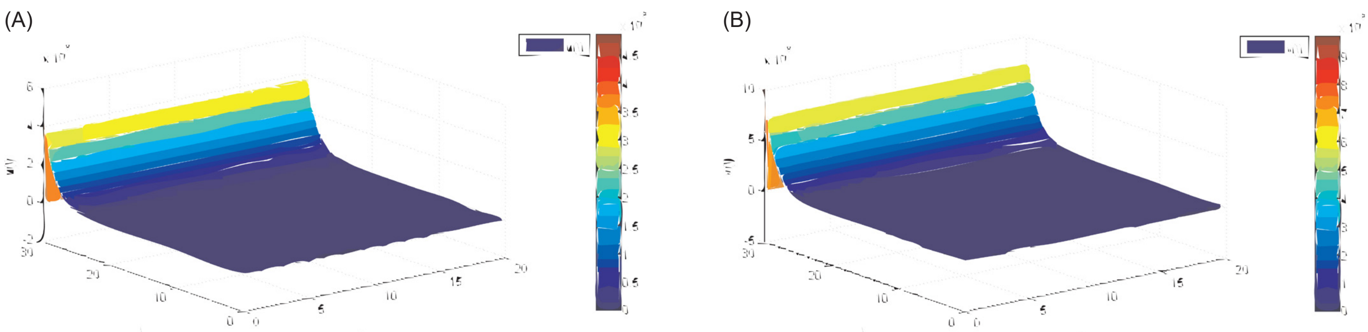

Figure 1: Numerical solution of (8) for $(a_1, a_2, \sigma, \mu, \nu, p, q, r, s) = (0.02, 0.03, 1, 0.1, 0.18, 1, 1, 1, 1)$, $N = 80$ and $c = 0.85$.

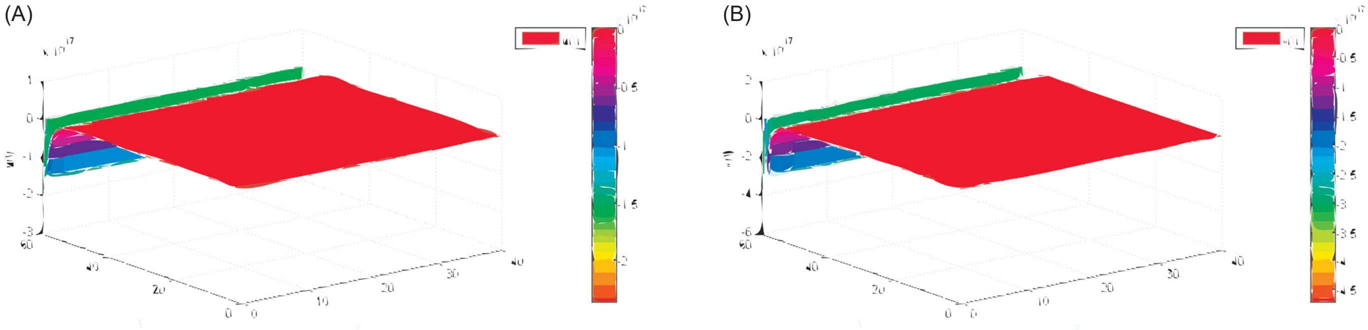

Figure 2: Numerical solution of (8) for (a_1, a_2, σ, μ, ν, p, q, r, s) = (0.03, 0.04, 1.2, 0.16, 0.5, 1, 1, 1, 1), N = 100 and c = 0.55.

Based on the previous assumptions, we can see that all of the solutions to model (8) eventually reach (u^*, v^*) = (32.8172, 8.1015). *This means that it is asymptotically stable. Anyhow, we can notice that the performed numerical results shown in Figures* 2A *and* 2B *agree with our past theoretical findings.*

5. Conclusion

In this paper, we provide a novel model of the reaction-diffusion Gierer-Meinhardt system based on the Caputo h-difference operator. To establish proper circumstances for the model's local and global asymptotic stability, the basic theory of discrete fractional models, the linearization technique, and the Lyapunov method are employed. Simulations of the fractional discrete Gierer-Meinhardt systems are offered to show the usefulness and applicability of the proposed theoretical conclusions.

The investigation of the stability of this system opens up the prospect of many different kinds of future research and study, involving analysis and modeling in a variety of fields, as well as unique aspects related to system chaos, stabilization, and synchronization of fractional discrete reaction diffusion systems.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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