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Generalized near approximations in bitopological spaces

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Abstract

The current study is aimed at the utilization of the concept of bitopological space in the definition of other concept such as generalized near approximations, generalized near boundary regions and generalized near accuracy as well as providing an indepth understanding of the properties of this concept in relation with other bitopological space. The study concluded by suggesting the concept of *generalized near approximations open way* for the construction of lower and upper approximations.

Keywords: Topologized spaces, Generalized near approximations, Bitopological spaces.

1. Introduction

The under mention generalized set are topologies are by set that aid in the introduction of the concept, bitopological space.

$$B_{\rho}\in = \{Y \in \Gamma \mid \rho(X, Y) \leq \in\}$$

and

$$B_{\delta}\in = \{y \in \Gamma \mid \delta(x, y) \leq \in\}$$

A bitopological space $(\Gamma, \lambda_1, \lambda_2)$ is signified by a non-empty set containing two (2) arbitrary topologies λ_1, λ_2 considering that ρ *and* δ *are* quasi-metric spaces of X with $\rho(x, y)$ is equal to $\delta(y, x)$. Since

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the introduction of bitopological space by Kelly in 1963, an enormous number of topological space have been embedded into bitopological space which includes properties such as separation axioms, compactness, connectedness, paracompactness, types of functions among many other properties and such several studies have followed the path of Kelly in his study of bitopological space. These authors include Birsan (1970), Fletcher et al. (1969), Bose et al (2008), Data (1972), Kilicman and salleh (2011), Hasan z.Hdeib (2016), Reiley (1970), Hamza Qoqazeh, Hasan z.Hdeib. (2018) and Hdeib and Fora (1982, 1983).

Consequently, The aim of this work present some properties of pairwise perfect and pairwise countably perfect mappings, and use them to obtain finite product theorems concerning pairwise expandable and almost pairwise expandable space, considering that *CL*(*A*) and *Int*(*A*) is the topological space of $A \subseteq X$ *in both the closure and interior of A. Taking into consideration that the subset A is a* bitopological space of Γ is equal to $(\Gamma, \lambda_1, \lambda_2)$, then the relative topology (subspace topology) on the set *A* inherited by λ will be denoted by λ_A. The cardinality of a set Δ will be denoted by Δ. R, Q, N, Z signifies the sets of real numbers, natural numbers, rational numbers, and integers respectively, ω_0, ω_1 will signify the first uncountable ordinal respectively, and *m* denotes an infinite cardinal in general. λ_u, λ_{dis}, λ_{cof}, λ_{coc} will denote the usual, the discrete, the cofinite and the co-countable topology respectively.

2. Preliminaries

Definition 2.1: The following conditions must be met for a paired topological space (Γ, λ) which consist of a set of *set* Γ and family λ in a subsets of Γ. The conditions are:

1. $\varphi \in \lambda$ *and* $\Gamma \in \lambda$.
2. *Under arbitrary union, λ signifies closed.*
3. *Under finite intersection λ signifies closed.*

Definition 2.2: *A bitopological space is a triple* $(\Gamma, \lambda_1, \lambda_2)$ *when it is inclusive of* a set Γ *and families* $\{\lambda_i; i = 1, 2\}$ *of subsets of* Γ, *thus,* the following conditions would be satisfied:

1. $\varphi \in \lambda_i$ *and* $\Gamma \in \lambda_i$ *for all* $i = 1, 2$.
2. *For an arbitrary union of all* $i = 1,2$, λ_i *is closed*
3. *For all* $i = 1, 2$, λ_i is closed under finite intersection.

Definition 2.3: [1] Let $(\Gamma, \lambda_1, \lambda_2)$ represent bitopological space while B be a subset of Γ, therefore B is pairwise close in Γ when λ_1–*closed and* λ_2–*closed. More* so, B would be known as open pairwise when Γ – ***A is pairwise closed in*** Γ.

In this study, bitopological space will be represented as $(\Gamma, \lambda_1, \lambda_2)$ and the element of Γ would be denoted as the points of the space, the ∈ of Γ represents the *points of the space*, the ⊆ of Γ in λ_i; $i = 1, 2$ in the space are called **Open pairwise** while in a bitopological space, the compliment of ⊆ Γ that are members of λ_i; $i = 1, 2$ will be known as closed pairwise. The members of all pairwise open sets of $(\Gamma, \lambda_1, \lambda_2)$ is signified by λ_i; $i = 1, 2$ and the entire members of pairwise closed sets of $(\Gamma, \lambda_1, \lambda_2)$ is signified by *PC*(Γ).

Considering the subset B of a space $(\Gamma, \lambda_1, \lambda_2)$, $Cl_\lambda i$ (B); $i = 1, 2$ *represents the closing of B and is given by* $PCl_\lambda i\ (B) = \cap\{F \subseteq \Gamma : B \subseteq F \text{ and } F \in C(\Gamma)\}$. *Evidently,* $Cl_\lambda i\ (B)$; $i = 1, 2$ *is the slightest pairwise of closed ⊆ of Γ which is inclusive of B considering that B is closed if* $B = Cl_\lambda i\ (B)$; $i = 1, 2$. $Int_\lambda i\ (B)$; $i = 1, 2$ *signifying that the interior of B and is given by* $Int_\lambda i\ (B) = \cup\{M \subseteq \Gamma : M \subseteq B \text{ and } M \in \lambda_i\}$; $i = 1, 2$ *Evidently,* $Int_\lambda i\ (B)$; $i = 1, 2$ *is the Maximum pairwise open subset of B. More so, B is open if* $B = Int_\lambda i\ (B)$; $i = 1, 2$. *The boundary of* $B \subseteq \Gamma$ *is signified by* $BN_\lambda i\ (B)$; $i = 1, 2$ *and is given by* $BN_\lambda i\ (B) = Cl_\lambda i\ (B)$ $Int_\lambda i\ (B)$; $i = 1, 2$. Kindly note, that some of the concept of Open set will be vital in the current study.

Definition 2.4: *The* ⊆ *B of a space* (Γ, λ) is known as:

1. *Semi-open* [2] *when* (*briefly, s–open*) *if* $B \subseteq Cl(Int(B))$.

2. *Pre-open* [3–4] (*briefly, p–open*) *if* $B \subseteq Int(Cl(B))$.
3. *α-open* [5] *if* $B \subseteq Int(Cl(Int(B)))$.
4. *β-open* [6] *if* $B \subseteq Cl(Int(Cl(B)))$.

Definition 2.5: *The* $\subseteq B$ *of a space* $(\Gamma, \lambda_1, \lambda_2)$ is termed:

1. *Pairwise Semi-open (briefly, Ps open) if* $B \subseteq Cl_\lambda i\ (Int_\lambda i\ (B))$; $i = 1, 2$.
2. *Pairwise Pre-open (briefly, Pp–open) if* $B \subseteq Int_\lambda i\ (Cl_\lambda i\ (A))$; $i = 1, 2$.
3. *Pairwise α-open (briefly,Pα open) if* $B \subseteq Int_\lambda i\ (Cl_\lambda i\ (Int_\lambda i\ (B)))$; $i = 1, 2$.
4. *Pairwise β-open (briefly,Pβ open) if* $B \subseteq Cl_\lambda i\ (Int_\lambda i\ (Cl_\lambda i\ (B)))$; $i = 1, 2$.

The complement of a Ps-open (represent Pp-open, Pα-open and Pβ-open) set is called Ps- closed (represent Pp- closed, Pα- closed and Pβ-closed) set.

The member of the entire Ps-open (represent Pp-open, Pα-open and Pβ-open) sets of $(\Gamma, \lambda_1, \lambda_2)$ *is denoted by PSO(Γ) (represent PPO(Γ), PαO(Γ) and PβO(Γ)). The member of the entire Ps-closed (represent Pp- closed, Pα-closed and Pβ-closed) sets of* $(\Gamma, \lambda_1, \lambda_2)$ *is signified by PSC(Γ) (represent PPC(Γ), PαC(Γ) and PβC(Γ)).*

The pairwise semi-closure (briefly, Psp-clouser) (resp. Pα-closure, Ppre- closure, Psemi-pre-closure of a subset B of $(\Gamma, \lambda_1, \lambda_2)$, *denoted by* ${}_PsCl_\lambda i\ (B)$; $i = 1, 2$ *(resp.* ${}_P\alpha Cl_\lambda i\ (B)$, ${}_PpCl_\lambda i\ (B)$, ${}_PspCl_\lambda i\ (B)$; $i = 1, 2$) *and defined to be the intersection of all semi-closed (represent Pα- closed, Pp-closed and Psp-closed) sets containing B. The pairwise semi-interior (briefly, Psp-interior) (resp. Pα- interior, Ppre- interior, Psemi-pre- interior of a subset B of* $(\Gamma, \lambda_1, \lambda_2)$, *denoted by* ${}_PsInt_\lambda i\ (B)$; $i = 1, 2$ *(represent* ${}_P\alpha Int_\lambda i\ (B)$, ${}_PpInt_\lambda i\ (A)$, ${}_PspInt_\lambda i\ (B)$; $i = 1, 2$) *and said to be the union of the entire Ps-open (represent Pα- open, Pp-open and Psp-open) sets shown in B*

Definition 2.6: *The* $\subseteq B$ *of a space* (Γ, λ) is define as:

(1) ***Generalized closed*** [7] (*temporarily, g-closed*) *if* $Cl(B) \subseteq G$ *whenever* $B \subseteq G$ and G is open in (Γ, λ).
(2) ***Generalized semi-closed*** [8] (*temporarily, gs-closed*) *if* ${}_sCl(B) \subseteq G$ *whenever* $B \subseteq G$ and G is open in (Γ, λ).
(3) ***Generalized semi-preclosed*** [9] (*temporarily, gsp -closed*) *if* ${}_{sp}Cl(B) \subseteq G$ whenever $B \subseteq G$ and G is open in (Γ, λ).
(4) ***α-generalized closed*** [10–11] (*temporarily, αg-closed*) *if* ${}_\alpha Cl(B) \subseteq G$ *whenever* $B \subseteq G$ and G is open in (Γ, λ).
(5) ***Generalized preclosed*** [12] (*temporarily, gp -closed*) *if* ${}_pCl(B) \subseteq G$ *whenever* $B \subseteq G$ and G is open in (Γ, λ).

Definition 2.7: *The* $\subseteq B$ *of a bitopological space* $(\Gamma, \lambda_1, \lambda_2)$ is define as:

(1) ***Generalized pairwise closed*** (*temporarily, gP-closed*) *if* $Cl_\lambda i\ (B) \subseteq G$ *for all* $i = 1, 2$ *whenever* $B \subseteq G$ *and G is pairwise open in* $(\Gamma, \lambda_1, \lambda_2)$.
(2) ***Semi-closed* generalized pairwise** (*temporarily, gPs-closed*) *if* ${}_{Ps}Cl_\lambda i\ (B) \subseteq G$ *for all* $i = 1, 2$ *whenever* $B \subseteq G$ *and G is pairwise open in* $(\Gamma, \lambda_1, \lambda_2)$.
(3) ***Semi-preclosed generalized pairwise*** (temporarily, *gPsp -closed*) *if* ${}_{Psp}Cl_\lambda i\ (B) \subseteq G$ for all $i = 1, 2$ whenever $B \subseteq G$ and G is pairwise open in $(\Gamma, \lambda_1, \lambda_2)$.
(4) ***Closed α-generalized pairwise*** (temporarily, *αgP-closed*) *if* ${}_{P\alpha}Cl_\lambda i\ (B) \subseteq G$ *for all* $i = 1, 2$ *whenever* $B \subseteq G$ *and G is pairwise open in* $(\Gamma, \lambda_1, \lambda_2)$.
(7) ***Preclosed generalized pairwise*** (temporarily, *gPp -closed*) *if* ${}_{Pp}Cl_\lambda i\ (B) \subseteq G$ *for all* $i = 1, 2$ *whenever* $B \subseteq G$ *and G is pairwise open in* $(\Gamma, \lambda_1, \lambda_2)$.

The complement of a gP-closed (represent gPs-closed, gPsp-closed, gPp-closed and αgP-closed) set is called gP-open (respresent gPs-open, gPsp-open, gPp- open and αgP-open). The member of the entire gP-open (resp. gPs-open, gPp-open, αgP -open and gPsp-open) sets of $(\Gamma, \lambda_1, \lambda_2)$ is signified by gPO(Γ) (respresent gPSO(Γ), gPPO(Γ), αgPO(Γ) and gPspO(Γ)). The member of the enire Pg-closed (respresent gPs-closed, gPp-closed, αgP-closed and gPsp-closed) sets of $(\Gamma, \lambda_1, \lambda_2)$ is signified by gPC(Γ) (respresent gPSC(Γ),gPpC(Γ) αgPC(Γ), and gPspC(Γ)). The generalized pairwise interior (briefly, gP-interior) of B is signified by ${}_g\mathrm{Int}_\lambda \mathrm{i}$ (B) ; i = 1, 2 and is defined by ${}_g\mathrm{Int}_\lambda \mathrm{i}\ (B) = \cup\{G \subseteq \Gamma : G \subseteq B$, G is a gP–sopen}; i = 1, 2, and the generalized near pairwise interior (briefly gjP-interior) of B is denoted by ${}_{gjP}\mathrm{Int}_\lambda \mathrm{i}$ (B) for all $j \in \{s, p, \alpha, \beta\}$, i = 1, 2 and is defined by ${}_{gjP}\mathrm{Int}_\lambda \mathrm{i}\ (B) = \cup\{G \subseteq \Gamma : G \subseteq B$, G is a gjP – open}.

The generalized pairwise closure (temporary *gP*-closure) of *B* is signified by ${}_gP\ Cl_\lambda i\ (B)$; *i* = 1, 2 and is defined by ${}_gP\ Cl_\lambda i\ (B) = \cap\{F \subseteq \Gamma : A \subseteq F$, *F* is a *gP*–closed set }; *i* = 1, 2, and the generalized near pairwise closure (temporary *gjP*-closure) of *B* is signified by ${}_gj\ P\ Cl_\lambda i\ (B)$ for all $j \in \{s, p, \alpha, \beta\}$; *i* = 1, 2 and is defined by ${}_{gjP}Cl_\lambda i\ (B) = \cap\{F \subseteq \Gamma : B \subseteq F$, *F* is a *gjP* closed set }.

The generalized pairwise boundary (temporary, gP-boundary) region of B is signified by ${}_gP\ BN_\lambda i$ (B) for all i = 1, 2 and is defined by ${}_gP\ BN_\lambda i\ (B) = {}_gP\ Cl_\lambda i\ (B) - {}_gP\ Int_\lambda i\ (B)$ for all i = 1, 2 and the generalized near pairwise boundary (temporary, gjP-boundary) region of B is denoted by ${}_gj\ P\ BN_\lambda i$ (B) for all $j \in \{s, p, \alpha, \beta\}$; i = 1, 2 and is defined by ${}_gj\ P\ BN_\lambda i\ (B) = {}_gj\ P\ Cl_\lambda i\ (B) - {}_gj\ P\ Int_\lambda i\ (B)$ for all i = 1, 2

The generalized pairwise exterior *(temporary, gP-exterior) of B is denoted by* $gP\ Ext_\lambda i\ (B)$ *for all i* = 1, 2 *and is defined by* ${}_gP\ Ext_\lambda i\ (B) = \Gamma - {}_gP\ Ext_\lambda i\ (B)$ *for all i* = 1, 2 *and* ***the generalized near pairwise*** *exterior (briefly gjP-exterior) of B is signified by* ${}_gj\ P\ Ext_\lambda i\ (B)$ *for all* $j \in \{s, p, \alpha, \beta\}$; *i* = 1, 2 *and is defined by*

$$gjP\ Ext_\lambda i\ (B) = \Gamma - {}_{gj}P\ Ext_\lambda i\ (B)$$

3. Generalization of Pawlak Approximation Space in Bitopological Spaces

Approximation space in a bitopological space is denoted by ...≪⌣⌐ ⊆ ∡⌢ with R signifying a distinct bitopological space $(\Gamma, \lambda_{k1}, \lambda_{k2})$ where λ_{ki}; *i* = 1, 2 is a member of the entire open pairwise sets in $(\Gamma, \lambda_{k1}, \lambda_{k2})$ and the base of λ_{ki}; *i* = 1, 2 is defined as Γ Considering that the lower represent the approximation upper of any subset $B \subseteq \Gamma$ which is exactly the pairwise interior (represent Closed pairwise) of ⊆ of *B*.

4. Generalized Lower and Near Upper Approximations in Bitopological Spaces

Definition 4.1: *Let K* = (Γ, *R*) *be an approximation space in bitopological with general relation R and* λ_{ki}; *i* = 1, 2 *are the topologies associated to K . Then the quadruple* $(\Gamma, R, \lambda_{k1}, \lambda_{k2})$ is known as topological approximation space in bitopological space.

Definition 4.2: *A topologized approximation space in bitopological space will be* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$, *If* $B \subseteq \Gamma$, as such the lower approximation (resp. upper approximation) of B in bitopological space is signified by

$$\underline{R}_P B = Int_\lambda i\ (B)(represent\ .\bar{R}\ PB = Cl_\lambda i\ (B));\ i = 1, 2.$$

Definition 4.3: *The generalized lower approximation (temporary, g-lower approximation) of B in bitopological space is signified by* $\underline{R}gP\ B$ *and is defined by* $\underline{R}gP\ B = {}_{gP}\ Int_\lambda i\ (B)$; *i* = 1, 2 when $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$ *is a topologized approximation space in bitopological space considering that* $B \subseteq \Gamma$.

Definition 4.4: *A topologized approximation space in bitopological space is represented by* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$. *If* $B \subseteq \Gamma$, the generalized near lower approximation (briefly, gj-lower approximation) of B in bitopological space is signified by $\underline{R}$gjP B and therefore:

$$\underline{R}_{gj}P\ B = {}_{gj}P\ Int_\lambda i\ (B),\ where\ j \in \{s, p, \alpha, \beta\},\ i = 1, 2.$$

Definition 4.5: *A topologized approximation space in bitopological space is represented by* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$. *If* $B \subseteq \Gamma$, *then the generalized upper approximation (briefly, g-upper approximation) of B in bitopological space is signified by* $\bar{R}gP\ B$ *and as such* $\bar{R}gP\ B = {}_g\ Cl_\lambda i\ (B)$; *i* = 1, 2

Definition 4.6: *A topologized approximation space in bitopological space is represented by* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$. *If* $B \subseteq \Gamma$, *then the generalized near upper approximation (briefly, g-upper approximation) of B in bitopological space is signified by* $\bar{R}$gjP B and therefore:

$$\bar{R}\, gjP\, B = _{gj}P\, Cl_{\lambda}i\, (B),\ \text{where } j \in \{s, p, \alpha, \beta\},\ i = 1, 2.$$

Theorem 4.1: *Let* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$ *be a topologized approximation space in bitopological space. If* $B \subseteq \Gamma$, *then* $\underline{R}_P B \subseteq \underline{R}_g P\, B \subseteq B \subseteq \bar{R}\, gP\, B \subseteq \bar{R}$ PB

Proof. $\underline{R}_P B = Int_{\lambda} i\, (B) = \cup\{G \in \lambda_i : G \subseteq B \subseteq \cup\{G \in gPO(\Gamma) : G \subseteq B = gP\, Int(B) = \underline{R}_g P\, B \subseteq B$, i=1,2 ,considering all pairwise open set is generalized pairwise open. $\to$ (1)

$$\bar{R}\, PB = Cl_{\lambda} i\, (B = \cap\{F \in PC(\Gamma) : B \subseteq F\} \supseteq \cap\{F \in gPC(\Gamma) : B \subseteq F\} = gP\, Cl_{\lambda} i\, (B) = \bar{R}\, gP\, B \supseteq B \to (2)$$

From (1) and (2) we get $\underline{R}_P B \subseteq \underline{R}_g P\, B \subseteq B \subseteq \bar{R}\, gP\, B \subseteq \bar{R}$ PB

Theorem 4.2. *A topologized approximation space in bitopological space is represented by* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$. *If* $B \subseteq \Gamma$, *then* $\underline{R}_P B \subseteq \underline{R}_{gj} P\, B \subseteq A \subseteq \bar{R}\, gjP\, B \subseteq \bar{R}\, PB$ *for all* $j = \{s, p, \alpha, \beta\}$

Proof. The aim is to prove the theorem as it pertains $j = \alpha$ and the other cases can be similarly proved. Now, $\underline{R}_P B = _P Int_{\lambda} i = \cup\{G \in \lambda_i : G \subseteq B\} \subseteq \cup\{G \in \alpha gPO(\Gamma) : G \subseteq B\} = _{g\alpha}P\, Int_{\lambda} i\, (B) = \underline{R}_{g\alpha} P\, B \supseteq B$ for all $i = 1, 2$, considering every pairwise open set is generalize α- pairwise open. $\to$ (1)

$$\bar{R}\, PA = _P Cl_{\lambda} i\, (B) = \cap\{F \in PC(\Gamma) : B \subseteq F\} \supseteq \cap\{F \in g\alpha PC(\Gamma) : B \subseteq F\} = g\alpha P\, Cl_{\lambda} i\, (A) = \bar{R}\, g\alpha P\, B \supseteq B \text{ for all } i = 1, 2 \to (2)$$

As seen in (1) and (2) we get $\underline{R}_P B \subseteq \underline{R}_{g\alpha} P\, B \subseteq A \subseteq \bar{R}\, g\alpha P\, B \subseteq \bar{R}\, PB$

Theorem 4.3: *Let* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$ *be a topologized approximation space in bitopological space. If* B $\subseteq \Gamma$, then the implications between lower ap- proximation in bitopological space and gPj-lower approximations of B for the entire j $\in$ {s, p, α, β} are given by:

1. $\underline{R}_P B \subseteq \underline{R}_g P\, B$
2. $\underline{R}_g P\, B \subseteq \underline{R}_{g\alpha} P\, B$
3. $\underline{R}_{g\alpha} P\, \boldsymbol{B} \subseteq \underline{R}_{gs} P\, B$
4. $\underline{R}_{gs} P\, B \subseteq \underline{R}_{g\beta} P\, B$
5. $\underline{R}_{g\alpha} P\, B \subseteq \underline{R}_{gp} P\, B$
6. $\underline{R}_{gp} P\, B \subseteq \underline{R}_{g\beta} P\, B$

Proof.

1. $\underline{R}_P B = _P Int_{\lambda} i\, (B) = \cup\{G \in \lambda_i : G \subseteq B\} \subseteq \cup\{G \in gPO(\Gamma) : G \subseteq B\}$

$$\underline{R}_P B = _P Int_{\lambda} i\, (B) \subseteq _g P\, Int_{\lambda} i\, (\boldsymbol{B}) = \underline{R}_g \text{P B},$$

2. $\underline{R}_g P\, B = _g P\, Int_{\lambda} i\, (B) = \cup\{G \in gPO(\Gamma) : G \subseteq \boldsymbol{B} \subseteq \cup\{G \in g\alpha PO(\Gamma) : G \subseteq B\}$

$$\underline{R}_g P\, B = _g P\, Int_{\lambda} i\, (B) \subseteq _{g\alpha} P\, Int_{\lambda} i\, (B) = \underline{R}_{g\alpha} \text{P B},$$

3. $\underline{R}_{g\alpha} P\, B = _{g\alpha} P\, Int_{\lambda} i\, (B) = \cup\{G \in gP\alpha O(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\} \subseteq \cup\{\mathrm{G} \in gPsO(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\}$

$$\underline{R}_{g\alpha} P\, B = _{g\alpha} P\, Int_{\lambda} i\, (B) \subseteq _g Ps\, Int_{\lambda} i\, (B) = \underline{R}_g \text{PsB},$$

4. $\underline{R}_g PsB = _g Ps\, Int_{\lambda} i\, (B) = \cup\{G \in gPsO(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\} \subseteq \cup\{\mathrm{G} \in g\beta PO(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\}$

$$\underline{R}_g PsA = _g Ps\, Int_{\lambda} i\, (B) \subseteq _{g\beta} P\, Int_{\lambda} i\, (B) = \underline{R}_{g\beta} \text{P B},$$

5. $\underline{R}_{g\alpha} P\, A = _{g\alpha} P\, Int_{\lambda} i\, (B) = \cup\{G \in gP\alpha O(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\} \subseteq \cup\{\mathrm{G} \in gPpO(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\}$

$$\underline{R}_{g\alpha} P\, A = _{g\alpha} P\, Int_{\lambda} i\, (B) \subseteq _g Pp\, Int_{\lambda} i\, (B) = \underline{R}_g \text{PpB},$$

6. $\underline{R}_g PpA = _g Pp\, Int_{\lambda} i\, (B) = \cup\{G \in gPpO(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\} \subseteq \cup\{\mathrm{G} \in g\beta PO(\Gamma) : \mathrm{G} \subseteq \mathrm{B}\}$

$$\underline{R}_g PpA = _g Pp\, Int_{\lambda} i\, (B) \subseteq _{g\beta} P\, Int_{\lambda} i\, (B) = \underline{R}_{g\beta} \text{PB},$$

Where $i = 1, 2$.

Theorem 4.4: *If* $A \subseteq \Gamma$ *Whent* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$ is a topological approximation space in bitopological space., then the implications between upper approximation in bitopological space and gPj-upper approximations in bitopological space of B for the entire j ∈ {s, p, α, β} are defined by:

1. $\bar{R}\, gP\, B \subseteq \bar{R}$ PB
2. $\bar{R}\, g\alpha P\, B \subseteq \bar{R}$ gP B
3. $\bar{R}\, gsP\, B \subseteq \bar{R}$ gαP B
4. $\bar{R}\, gpP\, B \subseteq \bar{R}$ gαP B
5. $\bar{R}\, g\beta P\, B \subseteq \bar{R}$ gsP B
6. $\bar{R}\, g\beta P\, A \subseteq \bar{R}$ gpP A

Proof.

1. $\bar{R}\ PB =_P Cl_\lambda i\ (B) = \cap\{F \in C(\Gamma) : A \subseteq F\} \supseteq \cap\{F \in gPC(\Gamma) : B \subseteq F\}$
$$\bar{R}\ PB \supseteq {}_{gP}Cl_\lambda i\ (A) = \bar{R} \text{ gP B},$$
2. $\bar{R}\ gP\ B =_g P\ Cl_\lambda i\ (B) = \cap\{F \in gPC(\Gamma) : B \subseteq F\} \supseteq \cap\{F \in g\alpha PC(\Gamma) : \mathrm{B} \subseteq \mathrm{F}\}$
$$\bar{R}\ gP\ B \supseteq_g P\ Cl_\lambda i\ (B) = \bar{R} \text{ g}\alpha\text{P B},$$
3. $\bar{R}\ g\alpha P\ B =_{g\alpha} P\ Cl_\lambda i\ (B) = \cap\{F \in g\alpha PC(\Gamma) : B \subseteq F\} \supseteq \cap\{F \in gsPC(\Gamma) : \mathrm{B} \subseteq \mathrm{F}\}$
$$\bar{R}\ g\alpha P\ B \supseteq_{gs} P\ Cl_\lambda i\ (B) = \bar{R} \text{ gsP B},$$
4. $\bar{R}\ g\alpha P\ B =_{g\alpha} P\ Cl_\lambda i\ (B) = \cap\{F \in g\alpha PC(\Gamma) : B \subseteq F\} \supseteq \cap\{F \in gpPC(\Gamma) : \mathrm{B} \subseteq \mathrm{F}\}$
$$\bar{R}\ g\alpha P\ B \supseteq_g Pp\ Cl_\lambda i\ (B) = \bar{R} \text{ gPpB},$$
5. $\bar{R}\ gsP\ B =_{gs} P\ Cl_\lambda i\ (B) = \cap\{F \in gsPC(\Gamma) : B \subseteq F\} \supseteq \cap\{F \in g\beta PC(\Gamma) : \mathrm{B} \subseteq \mathrm{F}\}$
$$\bar{R}\ gsP\ B \supseteq_{g\beta} P\ Cl_\lambda i\ (B) = \bar{R} \text{ g}\beta\text{P B},$$
6. $\bar{R}\ gPpB =_g Pp\ Cl_\lambda i\ (B) = \cap\{F \in gPpC(\Gamma) : B \subseteq F\} \supseteq \cap\{F \in g\beta PC(\Gamma) : \mathrm{B} \subseteq \mathrm{F}\}$
$$\bar{R}\ gPpB \supseteq_{g\beta} P\ Cl_\lambda i\ (B) = \bar{R} \text{ g}\beta\text{P B},$$

Where $i = 1, 2$.

5. Near Boundary Regions and Generalized Accuracy in Bitopological Spaces

This section shall be centered on exploring the near boundary regions and generalized accuracy utilizing different rules in a generalized approximation space in a bitopological space considering also the general relations.

Definition 5.1: *If* $B \subseteq \Gamma$ *when* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$, which represent the *topologized approximation space in bitopological space., then the generalized near boundary (briefly, gj-boundary) region of B in bitopological space is signified by BNRgjP (B)* and is defined by

$$BNRgjP\ (B) = \bar{R}\ gjP\ (B) - \underline{R}_{gj}P\ (B), \text{ where j} \in \{\text{s, p, }\alpha, \beta\}.$$

Definition 5.2: *A topologized approximation space in bitopological space is represented by* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$. *If* $B \subseteq \Gamma$*, therefore, the positive generalized near region of B (briefly, gj- positive) will be represented by POSRgj (B) and therefore* $POSRgj\ (B) = \underline{R}_{gj}(B)$, where j ∈ {s, p, αβ}.

Definition 5.3: *A topologized approximation space in bitopological space is represented by* $K = (\Gamma, R, \lambda_{k1}, \lambda_{k2})$. *If* $B \subseteq \Gamma$*, therefore, the near negative generalized region of B (briefly, gj-negative) in bitopological space is signified by NEGRgjP (B)*, therefore:

$$NEGRgjP\ (B) = \Gamma - \bar{R}\ gjP\ (B), \text{ where j} \in \{\text{s, p, }\beta\}.$$

Theorem 5.1: A topologized approximation space in bitopological space is represented by K = (Γ, R, λ_{k1}, λ_{k2}). *If* $B \subseteq \Gamma$, *then* $BNRgjP(B) \subseteq BNRP(B)$, *for the entire* $j \in$ {s, p, α, β}

6. Generalized Near Exact Sets and Generalized Near Rough in Bitopological Spaces

This section will be centered on the introduction of the concept of generalized near exact sets and generalized near rough in bitopological spaces.

Definition 6.1: *A bitopological space can be represented as* $K = (\Gamma, \lambda_1, \lambda_2)$ *and* $B \subseteq \Gamma$. therefore

1. *B is totally* gj_P*-definable* (gj_P *-exact*) *set if* ${}_{gj}P\, Int_\lambda i\,(B) = B = {}_{gj}P\, Cl_\lambda i\,(B)$,
2. *B is internally* gj_P*-definable set if* $B = {}_{gj}P\, Int_\lambda i\,(B)$,
3. *B is externally* gj_P*-definable set if* $B = {}_{gj}P\, Cl_\lambda i\,(B)$,
4. *B is* gj_P*-indefinable set if* $A / = {}_{gj}P\, Int_\lambda i\,(B)$, B where j ∈ {s, p, α, β}, i = 1, 2. ${}_{gj}Cl_\lambda i\,(B)$,

Theorem 6.1: *A bitopological space can be represented as* $K = (\Gamma, \lambda_1, \lambda_2)$ *and* $B \subseteq \Gamma$. *therefore. If B* is an exact set then it is gj_P-exact for the entire j being an element of {s, p, α, β}.

Proof: The aim is to prove the theorem as it pertains $j = \beta$, and any other similar cases.

Let B be exact set, then ${}_PCl_\lambda i\,(B) = B = {}_P\, Int_\lambda i\,(B)$ for all $i = 1, 2$. Now,

$$ {}_PCl_\lambda i\,(B) \cap \{F \subseteq \Gamma : B \subseteq F,\, F \in PC(\Gamma)\} \supseteq \cap\{F \subseteq \Gamma : B \subseteq F,\, F \in_{g\beta} P\, PC(\Gamma) $$

Also,

$$ {}_PInt_\lambda i\,(B) = \cup\{G \subseteq \Gamma,\, G \in \lambda_i\} \subseteq \cup\{G \subseteq \Gamma : G \subseteq B,\, G \in_{g\beta} P\, PO(\Gamma)\} = {}_{g\beta}P\, Int_\lambda i\,(B). $$

Therefore, ${}_PCl_\lambda i\,(B) \supseteq_{g\beta} P\, Cl_\lambda i\,(B) \supseteq A \supseteq_{g\beta} P\, Int_\lambda i\,(B) \supseteq_P Int_\lambda i\,(B)$. Since B is exact we get ${}_{g\beta}P\, Cl_\lambda i\,(B) = B = {}_{g\beta}P\, Int_\lambda i\,(B)$;

Hence B is $g\beta_P$ - exact.

Conclusions

The study utilized the concept of bitopological space in the introduction of other concept which include generalized near boundary regions, generalized near rough into a bitopological space. The significance is to improve the concept of bitopological space.

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