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Applications of higher-order q-derivative operator for a new subclass of meromorphic multivalent q-starlike functions related with the janowski functions

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Abstract

In this paper, we expand on the notion of the q-derivative (or q-difference) operator for meromorphic multivalent functions, define the higher-order q-derivative operator for meromorphic multivalent functions associated with quantum calculus, and introduce new subclasses of meromorphic multivalent q-starlike functions in connection with Janowski functions. We investigate a number of useful properties of the Janowski functions and higher-order q-derivative operator for a new class of meromorphic multivalent q-starlike functions. Among the many potential uses of this class that we investigate are coefficient estimates, distortion theorems, partial sums, the radius of starlikeness, and a few other well-established results.

Key words and phrases. q-Calculus, the q-Derivative operator, Multivalent meromorphic q-Starlike functions, Janowski functions, Partial sums

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1. Introduction and Definitions

The $\mathcal{M}(\vartheta)$ is a set of all analytic functions h_ϑ that are meromorphic multivalent in the punctured open unit disc

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\},$$

and every $h\vartheta \in \mathrm{M}(\vartheta)$ is of the form:

$$h_\vartheta(z) = \frac{1}{z^\vartheta} + \sum_{v=0}^{\infty} a_{v+\vartheta} z^{v+\vartheta} \quad (\vartheta \in \mathbb{N} = \{1,2,\ldots\}). \tag{1.1}$$

We noticed that for $\vartheta = 1$, we have

$$\mathcal{M}(1) := \mathcal{M}.$$

Many authors introduced and studied several different subclasses of meromorphic univalent function class M, see for (example [1–5]).

A function $h_\vartheta \in \mathcal{M}(\vartheta)$ is known as meromorphic multivalent starlike whenever it satisfies the inequality

$$\mathcal{MS}^*(\vartheta) = \left\{ h_\vartheta \in \mathcal{M}(\vartheta) : \Re\left(-\frac{z h'_\vartheta(z)}{h_\vartheta(z)} \right) > 0 \right.$$

and let $\mathcal{MS}^*(\vartheta, \alpha)$ represent the class of meromorphic multivalent starlike functions of order α, $(0 \le \alpha < 1)$ whenever it satisfies the inequality

$$\mathcal{MS}^*(\vartheta,\alpha) = \left\{ h_\vartheta \in \%\mathcal{M}(\vartheta) : \Re\left(-\frac{z h'_\vartheta(z)}{h_\vartheta(z)} \right) > \alpha \right. .$$

Note that

$$\mathcal{MS}^*(\vartheta, 0) = \mathcal{MS}^*(\vartheta).$$

Numerous authors have done substantial research on these classes, for details (see [6–9]). Now we recall some basic notations and fundamental concepts of q-calculus operator theory and definitions, which will be helpful for the understanding of this article. We assume throughout this investigation that

$$q \in (0, 1), \quad -1 \le Y_2 < Y_1 \le 1, \quad \text{and} \quad \vartheta \in \mathrm{N} = \{1, 2, 3\ldots\}.$$

Definition 1. *(see [10]). Consider the q-number defined as:*

$$[\zeta]_q \begin{cases} \dfrac{1-q^\zeta}{1-q} & (\zeta \in \mathbb{C}) \\ \displaystyle\sum_{k=0}^{v-1} q^k & (\zeta = v \in \mathbb{N}) \end{cases}$$

and for any non-negative integer v

$$[v]_q! = \begin{cases} [v]_q [v-1]_q [v-2]_q \ldots [2]_q [1]_q, & v \ge 1 \\ 1 & v = 0. \end{cases}$$

Definition 2. (*see [11] and [12]*) . *Let* A *is the set of all analytic functions and* $h \in$ A. *The q-derivative (or q-difference)* D_q *operator is defined by*

$$\left(D_q h\right)(z) = \begin{cases} \dfrac{h(z) - h(qz)}{(1-q)z} & (z \neq 0) \\ h'(z) & (z = 0). \end{cases} \tag{1.2}$$

We observed from equation (1.2) that

$$\lim_{q \to 1-} (D_q h)(z) = h'(z).$$

For $h \in$ A *and from equation (1.2), we have*

$$(D_q h)(z) = 1 + \sum_{v=2}^{\infty} [v]_q a_v z^{v-1}.$$

Here analogous to Definition 3, Mahmood et al. [13] extend the idea of q-difference operator for $h_\vartheta \in$ M(ϑ) given in (1.1) and defined a new class $\mathcal{MS}^*_{q,\vartheta}[Y_1, Y_2]$ of meromorphic multivalent functions.

Definition 3. *(see [13]). For* $h \in$ M. *The q-derivative (or q-difference)* D_q *operator for a sub-collection of* $\mathbb{C}$ *is defined by*

$$(D_q h)(z) = \frac{h(z) - h(qz)}{(1-q)z}. \tag{1.3}$$

For $h \in \mathcal{M}$ *and from (1.3), we have*

$$= \frac{-1}{qz^2} + \sum_{v=0}^{\infty} [v]_q a_v z^{v-1}, \quad \forall z \in \mathcal{U}^*. \tag{1.4}$$

Definition 4. *Furthermore, on account of* (1.1) *and* (1.3) , *We generalize the concept of a q-difference operator for* $h_\vartheta \in \mathcal{M}(\vartheta)$ *such that*

$$\left(D_q h_\vartheta\right)(z) = \frac{-1}{q^\vartheta}[\vartheta]_q z^{-\vartheta-1} + \sum_{v=0}^{\infty} [v+\vartheta]_q a_{v+\vartheta} z^{v+\vartheta-1}, \tag{1.5}$$

$$\left(D_q^2 h_\vartheta\right)(z) = \left(\frac{-1}{q^\vartheta}\right)\left(\frac{-1}{q^{\vartheta+1}}\right)[\vartheta]_q [\vartheta+1]_q z^{-\vartheta-2}$$

$$+ \sum_{v=0}^{\infty} [v+\vartheta]_q [v+\vartheta-1]_q a_{v+\vartheta} z^{v+\vartheta-2},$$

$$\vdots$$

$$\left(D_q^\vartheta h_\vartheta\right)(z) = \left(\frac{-1}{q^\vartheta}\right)\left(\frac{-1}{q^{\vartheta+1}}\right)\cdots\left(\frac{-1}{q^{2\vartheta-1}}\right)[\vartheta]_q [\vartheta+1]_q \ldots [2\vartheta-1]_q z^{-2\vartheta}$$

$$+ \sum_{v=0}^{\infty} [v+\vartheta]_q [v+\vartheta-1]_q \ldots [v+1]_q a_{v+\vartheta} z^{v},$$

$$\left(D_q^{\vartheta} h_{\vartheta}\right)(z) = \left(\frac{-1}{q^{\vartheta}}\right)\left(\frac{-1}{q^{\vartheta+1}}\right)\cdots\left(\frac{-1}{q^{2\vartheta-1}}\right)\left[\vartheta\right]_q\left[\vartheta+1\right]_q\cdots\left[2\vartheta-1\right]_q z^{-2\vartheta} + \sum_{\upsilon=0}^{\infty}\frac{[\upsilon+\vartheta]_q\,!}{\left[\upsilon\right]_q!}a_{\upsilon+\vartheta}z^{\upsilon}, \tag{1.6}$$

$$\left(D_q^{\vartheta} h_{\vartheta}\right)(z) = \left(\frac{-1}{q^{\vartheta}}\right)\left(\frac{-1}{q^{\vartheta+1}}\right)\cdots\left(\frac{-1}{q^{2\vartheta-1}}\right)\left[\vartheta\right]_q\left[\vartheta+1\right]_q\cdots\left[2\vartheta-1\right]_q z^{-2\vartheta} + \sum_{\upsilon=1}^{\infty}\frac{[\upsilon-1+\vartheta]_q\,!}{\left[\upsilon-1\right]_q!}a_{\upsilon+\vartheta-1}z^{\upsilon-1} \tag{1.7}$$

and $(D_q h_\vartheta)(z)$ *is the* ϑ*-th time* q*-derivative of* $h_\vartheta(z)$*.*

Now, for each $h_\vartheta \in \mathrm{M}(\vartheta)$, the expression in (1.1) when differentiated s times with respect to z yields

$$\left(D_q^{s} h_{\vartheta}\right)(z) = \lambda_1 z^{-\vartheta-s} + \sum_{\upsilon=0}^{\infty}\varphi_{\upsilon+\vartheta}a_{\upsilon+\vartheta}z^{\upsilon+\vartheta-s}, \tag{1.8}$$

where

$$\begin{aligned}\varphi_{\upsilon+\vartheta} &= \frac{\left[\upsilon+\vartheta\right]_q!}{\left[\upsilon+\vartheta-s\right]_q!},\\ \lambda_1 &= \left(\frac{-1}{q^{\vartheta}}\right)\left(\frac{-1}{q^{\vartheta+1}}\right)\cdots\left(\frac{-1}{q^{2\vartheta-1}}\right)\left[\vartheta\right]_q\left[\vartheta+1\right]_q\cdots\left[2\vartheta-1\%\right]_q z^{-2\vartheta},\end{aligned} \tag{1.9}$$

for $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Taking $\vartheta = 1$ in (1.7) then we have the q-derivative (or q-difference) D_q for $h \in \mathrm{M}$ which is given by (1.4).

Recently, q-calculus has attracted more attention from researchers due to its applications in mathematics and physics. Ismail et al. article [14] first described the q-extension of the class of q-starlike functions. Numerous well-respected academics have since carried out ground breaking work in the field of Geometric Function Theory. In particular, the q-Mittag-Leffler functions were investigated by Srivastava and Bansal [15, 16] and in [17]. The authors of [18, 19] also explored the class of q-starlike functions and looked into a third Hankel determinant. Using q-calculus operator theory, Srivastava et al. have recently published a series of studies (for example, [20–24]). In addition, many mathematicians have investigated operator theory in the q-calculus within the framework of Geometric Function Theory, for examples, [25–31].

Definition 5. *A function* $h_\vartheta \in \mathrm{M}(\vartheta)$ *be in the class* $\mathcal{MS}^*_{q,\vartheta}[Y_1, Y_2]$ *if and only if*

$$\left|\frac{\left(Y_2-1\right)\left(-\dfrac{z^s\left(D_q^s h_\vartheta\right)(z)}{h_\vartheta(z)}\right)-\left(Y_1-1\right)}{\left(Y_2+1\right)\left(-\dfrac{z^s\left(D_q^s h_\vartheta\right)(z)}{h_\vartheta(z)}\right)-\left(Y_1+1\right)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

It can be observed easily that

$$\mathcal{MS}^*_{q,1}\,(Y_1,\ Y_2) = \mathcal{MS}^*_q(Y_1,\ Y_2)\ .$$

Mahmood *et al.* have introduced and investigated this class in [13].

It is clear that

$$\lim_{q\to 1-} \mathcal{MS}^*_{q,1}\left[Y_1, Y_2\right] = \mathcal{MS}^*\left[Y_1, Y_2\right]$$

where $\mathcal{MS}^*\ [Y_1,\ Y_2]$, Ali *et al.* introduced and investigated this class in [32].

For $q \to 1-$, $Y_1 = 1$ and $Y_2 = -1$, then

$$\lim_{q\to 1-} \mathcal{MS}^*_{q,1}\left[1,-1\right] = \%\mathcal{MS}^*,$$

where $\mathcal{MS}^*$ denote the class of meromorphic starlike function.

In this section, we explore a sufficient condition for $h_\vartheta \in \mathcal{MS}^*_{q,\vartheta}[Y_1,\ Y_2]$ that will be utilized in the exploration of subsequent outcomes. We will also study the ratio between the series of partial sums

$$h_{\vartheta,k}(z) = \frac{1}{z^\vartheta} + \sum_{v=0}^{k} a_{v+\vartheta} z^{v+\vartheta} \qquad (k \in \mathbb{N}) \tag{1.10}$$

and the function h_ϑ of the kind provided by (1.1) , when the coefficients are sufficiently small.

2. Main Result

2.1. Coefficient Estimates

Theorem 1. *Let h_ϑ is a function of type (1.1), then f belongs to the class $\mathcal{MS}^*_{q,\vartheta}[Y_1,\ Y_2]$, if*

$$\sum_{v=0}^{\infty} \Lambda^{v}_{q,\vartheta}\left(Y_1, Y_2\right)\left|a_{v+\vartheta}\right| \le \Upsilon_{q,\vartheta}\left(Y_1, Y_2\right), \tag{2.1}$$

where

$$\Lambda^{v}_{q,\vartheta}\left(Y_1, Y_2\right) = 2\left(\varphi_{v+\vartheta} + 1\right) + \left|\left(Y_2 + 1\right)\varphi_{v+\vartheta} - \left(Y_1 - 1\right)\right| \tag{2.2}$$

and

$$\Upsilon_{q,\vartheta}\,(Y_1,\ Y_2) = \left|\,(Y_2 + 1)\,\lambda_1 + (Y_1 + 1)\,\right| - 2\,(\lambda_1 + 1)\,, \tag{2.3}$$

where λ_1 is given by (1.9).

Proof. Suppose that (2.1) is satisfy, then it is enough to prove that

$$\left|\frac{\left(Y_2 - 1\right)\left(-\dfrac{z^s\left(D_q^s h_\vartheta\right)(z)}{h_\vartheta(z)}\right) - \left(Y_1 - 1\right)}{\left(Y_2 + 1\right)\left(-\dfrac{z^s\left(D_q^s h_\vartheta\right)(z)}{\%h_\vartheta(z)}\right) - \left(Y_1 + 1\right)} - \frac{1}{1-q}\right| < \frac{1}{1-q}.$$

Now we have

$$\left|\frac{(\Upsilon_2-1)\left(-\dfrac{z^s\left(D_q^s h_\vartheta\right)(z)}{h_\vartheta(z)}\right)-(\Upsilon_1-1)}{(\Upsilon_2+1)\left(-\dfrac{z^s\left(D_q^s h_\vartheta\right)(z)}{h_\vartheta(z)}\right)-(\Upsilon_1+1)}-\frac{1}{1-q}\right|$$

$$\leq\left|\frac{-(\Upsilon_2-1)z^s\left(D_q^s h_\vartheta\right)(z)-(\Upsilon_1-1)h_\vartheta(z)}{-(\Upsilon_2+1)z^s\left(D_q^s h_\vartheta\right)(z)-(\Upsilon_1+1)h_\vartheta(z)}-1\right|+\frac{q}{1-q}$$

$$=2\left|\frac{z^s\left(D_q^s h_\vartheta\right)(z)+h_\vartheta(z)}{-(\Upsilon_2+1)z^s\left(D_q^s h_\vartheta\right)(z)-(\Upsilon_1+1)h_\vartheta(z)}\right|+\frac{q}{1-q}$$

$$=2\left|\frac{(\lambda_1+1)+\sum_{\upsilon=0}^{\infty}(1+\varphi_{\upsilon+\vartheta})a_{\upsilon+\vartheta}z^{\upsilon+\vartheta}}{-(\Upsilon_2+1)\lambda_1-(\Upsilon_1+1)-\sum_{\upsilon=0}^{\infty}\left((\Upsilon_2+1)\varphi_{\upsilon+\vartheta}+(\Upsilon_1+1)\right)a_{\upsilon+\vartheta}z^{\upsilon+\vartheta}}\right|+\frac{q}{1-q}$$

$$\leq 2\left(\frac{(\lambda_1+1)+\sum_{\upsilon=0}^{\infty}(1+\varphi_{\upsilon+\vartheta})|a_{\upsilon+\vartheta}|}{\left|(\Upsilon_2+1)\lambda_1+(\Upsilon_1+1)\right|-\sum_{\upsilon=0}^{\infty}\left|(\Upsilon_2+1)\varphi_{\upsilon+\vartheta}-(\Upsilon_1-1)\right||a_{\upsilon+\vartheta}|}\right)+\frac{q}{1-q}. \tag{2.4}$$

The inequality (2.4) is bounded by $\frac{1}{1-q}$ if

$$\sum_{\upsilon=0}^{\infty}\Lambda_{q,\vartheta}^{\upsilon}(\Upsilon_1,\upsilon)|a_{\upsilon+\vartheta}|<\Upsilon_{q,\vartheta}(\Upsilon_1,\Upsilon_2),$$

where $\Lambda_{q,\vartheta}^{\upsilon}(\Upsilon_1,\Upsilon_2)$ and $\Upsilon_{q,\vartheta}(\Upsilon_1,\Upsilon_2)$ are given by (2.2) and (2.3) respectively. The proof of Theorem 1 is thus concluded.

Corollary 1. *Let h_ϑ is a function of type (1.1), then it will be in the class $\mathcal{MS}_{q,\vartheta}^*[\Upsilon_1,\Upsilon_2]$, then*

$$a_{\upsilon+\vartheta}\leq\frac{\Upsilon_{q,\vartheta}(\Upsilon_1,\Upsilon_2)}{\Lambda_{q,\vartheta}^{\upsilon}(\Upsilon_1,\Upsilon_2)}. \tag{2.5}$$

Equality hold for the function

$$h_{0,\upsilon}(z)=\frac{1}{z^\vartheta}+\frac{\Upsilon_{q,\vartheta}(\Upsilon_1,\Upsilon_2)}{\Lambda_{q,\vartheta}^{\upsilon}(\Upsilon_1,\Upsilon_2)}z^{\upsilon+\vartheta-1},$$

where $\Upsilon_{q,\vartheta}(\Upsilon_1,\Upsilon_2)$ and $\Lambda_{q,\vartheta}^{\upsilon}(\Upsilon_1,\Upsilon_2)$ are given by (2.2) *and* (2.3) *respectively.*

Theorem 1 has a well-known corollary that was first proposed in [13] for the case when $\vartheta = 1$.

Corollary 2. *[13]. Let h is a function function of $h\in\mathcal{M}$ be in the class $\mathcal{MS}_q^*[\Upsilon_1,\Upsilon_2]$, if*

$$\sum_{\upsilon=1}^{\infty}\Lambda(\upsilon,\Upsilon_1,\Upsilon_2,q)|a_\upsilon|\leq\Upsilon(\Upsilon_1,\Upsilon_2,q),$$

where

$$\Lambda\,(v,\,Y_1,\,Y_2,\,q) = 2\,(|\,v\,|_q + 1) + |\,(Y_2 + 1)\,|\,v\,|_q - (Y_1 - 1)\,|\,q$$

and

$$\Upsilon(Y_1,\,Y_2,\,q) = |\,(Y_2 + 1) - (Y_1 + 1)\,q\,| + 2\,(1 - q)\,.$$

2.2. Distortion Inequalities

Theorem 2. *If $h_\vartheta \in \mathcal{MS}^*_{q,\vartheta}[Y_1,\,Y_2]$, then*

$$\frac{1}{r^\vartheta} - \frac{\Upsilon_{q,\vartheta}(Y_1,Y_2)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} r^\vartheta \le \left|h_\vartheta(z)\right| \le \frac{1}{r^\vartheta} + \frac{\Upsilon_{q,\vartheta}(Y_1,Y_2)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} r^\vartheta.$$

Equality hold for the function

$$h_{0,1}(z) = \frac{1}{z^\vartheta} + \frac{\Upsilon_{q,\vartheta}(Y_1,Y_2)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} z^\vartheta \quad \text{at } z = ir,$$

with $\Lambda^v_{q,\vartheta}(Y_1,\,Y_2)$ and $\Upsilon_{q,\vartheta}(Y_1,\,Y_2)$ are given in (2.2) *and* (2.3) *respectively.*

Proof. Let $h_\vartheta \in \mathcal{MS}^*_{q,\vartheta}[Y_1,\,Y_2]$. Then in the view of Theorem 1, we have

$$\Lambda^1_{q,\vartheta}(Y_1,Y_2)\sum_{v=1}^{\infty}\left|a_{v+\vartheta}\right| \le \sum_{v=1}^{\infty}\Lambda^v_{q,\vartheta}(Y_1,Y_2)\left|a_{v+\vartheta}\right| < \Upsilon_{q,\vartheta}(Y_1,Y_2),$$

which yields

$$\left|h_\vartheta(z)\right| \le \frac{1}{r^\vartheta} + \sum_{n=1}^{\infty}\left|a_{n+\vartheta}\right| r^{n-\vartheta} \le \frac{1}{r^\vartheta} + r^\vartheta\sum_{n=1}^{\infty}\left|a_{n+\vartheta}\right| \le \frac{1}{r^\vartheta} + \frac{\Upsilon_{q,\vartheta}(Y_1,Y_2)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} r^\vartheta.$$

Similarly, we have

$$\left|h_\vartheta(z)\right| \ge \frac{1}{r^\vartheta} - \sum_{v=1}^{\infty}\left|a_{v+\vartheta}\right| r^{v-\vartheta} \ge \frac{1}{r^\vartheta} - r^\vartheta\sum_{v=1}^{\infty}\left|a_{v+\vartheta}\right| \ge \frac{1}{r^\vartheta} - \frac{\Upsilon_{q,\vartheta}(Y_1,Y_2)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} r^\vartheta.$$

Thus its complete the proof of Theorem 2.

Theorem 2 has a well-known corollary that was first proposed in [13] for the case when $\vartheta = 1$.

Corollary 3. *[13]. If $h \in \mathcal{MS}^*_q\,[Y_1,\,Y_2]$, then*

$$\frac{1}{r} - \frac{\Upsilon(Y_1,Y_2,q)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} r \le \left|h_\vartheta(z)\right| \le \frac{1}{r} + \frac{\Upsilon(Y_1,Y_2,q)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} r.$$

Equality hold for the function

$$h_1(z) = \frac{1}{z} + \frac{\Upsilon(Y_1,Y_2,q)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)} z \quad \text{at } z = ir.$$

Theorem 3. *If $h_\vartheta \in \mathcal{MS}^*_{q,\vartheta}[Y_1, Y_2]$, then*

$$\frac{1}{r^{\vartheta+1}}-\frac{(\vartheta+1)\Upsilon_{q,\vartheta}(Y_1,Y_2)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)}\leq|h'_\vartheta(z)|\leq\frac{1}{r^{\vartheta+1}}+\frac{(\vartheta+1)\Upsilon_{q,\vartheta}(Y_1,Y_2)}{\Lambda^1_{q,\vartheta}(Y_1,Y_2)},\quad(|z|=r)$$

and $\Lambda^v_{q,\vartheta}(Y_1, Y_2)$ and $\Upsilon_{q,\vartheta}(Y_1, Y_2)$ are given by (2.2) *and* (2.3) *respectively.*

Proof. The proof of Theorem 3 can easily obtain by using the same steps of Theorem 2.

Theorem 3 has a well-known corollary that was first proposed in [13] for the case when $\vartheta = 1$.

Corollary 4. *[13]. If $h \in \mathcal{MS}^*_q[Y_1, Y_2]$, then*

$$\frac{1}{r^2}-\frac{2\Upsilon_q(Y_1,Y_2)}{\Lambda_q(Y_1,Y_2)}\leq|h'(z)|\leq\frac{1}{r^2}+\frac{2\Upsilon_q(Y_1,Y_2)}{\Lambda_q(Y_1,Y_2)},\quad(|z|=r).$$

2.3. Partial Sums for the function class $MS^*_{q,\vartheta}[Y_1, Y_2]$

Here, we examine the relation-ship between the series of partial sums and a function of the type (1.1). We will investigate sharp lower bounds for

$$Re\left(\frac{h_\vartheta(z)}{h_{\vartheta,k}(z)}\right),\left(\frac{h_{\vartheta,Y_2}(z)}{h_\vartheta(z)}\right),Re\left(\frac{D_qh_\vartheta(z)}{D_qh_{\vartheta,k}(z)}\right)\text{and }Re\left(\frac{(D_qh_{\vartheta,k})(z)}{(D_qh_\vartheta)(z)}\right).$$

Partial sums of $h_{\vartheta,k}$ are represented by

$$h_{\vartheta,k}(z)=\frac{1}{z^\vartheta}+\sum_{v=0}^{k}a_{v+\vartheta}z^{v+\vartheta}.$$

Theorem 4. *A function h_ϑ of the type (1.1) satisfies condition (2.1), then*

$$Re\left(\frac{h_\vartheta(z)}{h_{\vartheta,k}(z)}\right)\geq1-\frac{1}{\chi_{k+\vartheta+1}}\qquad(\forall z\in\mathbb{U})\tag{2.6}$$

and

$$Re\left(\frac{h_{\vartheta,k}(z)}{h_\vartheta(z)}\right)\geq\frac{\chi_{k+\vartheta+1}}{1+\chi_{k+\vartheta++1}}\qquad(\forall z\in\mathbb{U}),\tag{2.7}$$

where

$$\chi_{k+\vartheta}=\frac{\Lambda^k_{q,\vartheta}(Y_1,Y_2)}{\Upsilon_{q,\vartheta}(Y_1,Y_2)}\tag{2.8}$$

and $\Lambda^k_{q,}\vartheta(Y_1, Y_2)$ and $\Upsilon_{q,\vartheta}(Y_1, Y_2)$ are defined in (2.2) *and* (2.3) *respectively.*

Proof. We build up a proof of the inequality (2.6) by assuming that

$$\chi_{k+\vartheta+1}\left[\frac{h_\vartheta(z)}{h_{\vartheta,j}(z)}-\left(1-\frac{1}{\chi_{k+\vartheta+1}}\right)\right]=\frac{1+\sum_{v=0}^{k}a_{v+\vartheta}z^{v+\vartheta-1}+\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}a_{v+\vartheta}z^{v+\vartheta+1}}{1+\sum_{v=0}^{k}a_{v+\vartheta}z^{v+\vartheta+1}}=\frac{1+q_1(z)}{1+q_2(z)}.$$

If, we set

$$\frac{1+q_1(z)}{1+q_2(z)}=\frac{1+w(z)}{1-w(z)},$$

then, after a little more simplifying, we have

$$w(z)=\frac{q_1(z)-q_2(z)}{2+q_1(z)+q_2(z)}.$$

Thus, we find that

$$w(z)=\frac{\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}a_{v+\vartheta}z^{v+\vartheta-1}}{2+2\sum_{v=0}^{k}a_{v+\vartheta}z^{v+\vartheta+1}+\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}a_{v+\vartheta}z^{v+\vartheta+1}}$$

and

$$|w(z)|\leq\frac{\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}|a_{v+\vartheta}|}{2-2\sum_{v=0}^{k}|a_{v+\vartheta}|-\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}|a_{v+\vartheta}|}.$$

Now one can see that

$$|\,w\,(z)\,|\,\leq 1$$

if and only if

$$2\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}|a_{v+\vartheta}|\leq 2-2\sum_{v=0}^{k}|a_{v+\vartheta}|,$$

which implies that

$$\sum_{v=0}^{k}|a_{v+\vartheta}|+\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}|a_{v+\vartheta}|\leq 1. \tag{2.9}$$

Finally, The proof of (2.6), only requires us to demonstrate that the L.H.S of (2.9) is bounded above by $\sum_{v=0}^{\infty}\chi_{v+\vartheta}|a_{v+\vartheta}|$, which is equal to

$$\sum_{v=0}^{k}(1-\chi_{v+\vartheta})|a_{v+\vartheta}|+\sum_{v=k+1}^{\infty}(\chi_{k+\vartheta+1}-\chi_{v+\vartheta})|a_{v+\vartheta}|\geq 0. \tag{2.10}$$

Now we have finished the demonstration of inequality in (2.6).

Next to prove the inequality (2.7), we fixed

$$(1+\chi_{k+\vartheta})\left(\frac{h_{\vartheta,k}(z)}{h_\vartheta(z)}-\frac{\chi_{k+\vartheta}}{1+\chi_{k+\vartheta}}\right)=\frac{1+\sum_{v=0}^{k}a_{v+\vartheta}z^{v+\vartheta-1}-\chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}a_{v+\vartheta}z^{v+\vartheta-1}}{1+\sum_{v=0}^{\infty}a_{v+\vartheta}z^{v+\vartheta-1}}$$

$$=\frac{1+w(z)}{1-w(z)},$$

where

$$\left|w(z)\right| \leq \frac{\left(1+\chi_{k+\vartheta+1}\right)\sum_{v=k+1}^{\infty}\left|a_{v+\vartheta}\right|}{2-2\sum_{v=0}^{k}\left|a_{v+\vartheta}\right|-\left(\chi_{k+\vartheta+1}-1\right)\sum_{v=k+1}^{\infty}\left|a_{v+\vartheta}\right|} \leq 1. \tag{2.11}$$

The inequality (2.11) is equivalent to

$$\sum_{v=0}^{k}\left|a_{v+\vartheta}\right| + \chi_{k+\vartheta+1}\sum_{v=k+1}^{\infty}\left|a_{v+\vartheta}\right| \leq 1. \tag{2.12}$$

We have now finished the proof of (2.7) by establishing that the L.H.S in (2.12) is bounded above by $\sum_{v=0}^{\infty}\chi_{v+\vartheta}\left|a_{v+\vartheta}\right|$. This concludes the proof of Theorem 4.

Theorem 5. *If h_ϑ of the form* (1.1) *hold the condition* (2.1)*, then*

$$Re\left(\frac{\left(D_q h_\vartheta\right)(z)}{\left(D_q h_{\vartheta,k}\right)(z)}\right) \geq 1-\frac{\left[k+\vartheta\right]_q}{\chi_{k+\vartheta+1}} \qquad \left(\forall z \in \mathbb{U}\right)$$

and

$$Re\left(\frac{\left(D_q h_{\vartheta,k}\right)(z)}{\left(D_q h_\vartheta\right)(z)}\right) \geq \frac{\chi_{k+\vartheta+1}}{\chi_{k+\vartheta+1}+\left[k+\vartheta\right]_q} \qquad \left(\forall z \in \mathbb{U}\right),$$

where $\chi_{k+\vartheta}$ is given by (2.8) .

Proof. In this case, we do not detail how we came to prove Theorem 5. It is analogues cab be found in Theorem 4.

2.4. Radius of Starlikeness

In the Theorem 6 we obtain the radius of starlikeness for the class $\mathcal{MS}^{*}_{q,\vartheta}[Y_1, Y_2]$, when h_ϑ given by (1.1) is meromorphically starlike of order $\alpha(0 \leq \alpha < 1)$ in $|z| < r$.

Theorem 6. *Let the function h_ϑ defined by* (1.1) *will belong in the class $\mathcal{MS}^{*}_{q,\vartheta}[Y_1, Y_2]$. Then, if*

$$\inf_{v\geq 1}\left[\frac{(1-\alpha)\Lambda^{v}_{q,\vartheta}\left(Y_1,Y_2\right)}{\left(\left[v+\vartheta+\alpha\right]_q+(1-\alpha)\right)\Upsilon_{q,\vartheta}\left(Y_1,Y_2\right)}\right]^{\frac{1}{v+\vartheta}} = r$$

is positive, then h_ϑ is ϑ-valently meromorphically starlike of order α in $|z| \leq r$.

Proof. To prove the Theorem 6, we have to show that

$$\left|\frac{zD_q h_\vartheta(z)}{h_\vartheta(z)}+1\right| \leq 1-\alpha \qquad \left(0 \leq \alpha < 1\right) \text{ and } |z| \leq r_1.$$

We have

$$\left|\frac{zD_q h_\vartheta(z)}{h_\vartheta(z)}+1\right| = \left|\frac{\sum_{\upsilon=0}^{\infty}\left[\upsilon+\vartheta+\alpha\right]_q a_{\upsilon+\vartheta} z^{\upsilon+\vartheta}}{\frac{1}{z^\vartheta}+\sum_{\upsilon=0}^{\infty} a_{\upsilon+\vartheta} z^{\upsilon+\vartheta}}\right| \le \frac{\sum_{\upsilon=0}^{\infty}\left[\upsilon+\vartheta+\alpha\right]_q \left|a_{\upsilon+\vartheta}\right|\left|z\right|^{\upsilon+\vartheta}}{1-\sum_{\upsilon=0}^{\infty}\left|a_{\upsilon+\vartheta}\right|\left|z\right|^{\upsilon+\vartheta}}. \tag{2.13}$$

Hence (2.13) holds true if

$$\sum_{\upsilon=0}^{\infty}\left[\upsilon+\vartheta+\alpha\right]_q \left|a_{\upsilon+\vartheta}\right|\left|z\right|^{\upsilon+\vartheta} \le (1-\alpha)\left(1-\sum_{\upsilon=0}^{\infty}\left|a_{\upsilon+\vartheta}\right|\left|z\right|^{\upsilon+\vartheta}\right). \tag{2.14}$$

We may express the inequality (2.14) as:

$$\sum_{\upsilon=0}^{\infty}\left(\frac{\left[\upsilon+\vartheta+\alpha\right]_q+(1-\alpha)}{1-\alpha}\right)\left|a_{\upsilon+\vartheta}\right|\left|z\right|^{\upsilon+\vartheta} \le 1. \tag{2.15}$$

With the help of (2.1), the inequality (2.15) is true if

$$\left(\frac{\left[\upsilon+\vartheta+\alpha\right]_q+(1-\alpha)}{1-\alpha}\right)\left|z\right|^{\upsilon+\vartheta} \le \frac{\Lambda^{\upsilon}_{q,\vartheta}\left(Y_1,Y_2\right)}{\Upsilon_{q,\vartheta}\left(Y_1,Y_2\right)}. \tag{2.16}$$

Solving (2.16) for $|z|$, we have

$$\left|z\right| \le \left(\frac{(1-\alpha)\Lambda^{\upsilon}_{q,\vartheta}\left(Y_1,Y_2\right)}{\left[\upsilon+\vartheta+\alpha\right]_q+(1-\alpha)\Upsilon_{q,\vartheta}\left(Y_1,Y_2\right)}\right)^{\frac{1}{\upsilon+\vartheta}}. \tag{2.17}$$

In the view of (2.17), Theorem 6 is now completed.

3. Conclusion

In this article, we used the concepts of q-calculus notations and introduced a higher-order q-derivative operator for multivalent meromorphic. We used this newly defined operator and Janowski functions to establish a new class of meromorphic multivalent q-starlike functions. Furthermore, we investigated some useful properties, such as coefficient estimates, distortion theorems, partial sums, and the radius of starlikeness for the functions belonging to the newly defined class of meromorphic multivalent q-starlike functions. We also highlighted a number of well established consequences of our main findings.

Further mathematical work may be done using the operator of this article and the subordinations approach, which enables for the definition of several further subclasses for meromorphic functions. For these classes, a number of new properties can be investigated, such as Feketo-Szego inequality, Hankel determinant, Upper bound, subordination results, etc.

Data Availability Statement

No data were used to support this study.

Conflict of interest

The author declare that they have no competing interests.

Authors contributions

All authors contributed equally to the writing of this paper.

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