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Geometric characterization of pointwise slant curves

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Abstract

In the present paper we study the characteristics of pointwise slant curves in a normal almost contact semi-Riemannian three-manifold N^3 . These curves are characterized by the pseudo-Riemannian scalar product between the normal vector at the curve and the reeb vector field of manifold $N3$. In this class of manifolds, curvature and torsion of such curves are determined. The Lancret of slant curves in manifold N^3 is obtained. Additionally, pointwise slant curves with proper mean curvature are characterized.

Key words and phrases. Semi-Riemannian metrics, Partial differential equations, Mathematical operators, Contact structure, Slant curve, Legendre curve, Lancret

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1. Introduction

The study of slant curves in contact three-manifolds was started by the authors in [1]. According to [2], slant curves are the generalization of Legendre curves. More precisely, let $(N^3; \varphi, \xi, \eta, g)$ be an almost contact Riemannian three-manifold. Then a smooth unit speed curve $\upsilon : J \to N^3$ is called *slant curve* if $g(\xi, \upsilon'(s)) = \cos\vartheta(s) = constant$, where J is an open interval and $\vartheta: I \to [0, 2\pi)$ is called

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structural angle (or contact angle). In light of Theorem 3.1 of [1], we obtain for a non-geodesic slant curve υ in Sasakian three-manifolds that

$$\text{Lancret}_{\pm}(\upsilon) = \frac{\pm\tau - 1}{\kappa}. \tag{1.1}$$

The equation (1.1) signifies that the curve υ is a Legendre helix if and only if the absolute value of its torsion is equal to 1, that is, $|\tau| = 1$. Several authors exhaustively studied and analyzed the geometry of slant curves (see [3–7]). Recently, in [8], the present authors defined pointwise slant curves (abbreviated as PS curves) as a natural generalization of slant curves. In this paper, we investigate how these curves are characterized in N^3, where N^3 is a normal almost contact semi-Riemannian three-manifold (abbreviated as a. c. s. three-manifold).

The organizational structure of the paper is as follows: the basics of almost contact semi-Riemannian manifolds are given in Sect. 2 and Sect. 3. The characterizations of PS curves in N^3 are obtained in Sect. 4. The curvature and torsion of PS curves which are not geodesic in N^3 are determined in Sect. 5. We derive the necessary and sufficient condition for the PS curve (which is not geodesic) having proper mean curvature vector $\mathbb{H}$. Examples are also constructed for illustration.

2. Preliminaries

Let the manifold N^{2n+1} of dimension $(2n + 1)$ be C^∞ and paracompact. Let the Lie algebra of vector fields on N^{2n+1} is denoted by $\Xi(N^{2n+1})$ and $\Gamma(F)$ denotes $\Xi(N^{2n+1})$-module of sections of vector bundle F over the manifold.

The manifold N^{2n+1} is referred to as an *almost contact manifold* if the structure group $\mathrm{GL}_{2n+1}\mathbb{R}$ of TN^{2n+1} (tangent bundle) is reducible to $\mathrm{U}(n) \times \{1\}$. Equivalently, if there exists (φ, ξ, η)-*structure* satisfying

$$\varphi^2 + I = \eta \otimes \xi \quad \text{and } \eta(\xi) = 1, \tag{2.1}$$

where the vector field ξ is called *characteristic* or *Reeb vector field*, φ is an endomorphism, $\mathcal{I}$ denotes the identity, and η is a 1–form such that $\eta\wedge(d\eta)^n \neq 0$ known as *contact form*; d is one of the mathematical operators called the exterior differential operator. It is simple to deduce from equation (2.1) that $\eta \circ \varphi = \varphi\xi = 0$ and $\mathrm{rank}(\varphi) = 2n$ [9].

A semi-Riemannian metric g is called compatible with the (φ, ξ, η)-*structure* if

$$g(\varphi\cdot, \varphi\cdot) + \varepsilon\eta(\cdot)\eta(\cdot) = g(\cdot, \cdot),$$

where g has the signature $(2q + 1, 2n - 2q)$ or $(2q, 2n - 2q + 1)$ depending on whether ξ is spacelike or timelike, respectively, and $\varepsilon^2 = 1$. $(N^{2n+1}; \varphi, \xi, \eta, g)$ is known as an *almost contact semi-Riemannian* $(2n + 1)$–*manifold* (abbreviated as *a. c. s.* $(2n + 1)$–*manifold*). Here, $g(\xi, \xi) = \varepsilon$ and $\eta(X) = \varepsilon g(X, \xi)$. This implies that ξ is never lightlike. Let Φ denotes the *fundamental* 2–*form* then it is given by $\Phi(\cdot, \cdot) = \varepsilon g(\cdot, \varphi\cdot)$. Let the manifold N^{2n+1} further satisfies $d\eta = \Phi$, then it is called a *contact semi-Riemannian manifold*. Let $\mathfrak{h}$ denotes the tensor field defined by $\mathfrak{h} = (1/2)\pounds_\xi\varphi$. Then this tensor field plays a crucial role in N^{2n+1}, where $\pounds$ denotes the operator of Lie-derivative. Here $\mathfrak{h}$ is self-adjoint and satisfies

$$\nabla_{\xi\varphi} = 0, \quad \nabla\xi = -\varphi \circ \mathfrak{h} - \varepsilon\varphi\, \mathfrak{h}(\xi) = \mathrm{trace}(\mathfrak{h}) = 0, \quad \varphi \circ \mathfrak{h} = -\mathfrak{h} \circ \varphi,$$

where ∇ is a Levi-Civita connection. For more details about the geometry of the contact semi-Riemannian manifold, we refer to [10–12].

Let us consider the product manifold $N^{2n+1} \times \mathbb{R}$: $\left(Z, \varsigma\frac{d}{dt}\right)$ is an arbitrary tangent vector, ς is a smooth function on $N^{2n+1} \times \mathbb{R}$, t is the standard coordinate on $\mathbb{R}$, and $Z \in \Gamma(TN^{2n+1})$.

The almost complex structure J on this direct product is given as follows:

$$J\left(Z,\varsigma\frac{d}{dt}\right)=\left(\varphi Z-\varsigma\xi,\eta(Z)\frac{d}{dt}\right).$$

Then N^{2n+1} is called *normal* if and only if

$$d\eta(\cdot,\cdot)\xi+\frac{1}{2}[\varphi,\varphi](\cdot,\cdot)=0,$$

where $[\varphi, \varphi]$ denotes the *Nijenhuis torsion*, and it is given as follows:

$$[\varphi, \varphi](\cdot, \cdot) = [\varphi\,\cdot, \varphi\,\cdot] - \varphi[\varphi\cdot, \cdot] + \varphi^2\,[\cdot, \cdot] - \varphi[\cdot, \varphi\cdot]$$

(see [9, 11]).

3. Normal a. c. s. Three-Manifolds

In the present paper, we restrict ourselves to dimension three. Analogous to [13], we give some results related to this case. If we consider N^3 to be an a. c. s. three-manifold, we find that

$$(\nabla_{Z1}\,\varphi)Z_2 = -\eta(Z_2\,)\varphi\nabla Z_1\xi + \varepsilon g(\varphi\nabla Z_1\xi,\, Z_2)\xi, \tag{3.1}$$

where Z_1 , $Z_2 \in \Gamma(\mathcal{T}N^3\,)$.

Proposition 3.1. *In an a. c. s. three-manifold N^3 , the following conditions are mutually equivalent:*

(i) *manifold $\mathbb{M}^3$ is normal;*
(ii) *$\varphi\nabla Z\xi = \nabla\varphi Z\xi$;*
(iii) *$\nabla Z\xi = \varepsilon\beta(Z - \eta(Z)\xi) - \varepsilon\alpha\varphi Z$.*

Here $Z \in \Gamma(\mathcal{T}N^3)$, α and β being smooth functions on N^3 for which we have

$$2\alpha = \text{trace } \{Z \rightarrow \varphi\nabla_Z\xi\}\ ,\ 2\beta = \text{trace } \{Z \rightarrow \nabla Z\xi\}.$$

From equation (3.1) and Proposition 3.1, we find that

$$(\nabla_{Z_1}\varphi)Z_2 = \beta(g(\varphi Z_1,\, Z_2)\xi - \varepsilon\eta(Z_2)\varphi Z_1) + \alpha(g(Z_1,\, Z_2)\xi - \varepsilon\eta(Z_2)Z_1. \tag{3.2}$$

Moreover, manifold N^3 satisfies

$$\xi(\alpha) + 2\varepsilon\alpha\beta = 0.$$

Therefore, $\beta = 0$ if α is a non-zero constant. Analogous to [5], N^3 is called

- *Cosymplectic semi-Riemannian manifold* if $\alpha = \beta = 0$;
- *quasi-Sasakian semi-Riemannian manifold* if $\beta = 0$ and $\xi(\alpha) = 0$;
- β-*Kenmotsu semi-Riemannian manifold* if $\alpha = 0$ and β is a non-zero constant.

In addition, N^3 is said to be a *Sasakian semi-Riemannian manifold* if α = 1, β = 0 and *Kenmotsu semi-Riemannian manifold* if $\alpha = 0$, $\beta = 1$. Now, we give examples of normal a. c. s. three-manifolds.

Example 3.2. Consider the standard Cartesian coordinates on $\mathbb{R}^3_1$ as (x, y, z), 1-form η is given by $\eta = ydx + dz$, $\xi = \partial z$ and the endomorphism φ is defined by $\varphi\partial_x = \partial_y$, $\varphi\partial_y = y\partial_z - \partial_x$, $\varphi\partial_z = 0$, where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial z}$. Then $\varphi^2 + \mathcal{I} = \eta \otimes \xi$ and $\eta(\xi) = 1$ are obtained. Therefore, the (φ, ξ, η)-structure is almost contact. Further, by simple computations, we find that the (φ, ξ, η)-structure is normal.

Let $\mathcal{N}^3_\varepsilon := \mathbb{R}^2_1 \times \mathbb{R}_+ \subset \mathbb{R}^3_1$ and normal a. c. s. structure (φ, ξ, η, g) is restricted to $\mathcal{N}^3_\varepsilon$, where $g = \varepsilon\eta \otimes \eta + z^2\,(dx^2 + dy^2)$. Then, we have $g(\partial_x , \partial_x) = \varepsilon y^2 + z^2$, $g(\partial_y, \partial_y) = z^2$, $g(\partial_z , \partial_z) =\varepsilon$, $g(\partial_x, \partial_y) = g(\partial_y, \partial_x) = 0$,

$g(\partial_x, \partial_z) = g(\partial_z, \partial_x) = \varepsilon y$, $g(\partial_y, \partial_z) = g(\partial_z, \partial_y) = 0$, where $\varepsilon^2 = 1$. Using φ and g, we have $g(\varphi Z_1, \varphi Z_2) = g(Z_1, Z_2) - \varepsilon\eta(Z_1)\eta(Z_2)$ and $\eta(Z_1) = \varepsilon g(Z_1, \xi)$, and thus $(\mathcal{N}^3_\varepsilon; \varphi, \xi, \eta, g)$ is a normal a. c. s. three-manifold. For ∇ with respect to g, we have

$$\nabla_{\partial_x}\partial_x = \frac{y}{z}\partial_x - \frac{\varepsilon y}{z^2}\partial_y - \left(\frac{z^2+\varepsilon y^2}{\varepsilon z}\right)\partial_z, \nabla_{\partial_x}\partial_y = \nabla_{\partial y}\partial_x = \frac{\varepsilon y}{2z^2}\partial_x + \left(\frac{z^2-\varepsilon y^2}{2z^2}\right)\partial_z,$$

$$\nabla_{\partial_x}\partial_z = \nabla_{\partial_z}\partial_x = \frac{1}{z}\partial_x - \frac{\varepsilon}{2z^2}\partial_y - \frac{y}{z}\partial_z, \nabla_{\partial_y}\partial_y = \frac{y}{z}\partial_x - \left(\frac{z^2+\varepsilon y^2}{\varepsilon z^2}\right)\partial_z,$$

$$\nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_y = \frac{\varepsilon}{2z^2}\partial_1 + \frac{1}{z}\partial_y - \frac{\varepsilon y}{2z^2}\partial_z, \nabla_{\partial_z}\partial_z = 0.$$

Using the expressions above and the equation (3.2), we have $\beta = \frac{\varepsilon}{z}$ and $\alpha = \frac{1}{2z^2}$.

Example 3.3. Let $\mathcal{K}^3_\varepsilon := \mathbb{R}^3_1$ with (x, y, z) as standard Cartesian coordinates, $\eta = dz$, $\xi = \partial z$, endomorphism φ satisfies: $\varphi\partial_x = \partial_y$, $\varphi\partial_y = -\partial_x$, $\varphi\partial_z = 0$ and metric tensor is given as $g = \varepsilon\eta \otimes \eta + \exp(2z)(dx^2 + dy^2)$, where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial z}$. Then, by making straightforward computations, we find that $(\mathcal{K}^3_\varepsilon; \varphi, \xi, \eta, g)$ is a normal a. c. s. three-manifold. For ∇ with respect to this g, we have

$$\begin{aligned}&\nabla_{\partial_x}\partial_x = \nabla_{\partial_y}\partial_y = -\varepsilon\exp(2z)\partial_z, \nabla_{\partial_x}\partial_z = \nabla_{\partial_z}\partial_x = \partial_x,\\ &\nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_y = \partial_y, \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \nabla_{\partial_z}\partial_z = 0.\end{aligned} \tag{3.3}$$

By the virtue of partial differential equations (3.3) and equation (3.2), we obtain $\beta = \varepsilon$ and $\alpha = 0$. Therefore, $\mathcal{K}^3_\varepsilon$ is a 3-dimensional ε-Kenmotsu manifold. Furthermore, $\mathcal{K}^3_\varepsilon$ is a warped product $\mathbb{R} \times_f \mathbb{F}$ where warping function f is given by $f(z) = \exp(\epsilon z)$.

For more information on warped geometry, we may refer to [14].

4. Pointwise Slant Curves

Let N^3 be a normal a. c. s. three-manifold with Levi-Civita connection ∇, $\upsilon : I \to N^3$ be an unit speed curve in N^3, I being an open interval. Then υ is called a *Frenet curve* if the Frenet frame $\{T := \upsilon', N, B\}$ of υ satisfies (*Frenet–Serret formulas*) [7, p. 968]:

$$\nabla_T T = \kappa N,\ \nabla_T N = \varepsilon\tau B - \kappa T \text{ and } \nabla_T B = -\tau N, \tag{4.1}$$

where $\kappa = |\nabla_T T|$ and τ are denote the curvature and torsion of υ, respectively. The vectors T, B and N are known as the tangent, binormal and principal normal of υ, respectively. The curve υ is called geodesic if $\nabla_{\upsilon'}\upsilon' = 0$ and it is not geodesic if $\kappa > 0$ everywhere on I.

Following [8], we give

Definition 4.1. Let N^3 be a normal a. c. s. three-manifold and $\upsilon : J \to N^3$ be a Frenet curve. Let $\rho : I \to I_1 \subseteq \mathbb{R}$ be a smooth function, where $I_1 = [-1, 1]$ or $I_1 = [0, \infty)$. Then υ is said to be a *pointwise slant curve* (abbreviated as *PS curve*) if $\eta(\upsilon') = \rho$. We call ρ, a *slant function*. In particular, υ is *slant curve* if ρ =constant [4] and if $\rho = 0$ it is *Legendre curve* ([2, 9]). The PS curve is said to be *proper*, if neither $\rho = 0$ nor ρ =constant.

Remark 4.2. For a PS curve υ in N^3, we have

$$\rho = \varepsilon g(\xi, \upsilon'(s)). \tag{4.2}$$

If the characteristic vector field is timelike, then $\rho = \sinh\vartheta_1(s)$, where angle $\vartheta_1 : I \to [0, \infty)$ is called *Lorentzian timelike* between υ' and characteristic vector field [15]. In this case, $\rho \in [0, \infty)$. Further, let

$\{\upsilon', \xi\}$ span a spacelike vector subspace, and if characteristic vector field is spacelike, then $\rho = \cos\vartheta_2(s)$, where $\vartheta_2 : I \to [0, 2\pi)$ is the contact angle of υ [4]. In this case, $\rho \in [-1, 1]$.

Using Definition 4.1, we find that a curve $\upsilon(s) = (\upsilon_1(s), \upsilon_2(s), \upsilon_3(s))$ in $\mathcal{N}^3_\varepsilon$ is a PS if and only if

$$\begin{cases} \upsilon_1'\upsilon_2 + \upsilon_3' = \rho, \\ \upsilon_3^2(\upsilon_1'^2 + \upsilon_2'^2) = 1 - \varepsilon\rho^2, \end{cases}$$

where ρ is a smooth function. It can be easily seen that

$$\upsilon' = \upsilon_1' \partial_1 + \upsilon_2'\partial_2 + (\rho - \upsilon_1'\upsilon^2)\partial_3 \text{ and } \varphi\upsilon' = -\upsilon_2'\partial_1 + \upsilon_1'\partial_2 + \upsilon_2\upsilon_2'\partial_3.$$

Here, it is important to mention that every unit speed curve in N^3 is not necessarily a PS curve, for instance, consider the following curve in $\mathcal{N}^3_{-1}$:

$$\gamma(s) = \left(-\sqrt{5}s, \tfrac{2}{\sqrt{5}}, 1\right), s \in \mathbb{R}.$$

Then we have $g(\gamma', \gamma') = 1$. Here $\rho = -2$, this implies that sinh $\vartheta 1(s) = -2$, which is not possible value for the above defined smooth function ρ.

After taking covariant differentiation of equation (4.2) along υ, we get

$$\rho' = \varepsilon g(\xi, \kappa N) + g(\upsilon', -\alpha\varphi\upsilon' + \beta(\upsilon' - \rho\xi)) = \beta(1 - \varepsilon\rho^2) + \kappa\eta(N). \tag{4.3}$$

The interpretation of ξ in terms of the Frenet frame of υ provides

$$\varepsilon\eta(N)^2 + \eta(B)^2 = 1 - \varepsilon\rho^2. \tag{4.4}$$

Using equations (4.3) and (4.4), we have the following characterization result for the PS curve:

Proposition 4.3. *Let $\upsilon : J \to N^3$ be a non-geodesic curve. Then υ is a PS curve if and only if*

$$\eta(N) = \frac{\rho' - \beta(1 - \varepsilon\rho^2)}{\kappa}. \tag{4.5}$$

Therefore, a necessary condition for υ to be a PS curve is

$$\varepsilon\eta(N)^2 \leq (1 - \varepsilon\rho^2), \tag{4.6}$$

only if $\rho \neq \pm 1$.

Remark 4.4.

(i) The characterization (4.5) (as well as (4.6)) is independent of the Sasakian part, *i.e.*, does not depend on α. Thus for a (semi-Riemannian) quasi-Sasakian manifold, this expression gives$\eta(N) = \frac{\rho'}{\kappa}$. Particularly, for ρ-slant curve υ which is not geodesic in quasi-Sasakian 3-manifold, we have $\eta(N) = 0$ [1].

(ii) From (4.6), equality yields $\eta(N)^2 = 1 - \rho^2$ (then $\eta(T) = \rho$ and $\eta(B) = 0$) only if ξ is spacelike. Particularly, υ is a Legendre curve with $\kappa = \beta \mid \upsilon$ and $N = -\xi$.

(iii) For a slant curve υ with $\varepsilon = 1$, equation (4.5) provides $\eta(N) = -\frac{\beta}{\kappa}\sin^2\vartheta_1$ [4].

Let υ be a PS curve in N^3. Consider υ', $\varphi\upsilon'$, ξ such that $g(\xi, \xi) = \varepsilon$, $g(\upsilon', \upsilon') = 1$, $g(\upsilon', \xi) = \varepsilon\rho$, $g(\varphi\upsilon' \varphi\upsilon') = 1 - \varepsilon\rho^2$ and $g(\upsilon', \varphi\upsilon') = g(\xi, \varphi\upsilon') = 0$; ρ being a smooth function. Then, the set $\{\upsilon', \varphi\upsilon', \xi\}$ is linearly independent, forms a basis of $\mathcal{T}_{\upsilon(s)}N^3$ for every $s \in I$ if and only if $m = \sqrt{|1 - \varepsilon\rho^2|} \neq 0$. Now, we can define orthonormal vector fields as:

$$B_1 = \upsilon', \qquad B_2 = \frac{\varphi\upsilon'}{m}, \qquad B_3 = \frac{\xi - \varepsilon\rho\upsilon'}{m}, \tag{4.7}$$

where $m = \sqrt{|1 - \varepsilon\rho^2|}$, $g(B_3, B_3) = \varepsilon$ and $g(B_1, B_1) = g(B_2, B_2) = 1$.

Here, $\{\upsilon', \varphi\upsilon', \xi\}$ is linearly dependent if and only if $\upsilon' = \varepsilon\varphi\upsilon' + \rho\xi$ or $\upsilon' = \rho\xi$. Furthermore, if $\{\upsilon', \varphi\upsilon', \xi\}$ is linearly dependent, then $|\rho| = 1$ and ξ is spacelike. This implies that υ is necessarily a geodesic. Therefore, we must have $m \neq 0$ for the non-geodesic curve υ. The decomposition of ξ with respect to $\{B_1, B_2, B_3\}$ is as follows:

$$\xi = \varepsilon(mB_3 + \rho B_1)\,. \tag{4.8}$$

Remark 4.5. We define the *Lancret coefficient* of a PS curve υ in N^3 which is not geodesic by

$$\text{Lancret }(\upsilon) = \frac{\rho}{m}. \tag{4.9}$$

The insight for above definition is that for $\varepsilon = 1$, the above expression yields $\text{Lancret}(\upsilon) = \dfrac{\cos\vartheta}{|\sin\vartheta|}$, where $\vartheta = constant$, analogous to contact geometry [3].

5. Main Results

Let $\nabla_{\upsilon'}B_1 = a_1B_1 + b_1B_2 + c_1B_3$, where a_1, b_1 and c_1 are any C^∞ functions. Then $a_1 = g(\nabla\upsilon'\, B_1, B_1) = 0$, $b_1 = g(\nabla\upsilon'\, B_1, B_2) = \delta m$ and $-\varepsilon c_1 = g(B_1, \nabla_{\upsilon'}B_3)$, where $\delta = \dfrac{1}{m^2}g(\nabla_{\upsilon'}\upsilon', \varphi\upsilon')$. Using equation (3.2), we get $g(B_1, \nabla_{\upsilon'}B_3) = \varepsilon\left(\beta m - \dfrac{\rho'}{m}\right)$. Thus, $\nabla_{\upsilon'}B_1 = \delta m B_2 - \left(\beta m - \dfrac{\rho'}{m}\right)B_3$. Analogy to this, we can find $\nabla_{\upsilon'}B_2$ and $\nabla_{\upsilon'}B_3$. This provides the following result:

Lemma 5.1. *Let $\upsilon : J \to N^3$ be a PS curve which is not geodesic. Then, we have*

$$\nabla_{\upsilon'}B_1 = m\delta B_2 + \left(\frac{\rho'}{m} - m\beta\right)B_3, \tag{5.1}$$

$$\nabla\upsilon' B_2 = -m\delta B_1 + (\alpha + \delta\rho)B_3, \tag{5.2}$$

$$\nabla_{\upsilon'}B_3 = \varepsilon\left(\left(m\beta - \frac{\rho'}{m}\right)B_1 - (\delta\rho + \alpha)B_2\right), \tag{5.3}$$

where

$$m = \sqrt{|1 - \varepsilon\rho^2|} \text{ and } \delta = \frac{1}{m^2}g(\nabla_{\upsilon'}\upsilon', \varphi\upsilon'). \tag{5.4}$$

Theorem 5.2. *Let $\upsilon : J \to N^3$ be a PS curve which is not geodesic. Then expression for curvature κ and torsion τ of υ are as follow:*

$$\begin{cases} \kappa = & m\sqrt{\left|\varepsilon\left(\beta - \dfrac{\rho'}{m^2}\right)^2 + \delta^2\right|}, \\ \tau = & \pm\left(\alpha + \rho\delta + \dfrac{(\beta\delta' - \beta'\delta) - 2\left(\dfrac{\rho'\delta'}{m^2}\right) + \left(\dfrac{\delta\rho'}{m^2}\right)'}{\varepsilon\left(\beta - \dfrac{\rho'}{m^2}\right)^2 + \delta^2}\right), \end{cases} \tag{5.5}$$

where m and δ *are given in equation* (5.4).

Proof. From equation (5.1) and by computation of length of $\nabla_{\upsilon'}\upsilon'$ i.e. $\|\nabla_{\upsilon'}\upsilon'\|$, we receive κ. In light of equations (4.1) and (5.1), we get

$$N = \frac{m\delta}{\kappa}B_2 - \left(\frac{m\beta}{\kappa} - \frac{\rho'}{m\kappa}\right)B_3.$$

Let $a = m\delta$ and $b = \left(m\beta - \frac{\rho'}{m}\right)$. Then

$$\begin{aligned}\nabla_{\upsilon'}N &= \left(-\frac{a^2}{\kappa} - \frac{\varepsilon b^2}{\kappa}\right)B_1 + \left(\frac{a'\kappa - a\kappa'}{\kappa^2} + \frac{\varepsilon b}{\kappa}\left(\alpha + \frac{a\rho}{m}\right)\right)B_2 + \left(-\frac{\kappa b' - \kappa' b}{\kappa^2} + \frac{a}{\kappa}\left(\frac{a\rho}{m} + \alpha\right)\right)B_3 \\ &= -\kappa B_1 + \left(\frac{\varepsilon b B_2 + a B_3}{\kappa}\right)\left(\alpha + \frac{\rho a}{m} + \frac{ba' - b'a}{\kappa^2}\right). \end{aligned} \tag{5.6}$$

Using Frenet–Serret formulas, we obtain from equation (5.6) that $\tau = \pm\left(\frac{\rho a}{m} + \frac{ba' - b'a}{\kappa^2} + \alpha\right)$
Hence, it completes the proof.

Now, we can give the following result as a corollary of the above theorem:

Corollary 5.3. *Let υ be a slant curve in N^3 which is not geodesic. Then the expressions for curvature and torsion of υ are as follows:*

$$\begin{cases} \kappa = m\sqrt{|\varepsilon\beta^2 + \delta^2|}, \\ \tau = \pm\left(\rho\delta + \alpha + \dfrac{\beta\delta' - \beta'\delta}{\delta^2 + \varepsilon\beta^2}\right). \end{cases} \tag{5.7}$$

Then the associated Lancret of υ for $\delta \neq 0$, is given by

$$\text{Lancret}_{\pm}(\upsilon) = \frac{(\beta'\delta - \beta\delta')\left|\delta^2 + \varepsilon\beta^2\right|^{-\frac{1}{2}} - (\alpha \mp \tau)\left|\delta^2 + \varepsilon\beta^2\right|^{\frac{1}{2}}}{\kappa\delta}, \tag{5.8}$$

where m and δ are given in equation (5.4).

Proof. Since $\rho' = 0$, *i.e.*, υ is slant. Thus, equation (5.5) leads to (5.7) and equation(5.8) follows directly from (4.9) and (5.7). Hence, it completes the proof.

Consider $\xi = p_1T + p_2N + p_3B$, where p_1, p_2, and p_3 are any smooth functions. By the virtue of equations (4.1), (4.7), (5.1)-(5.3), and (5.6), we can readily compute p_1, p_2, and p_3. This provides the following result:

Proposition 5.4. *Let $\upsilon : J \to N^3$ be a PS curve which is not geodesic. Then, the decomposition of ξ is expressed as follows*

$$\xi = \varepsilon(\rho\kappa T + (\rho' - m^2\beta)N + m^2 sgn(\tau)\delta B)/\kappa,$$

where $\{T, N, B\}$ denotes the Frenet frame of υ and m, δ are given in equation (5.4).

Now, we present the κ and τ values of a non-geodesic PS curve in some subclasses of N^3.

Corollary 5.5. *Let $\upsilon : J \to N^3$ be a PS curve which is not geodesic.*

(a) *Let N^3 be a Cosymplectic semi-Riemannian manifold then we have*

$$\begin{cases} \kappa = m\sqrt{\left|\varepsilon\left(\frac{\rho'}{m^2}\right)^2 + \delta^2\right|}, \\ \tau = \pm\left(\rho\delta - \dfrac{2\left(\frac{\rho'\delta'}{m^2}\right) - \left(\frac{\rho'\delta}{m^2}\right)'}{\varepsilon\left(\frac{\rho'}{m^2}\right)^2 + \delta^2}\right). \end{cases} \tag{5.9}$$

(b) *Let N^3 be a quasi-Sasakian semi-Riemannian manifold then we have*

$$\begin{cases} \kappa = m\sqrt{\left|\varepsilon\left(\frac{\rho'}{m^2}\right)^2 + \delta^2\right|}, \\ \tau = \pm\left(\alpha + \rho\delta - \dfrac{2\left(\frac{\rho'\delta'}{m^2}\right) - \left(\frac{\rho'\delta}{m^2}\right)'}{\varepsilon\left(\frac{\rho'}{m^2}\right)^2 + \delta^2}\right). \end{cases}$$

(c) *Let N^3 be a β-Kenmotsu semi-Riemannian manifold then we have*

$$\begin{cases} \kappa = m\sqrt{\left|\varepsilon\left(\beta - \frac{\rho'}{m^2}\right)^2 + \delta^2\right|}, \\ \tau = \pm\left(\rho\delta + \dfrac{\beta\delta' - 2\left(\frac{\rho'\delta'}{m^2}\right) + \left(\frac{\rho'\delta}{m^2}\right)'}{\varepsilon\left(\beta - \frac{\rho'}{m^2}\right)^2 + \delta^2}\right), \end{cases}$$

where m and δ are given in equation (5.4).

Below, we give certain proper PS curves in $\mathcal{N}_\varepsilon^3$:

Example 5.6. Let in $\mathcal{N}_1^3$

$$\upsilon^1(s) = (s, 0, \sin s),\ s \in (0, 2\pi).$$

Then the curve υ^1 is a proper PS curve in $\mathcal{N}_1^3$. Here, we have $\alpha(\upsilon^1(s)) = \frac{1}{2}\csc^2 s$, $\beta(\upsilon^1(s)) = \csc s$, $\delta(s) = -\cot s \csc s$, $\rho = \cos s$,

$$\kappa = \left(1 + 3\sin^2 s\right)^{\frac{1}{2}} \csc s \text{ and } \tau = \left|\frac{\cot^2 s}{2\kappa^2} - 1\right|.$$

The Euclidean image of υ^1 is depicted in Figure 1. Moreover, some of the particular cases of υ^1 are striking, these are portrayed in Figures 2, 3, and 4.

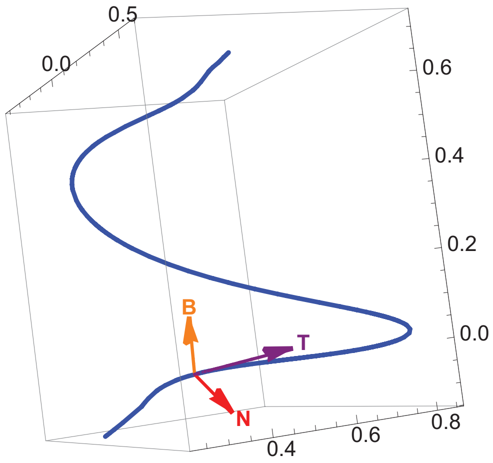

Figure 1: $\rho = \cos s$

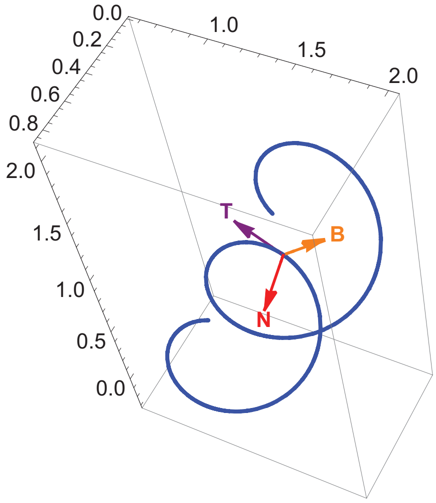

Figure 2: $\rho = \frac{\sqrt{3}}{2}$.

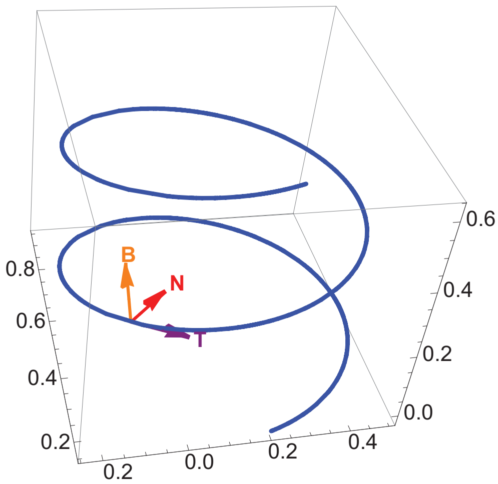

Figure 3: $\rho = \frac{1}{2}$.

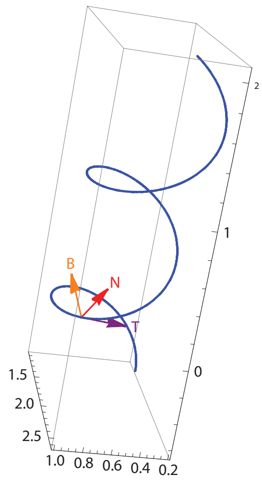

Figure 4: $\rho = 0$.

Example 5.7. Let in $\mathcal{N}^3_{-1}$

$$\upsilon^2(s) = (k,\ s,\ \cosh s), \quad s \in (0, \infty),\ k \in \mathbb{R}.$$

Then υ^2 is a proper PS curve in $\mathcal{N}^3_{-1}$. Here, we find $\beta(\upsilon_2\ (s)) = -\operatorname{sech} s$, $\delta(s) = \operatorname{sech} s(-s + \tanh s)$, $\alpha(\upsilon^2(s)) = \frac{1}{2}\operatorname{sech}^2 s$, $\rho = \sinh s$,

$$\kappa = \sqrt{|\,4 - (s - \tanh s)^2\,|} \ \textit{and}\ \tau = \left|(\tanh s)(s - \tanh s) - \frac{1}{2}\operatorname{sech}^2 s + \frac{2\tanh^2 s}{4 - (s - \tanh s)^2}\right|.$$

From Definition 4.1, the necessary and sufficient conditions for $\upsilon(s) = (\upsilon_1(s),\ \upsilon_2(s),\ \upsilon_3(s))$ to be a PS curve in $\mathcal{K}^3_\varepsilon$ are given by

$$\begin{cases} \upsilon_3' = \rho, \\ \upsilon_1'^2 + \upsilon_2'^2 = \exp(-2\upsilon_3)m^2. \end{cases} \tag{5.10}$$

Now, we have

$$\upsilon' = \upsilon_1'\partial_x + \upsilon_2'\partial_y + \rho\partial_z \text{ and } \varphi\upsilon' = -\upsilon_y'\partial_x 1 + \upsilon_1'\partial_y.$$

From (5.10), we find

$$\upsilon_1' = m\exp(-\upsilon_3)\cos\varsigma(s), \upsilon_2' = m\exp(-\upsilon_3)\cos\varsigma(s), \upsilon_3' = \rho,$$

where $\varsigma \in C^\infty(I)$. This leads to the following result:

Proposition 5.8. *Let $\upsilon : J \to \mathcal{K}_\varepsilon^3$ be a PS curve in $\mathcal{K}_\varepsilon^3$ which is not geodesic. Then υ can be expressed as follows:*

$$\upsilon(s) = \left(\int_{s_0}^{s} \Psi(t)\Omega(t)m\,dt, \int_{s_0}^{s} \rho(t)dt\right),$$

where $\Psi(s)$ = (cos $\varsigma(s)$, sin $\varsigma(s)$) is parametrization of circle $\mathbb{S}1$, $\Omega(s) = \exp\left(-\int_{s_0}^{s} \rho(t)dt\right)$ *and ς is a smooth function on I.*

By straightforward computations, we get

$$\begin{cases} \kappa &= m\sqrt{\left|\varepsilon\left(1 - \dfrac{\varepsilon\rho'}{m^2}\right)^2 + \varsigma'^2\right|}, \\ \tau &= \pm\left(\rho\varsigma' + \dfrac{\varepsilon\varsigma'' - 2\left(\dfrac{\varsigma''\rho'}{m^2}\right) + \left(\dfrac{\rho'\varsigma'}{m^2}\right)'}{\varsigma'^2 + \varepsilon\left(1 - \dfrac{\varepsilon\rho'}{m^2}\right)^2}\right). \end{cases}$$

Let us consider a PS curve $\upsilon : J \to N^3$ in N^3. Then we have

$$\mathbb{H} = \nabla_{\upsilon'}\upsilon',$$

where $\mathbb{H}$ denotes the *mean curvature vector field*. Then PS curve υ is known as a *curve with proper* $\mathbb{H}$, if we have a C^∞ function λ such that

$$\Delta\mathbb{H} = \lambda\mathbb{H}. \tag{5.11}$$

Here Δ denotes the Laplace operator and it is explicitly given by

$$\Delta = -\nabla_{\upsilon'}\nabla_{\upsilon'}.$$

If $\lambda = 0$ then PS curve υ is said to be a *curve with harmonic* $\mathbb{H}$ ([4, 6]). Using Frenet-Serret formulas, (5.11) can be rewritten as

$$-3\kappa\kappa'\upsilon' + \varepsilon(2\kappa'\tau + \kappa\tau')B + (\kappa'' - \kappa^3 - \varepsilon\kappa\tau^2)N = -\kappa\omega N.$$

This implies that the relation $\Delta\mathbb{H} = \lambda\mathbb{H}$ holds if and only if $\kappa\tau' = 0$ and $\varepsilon\tau^2\kappa + \kappa^3 = \lambda\kappa$. This yields the following remark:

Remark 5.9. Let υ be a PS curve in N^3 . Then it holds $\Delta\mathbb{H} = \lambda\mathbb{H}$ if and only if PS curve υ is either geodesic or helix satisfying

$$\varepsilon\tau^2 + \kappa^2 = \lambda. \tag{5.12}$$

In view of the above remark, it can be easily seen that the PS curve which is not geodesic with harmonic $\mathbb{H}$ does not exist in N^3. In light of equations (5.5), (5.11) and (5.12), we provide the following proposition which generalizes result 3.7 of [3] and result 3 of [4].

Proposition 5.10. *If we consider a PS curve* υ *in* N^3 *which is not geodesic. Then* υ *is having proper* $\mathbb{H}$ *if and only if it is helix such that* $\lambda = \left(\alpha + \rho\delta + \frac{\varrho}{\kappa^2}\right)\varepsilon + \kappa^2$, *where* δ *and* m *are given in* (5.4) *and* $\varpi = \left(\beta\delta' - \beta'\delta\right)m^2 - 2\delta'\rho' + m^2\left(\frac{\rho'}{m^2}\right)'$.

Example 5.11. Let in $\mathcal{N}_1^3$

$$\upsilon^3(s) = \left(1, \ln s, \frac{s}{\sqrt{2}}\right), s \in (0,\infty),$$

then υ^3 is a slant Frenet curve in $\mathcal{N}_1^3$. Here, we find $\rho = \frac{1}{\sqrt{2}}$ (that is, $\vartheta = \frac{\pi}{4}$), $\beta(\upsilon^3(s)) = \frac{\sqrt{2}}{s}$, $\alpha(\upsilon^3(s)) = \frac{1}{s^2}$, $\delta(s) = -\frac{\sqrt{2}}{s^2}\ln se \neq 0$,

$$\kappa = \frac{\sqrt{s^2 + (\ln se)^2}}{s^2} \text{ and } \tau = \left|\left(\frac{s^2}{(\ln se)^2 + 1} - 1\right)\frac{\ln s}{s^2}\right|.$$

Using (5.8), we find Lancret (υ^3) = 1. Now, we have that

$$\tau^2 + \kappa^2 = \frac{\left(2\ln se - (\ln s)^2\right)}{s^2\left(1 + (\ln se)^2\right)} + \frac{(2\ln s \ln se + 1)}{s^4} + \frac{(\ln s)^2}{\left(1 + (\ln se)^2\right)^2}.$$

From the above equation, we get $\tau^2 + \kappa^2$ = non-constant. Therefore, by the consequence of Proposition 5.10, we find υ^3 is without proper $\mathbb{H}$.

Example 5.12. Let in $\mathcal{N}_{-1}^3$

$$\upsilon^4(s) = \left(\sqrt{3}s, 1, \frac{2}{\sqrt{3}}\right), s \in \mathbb{R},$$

then υ^4 is slant curve in $\mathcal{N}_{-1}^3$. For υ^4, we have $\rho = \sqrt{3}$ (that is, $\vartheta = \operatorname{arcsin} h\sqrt{3}$), $\alpha(\upsilon^4(s)) = \frac{3}{8}$, $\delta(s) = -\frac{3\sqrt{3}}{4} \neq 0$, $\beta(\upsilon^4(s)) = -\frac{\sqrt{3}}{2}$, $\kappa = \frac{\sqrt{15}}{2}$ and $\tau = \frac{21}{8}$. Using equation (5.8), we get Lancret $(\upsilon^4) = \frac{\sqrt{3}}{2}$. Now, we obtain that $\kappa^2 - \tau^2$ =constant. Therefore, in light of Proposition 5.10, we have υ^4 is helix and having proper $\mathbb{H}$ with $\lambda = -\frac{201}{64}$.

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References

[1] Cho, J.T.; Inoguchi, J.; Lee, J.E. Slant curves in Sasakian 3-manifolds. *Bull. Austral. Math.Soc.* 2006, 74, 359–367.
[2] Baikoussis, C.; Blair, D.E. On Legendre curves in contact 3-manifolds. *Geom. Dedicata* 1994, 49, 135–142.
[3] Călin, C.; Crasmareanu, M. Slant curves in three-dimensional normal almost contact geometry. *Mediterr. J. Math.* 2013, 10, 1067–1077.
[4] Călin, C.; Crasmareanu, M.; Munteanu, M.I. Slant curves in 3-dimensional *f*-Kenmotsu manifolds. *J. Math. Anal. Appl.* 2012, 394, 400–407.
[5] Inoguchi, J.; Lee, J.E. Slant curves in 3-dimensional almost contact metric geometry. *Internat. Elect. J. Geom.* 2015, 8, 106–146.

[6] Inoguchi, J.; Lee, J.E. Almost contact curves in normal almost contact 3-manifolds. *J. Geom.* 2012, 103, 457–474.
[7] Welyczko J. Slant curves in 3-dimensional normal almost paracontact metric manifolds. *Mediterr. J. Math.* 2014, 11, 965–978.
[8] Sood, K.; Srivastava, K.; Srivastava, S.K. PS curves in quasi-paraSasakian 3-manifolds. *Mediterr. J. Math.* 2020, 17, 114. https://doi.org/10.1007/s00009-020-01554-y
[9] Blair, D.E. Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics 203, Birkhäuser Boston, Inc. Boston, MA, 2002.
[10] Perrone, D. Contact pseudo-metric manifolds of constant curvature and CR geometry, *Results Math.* 2014, 66, 213–225.
[11] Calvaruso, G.; Perrone, D. Contact pseudo-metric manifolds. *Differential Geom. Appl.* 2010, 28, 615–634.
[12] Deshmukh, S.; Belova, O.; Turki, N.B.; Vîlcu, G.E. Hypersurfaces of a Sasakian manifold-revisited. *Journal of Inequalities and Applications* 2021, 2021, 1-24. https://doi.org/10.1186/s13660-021-02584-0
[13] Olszak, Z. Normal almost contact manifolds of dimension three. *Ann. Pol. Math.* 1986, 47, 42–50.
[14] O'Neill, B. Semi-Riemannian geometry with applications to Relativity. Academic Press, New York, 1983.
[15] Ali, A.T.; Turgut, M. Position vector of a time-like slant helix in Minkowski 3-space. *J. Math. Anal. Appl.* 2010, 365, 559–569.