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On Graded W-2-absorbing second submodules

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Abstract

Let $\mathfrak{R}$ be a commutative graded ring with unity, $\mathfrak{I}$ be a graded $\mathfrak{R}$-module, W be a multiplicatively closed subset of homogeneous elements of $\mathfrak{R}$ and K be a graded submodule of $\mathfrak{I}$ such that $Ann_{\mathfrak{R}}(K) \cap W = \emptyset$. In this paper, we introduce the concept of graded W-2-absorbing second submodules of $\mathfrak{I}$ as a generalization of graded 2-absorbing second submodules. We say K is a graded W-2-absorbing second submodule of $\mathfrak{I}$, if there exists a fixed $s_\alpha \in W$ and whenever $r_g t_h K \subseteq H$, where r_g, $t_h \in h(\mathfrak{R})$ and H is graded submodule of $\mathfrak{I}$, then either $s_\alpha r_g K \subseteq H$ or $s_\alpha t_h K \subseteq H$ or $s_\alpha r_g t_h \in Ann_{\mathfrak{R}}(K)$. Several results concerning these classes of graded submodules are given.

Keywords and phrases: Graded W-2-Absorbing Second Submodules, Graded 2-Absorbing Second Submodules, Graded W-2-Absorbing Submodules

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1. Introduction

Throughout this article, we assume that $\mathfrak{R}$ is a commutative G-graded ring with identity and $\mathfrak{I}$ is a unitary graded $\mathfrak{R}$-module. Atani in [7] introduced the concept of graded prime submodules. Al-Zoubi, Abu-Dawwas, and Çeken in [2] introduced the concept of graded 2-absorbing ideals of graded commutative rings. Later on, Al-Zoubi and Abu-Dawwas in [1] extended graded 2-absorbing ideals to graded 2-absorbing submodules. In [14], the authors introduced and studied the concept of graded W-2-absorbing submodules as a generalization of graded 2-absorbing submodules. The notion of graded second sub-modules was introduced in [5] and studied in [3, 4, 6, 8]. Recently, Al-Zoubi and

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Al-Azaizeh in [3] introduced and studied the concepts of graded 2-absorbing second submodules. Here, we introduce the concept of graded W-2-absorbing second submodules over commutative graded rings as a generalization of graded 2-absorbing second submodules and investigate some properties of these classes of graded submodules.

2. Preliminaries

In this section we will give the definitions and results which are required in the next section.

Definition 2.1. (a) Let G be a group with identity e and $\Re$ be a commutative ring with identity $1_\Re$. Then $\Re$ is G-graded ring if there exist additive subgroups $\Re_g$ of $\Re$ indexed by the elements $g \in G$ such that $\Re = \bigoplus_{g\in G} \Re_g$ and $\Re_g\Re_h \subseteq \Re_{gh}$ for all $g, h \in G$. The elements of $\Re_g$ are called homogeneous of degree g. The set of all homogeneous elements of $\Re$ is denoted by $h(\Re)$, i.e. $h(\Re) = \bigcup_{g\in G} \Re g$, see [11].

(b) Let $\Re = \bigoplus_{g\in G} \Re_g$ be G-graded ring, an ideal P of $\Re$ is called a graded ideal if $P = \sum_{h\in G} P \cap \Re_h = \sum_{h\in G} P_h$. By $P \leq_G^{id} \Re$, we mean that P is a G-graded ideal of $\Re$. Also, by $P <_G^{id} \Re$, we mean that P is a proper G-graded ideal of $\Re$, see [11].

(c) A left $\Re$-module $\Im$ is said to be a G-graded $\Re$-module if $\Im = \bigoplus_{g\in G} \Im_g$ with $\Re_g\Im_h \subseteq \Im_{gh}$ for all $g, h \in G$, where $\Im_g$ is an additive subgroup of $\Im$ for all $g \in G$. The elements of $\Im_g$ are called homogeneous of degree g. The set of all homogeneous elements of $\Im$ is denoted by $h(\Im)$, i.e, $h(\Im) = \bigcup_{g\in G} \Im_g$. Note that $\Im_h$ is an $\Re_e$-module for every $h_\in G$, see [11].

(d) A submodule K of $\Im$ is called a graded submodule of $\Im$ if $K = \oplus_{h\in g}(K \cap \Im_h) := \oplus_{h\in g} K_h$. By $K \leq_G^{sub} \Im$, we mean that K is a G-graded submodule of $\Im$. Also, by $K <_G^{sub} \Im$, we mean that K is a proper G-graded submodule of $\Im$, see [11].

(e) If K is graded submodule of $\Im$, then $(K :_\Re \Im) = \{a \in \Re \mid a\Im \subseteq K\}$ is graded ideal of $\Re$, (see [7]). Furthermore, the annihilator of K in $\Re$ is denoted and defined by $Ann_\Re(K) = \{a \in \Re \mid aK = \{0\}\}$.

(f) A proper graded ideal K of $\Re$ is called a graded prime ideal if whenever $r,s \in h(\Re)$ with $rs \in K$, we have $r \in K$ or $s \in K$. The graded radical of a graded ideal P, denoted by $Gr(P)$, is the set of all $t = \sum_{g\in G} t_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $t_g^{ng} \in P$. Note that, if r is a homogeneous element, then $r \in Gr(P)$ if and only if $r^n \in P$ for some $n \in \mathbb{N}$, see [13].

(g) A proper graded submodule P of $\Im$ is called a graded prime submodule if whenever $a \in h(\Re)$ and $m \in h(\Im)$ with $am \in P$, then either $a \in (P :_\Re \Im)$ or $m \in P$, see [7].

(h) A non-zero graded submodule K of a $\Im$ is called graded second if for each $r \in h(\Re)$, the graded $\Re$-homomorphism $f : K \to K$ defined by $f(x) = rx$ is either surjective or zero. In other words, K is a graded second submodule of $\Im$ if $rK = K$ or $rK = 0$ for every $r \in h(\Re)$. This implies that $P = Ann_\Re(K)$ is a graded prime ideal of $\Re$ and K is called a P-graded second submodule. The graded second spectrum of $\Im$, denoted by $GSpec^s(\Im)$, is the set of all graded second submodules of $\Im$, see [6].

(i) A proper graded submodule K of $\Im$ is called a completely graded irreducible if $K = \cap_{\alpha\in\Lambda} K_\alpha$, where $\{K_\alpha\}_{\alpha\in\Lambda}$ is a family of graded submodule of $\Im$, implies that $K = K_\alpha$ for some $\alpha \in \Lambda$. Every proper graded submodule of $\Im$ is the intersection of all completely graded irreducible submodules containing it, see [3].

(j) A graded $\Re$-module $\Im$ is called graded comultiplication module (gr-comultiplication module) if for every graded submodule U of $\Im$, there exists a graded ideal P of $\Re$ such that $U = (0 :_\Im P)$, equivalently, for each graded submodule U of $\Im$, we have $U = (0 :_\Im Ann_\Re(U))$, see [5].

Definition 2.2. (a) A proper graded ideal J of $\Re$ is said to be a graded 2-absorbing (briefly, Gr-2^{abs}) ideal of $\Re$ if whenever $r, s, t \in h(\Re)$ with $rst \in J$, then $rs \in J$ or $rt \in J$ or $st \in J$.

(b) A non-zero graded submodule K of $\Im$ is called a graded 2-absorbing second (briefly, Gr-2^{abs}-second) submodule of $\Im$ if whenever $r, t \in h(\Re)$, C is a completely graded irreducible submodule of $\Im$, and $rtK \subseteq C$; then $rK \subseteq C$ or $tK \subseteq C$ or $rt \in Ann_R(K)$, see [3].

(c) A non-zero graded submodule K of $\Im$ is called a graded strongly 2-absorbing second (briefly, Gr-2^{abs}_{st}-second) submodule of $\Im$ if whenever $r, t \in h(\Re)$, C_1, C_2 are completely graded irreducible submodules of $\Im$, and $rtK \subseteq C_1 \cap C_2$, then $rK \subseteq C_1 \cap C_2$ or $tK \subseteq C_1 \cap C_2$ or $rt \in Ann_R(K)$, see [3].

(d) A nonempty subset W of a G-graded ring $\Re$ is called a multiplicatively closed subset (briefly, $m.c.s.$) of $\Re$ if $0 \notin W$, $1 \in W$ and $rt \in W$ for each $r, t \in W$.

(e) Let $W \subseteq h(\Re)$ be a m.c.s. of $\Re$ and K a graded submodule of $\Im$ such that $(K :_{\Re} \Im) \cap W = \emptyset$. We say that K is a graded W-2-absorbing (briefly, Gr-W-2^{abs}) submodule of $\Im$ if there exists a fixed $a_\alpha \in W$ and whenever $r_g s_h m_\lambda \in K$; where $r_g, s_h \in h(R)$ and $m_\lambda \in h(\Im)$, implies that $a_\alpha r_g s_h \in (K :_{\mathrm{R}} \Im)$ or $a_\alpha r_g m_\lambda \in K$ or $a_\alpha s_h m_\lambda \in K$. In particular, a graded ideal J of $\Re$ is called a graded W-2-absorbing (briefly, Gr-W-2^{abs}) ideal if J is a graded W-2-absorbing submodule of the graded $\Re$-module $\Re$, see [14].

(f) Let $W \subseteq h(\Re)$ be a m.c.s. of $\Re$ and $N \leq^{sub}_G \Im$ with $Ann_{\Re}(N) \cap W = \emptyset$. We say that N is a graded W-second (briefly, Gr-W-second) submodule of $\Im$ if there exists $w_\alpha \in W$, and whenever $rN \subseteq K$ for some $r \in h(\Re)$ and graded submodule K of $\Im$, then either $w_\alpha rN = 0$ or $w_\alpha N \subseteq K$.

Remark 2.3. Let K and H are two graded submodule of an graded $\Re$-module. To prove $K \subseteq H$, its enough to show that if J is a completely graded irreducible submodule of $\Im$ with $H \subseteq J$, then $K \subseteq J$, see [12].

3. Results

Definition 3.1. Let $W \subseteq h(\Re)$ be a $m.c.s.$ of $\Re$ and $K \leq^{sub}_G \Im$ with $Ann_{\Re}(K) \cap W = \emptyset$. We say that K is a graded W-2-absorbing second (briefly, Gr-W-2^{abs}-second) submodule of $\Im$, if there exists $w_\alpha \in W$ and whenever $r_g t_h K \subseteq H$, where $r_g, t_h \in h(\Re)$ and $H \leq^{sub}_G \Im$, then either $w_\alpha r_g K \subseteq H$ or $w_\alpha t_h K \subseteq H$ or $w_\alpha r_g t_h \in Ann_{\Re}(K)$. In particular, a graded ideal P of $\Re$ is said to be a graded W-2-absorbing second (briefly, Gr-W-2^{abs}-second) ideal if P is a Gr-W-2^{abs}-second submodule of $\Im$. By a Gr-W-2^{abs}-second module, we mean a graded module which is a Gr-W-2^{abs}-second submodule of itself.

Lemma 3.2. *Let $W \subseteq h(\Re)$ be a $m.c.s.$ of $\Re$ and $L = \bigoplus_{g\in G} Lg \leq^{id}_G \Re$. Let K be a Gr-W-2^{abs}-second submodule of $\Im$, then there exists $w_\alpha \in W$ and whenever $t_h \in h(\Re)$, $H \leq^{sub}_G \Im$ and $g \in G$ with $L_g t_h K \subseteq H$, then either $w_\alpha L_g K \subseteq H$ or $w_\alpha t_h K \subseteq H$ or $w_\alpha t_h L_g \subseteq Ann_{\Re}(K)$.*

Proof. Let $w_\alpha t_h K \nsubseteq H$ and $w_\alpha t_h L_g \nsubseteq Ann_{\Re}(K)$. Then there exists $b_g \in L_g$ with $w_\alpha t_h b_g K \neq 0$. Now $b_g t_h K \subseteq H$ and since K is a Gr-W-2^{abs}-second submodule of $\Im$, then $b_g w_\alpha K \subseteq H$. We show that $L_g w_\alpha K \subseteq H$. Let $c_g \in L_g$, then $(b_g + c_g)t_h K \subseteq H$, we get either $(b_g + c_g)w_\alpha K \subseteq H$ or $(b_g + c_g)t_h w_\alpha \in Ann_{\Re}(K)$. If $(b_g + c_g)w_\alpha K \subseteq H$, then since $b_g w_\alpha K \subseteq H$ we get $c_g w_\alpha K \subseteq H$. If $(b_g + c_g)t_h w_\alpha \in Ann_{\Re}(K)$, then since $w_\alpha t_h b_g K \neq 0$ we get $w_\alpha t_h c_g \notin Ann_{\Re}(K)$, but $c_g t_h K \subseteq H$ so $c_g w_\alpha K \subseteq H$. Thus $L_g w_\alpha K \subseteq H$. □

Lemma 3.3. *Let $W \subseteq h(\Re)$ be a $m.c.s.$ of $\Re$, $L = \bigoplus_{g\in G} Lg \leq^{id}_G \Re$ and $P = \bigoplus_{g\in G} P_g \leq^{id}_G \Re$. Let K be a Gr-W-2^{abs}-second submodule of $\Im$, then there exists $w_\alpha \in W$ and whenever $H \leq^{sub}_G \Im$ and $g, h \in G$ such that $L_g P_h K \subseteq H$, then either $w_\alpha L_g K \subseteq H$ or $w_\alpha P_h K \subseteq H$ or $w_\alpha L_g P_h \subseteq Ann_{\Re}(K)$.*

Proof. Let $w_\alpha L_g K \nsubseteq H$ and $w_\alpha P_h K \nsubseteq H$. We show that $w_\alpha L_g P_h \subseteq Ann_{\Re}(K)$. Let $r_g \in L_g$ and $t_h \in P_h$. By assumption there exists $x_g \in L_g$ such that $w_\alpha x_g K \nsubseteq H$. Since $x_g P_h K \subseteq H$, by Lemma 3.2, $w_\alpha x_g P_h \subseteq Ann_{\Re}(K)$, and hence $(L_g \backslash (H :_{h(\Re)} K))P_h w_\alpha \subseteq Ann_{\Re}(K)$. Similarly there exists $y_h \in (P_h \backslash (H :_{h(\Re)} K))$ with $w_\alpha L_g y_h \subseteq Ann_{\Re}(K)$ and $L_g(P_h \backslash (H :_{h(\Re)} K))w_\alpha \subseteq Ann_{\Re}(K)$. Hence $w_\alpha x_g y_h \in Ann_{\Re}(K)$, $w_\alpha x_g t_h \in Ann_{\Re}(K)$ and $w_\alpha r_g y_h \in Ann_{\Re}(K)$. Since $r_g + x_g = (r+x)_g \in L_g$ and $y_h + t_h = (y+t)_h \in P_h$, $(r_g + x_g)(y_h + t_h)K \subseteq H$. Thus, $w_\alpha(r_g + x_g) K \subseteq H$ or $w_\alpha(y_h + t_h)K \subseteq H$ or $w_\alpha(r_g + x_g)(y_h + t_h) \in Ann_{\Re}(K)$. If $w_\alpha(r_g + x_g)K \subseteq H$, then $w_\alpha r_g K \nsubseteq H$. So $r_g \in L_g \backslash (H :_{h(\Re)} K)$ and hence $w_\alpha r_g t_h \in Ann_{\Re}(K)$. Similarly if $w_\alpha(y_h + t_h)K \subseteq H$, then $w_\alpha r_g t_h \in Ann_{\Re}(K)$. If $w_\alpha(r_g + x_g)(y_h + t_h) \in Ann_{\Re}(K)$, then $w_\alpha(r_g y_h + r_g t_h + x_g y_h + x_g t_h) \in Ann_{\Re}(K)$ so $w_\alpha r_g t_h \in Ann_{\Re}(K)$. Thus $w_\alpha L_g P_h \subseteq Ann_{\Re}(K)$, as needed. □

Remark 3.4. Let U and V be two graded submodules of $\Re$. To prove $U \subseteq V$, its enough to show that if N is a completely graded irreducible submodule of $\Im$ such that $V \subseteq N$, then $U \subseteq N$, see([3], Lemma 2.2).

Theorem 3.5. *Let $W \subseteq h(\Re)$ be a m.c.s. of $\Re$. For $K \leq_G^{sub} \Im$ with $Ann_{\Re}(K) \cap W = \emptyset$ the following statement are equivalent*:

(i) *K is a Gr-W-2^{abs}-second submodule of $\Im$.*

(ii) *There exists $w_\alpha \in W$ with $w_\alpha^2 r_g t_h K = w_\alpha^2 r_g K$ or $w_\alpha^2 r_g t_h K = w_\alpha^2 t_h K$ or $w_\alpha^3 r_g t_h K = 0$ for each r_g, $t_h \in h(\Re)$.*

(iii) *There exists $w_\alpha \in W$* and whenever *$r_g t_h K \subseteq N_1 \cap N_2$ where r_g, $t_h \in h(\Re)$ and N_1, N_2 are completely graded irreducible submodules of $\Im$, implies either $r_g t_h w_\alpha K = 0$ or $w_\alpha r_g K \subseteq N_1 \cap N_2$ or $w_\alpha t_h K \subseteq N_1 \cap N_2$.*

(iv) *There exists $w_\alpha \in W$,* and *$L_g P_h K \subseteq H$ implies either that $w_\alpha L_g K \subseteq H$ or $w_\alpha P_h K \subseteq H$ or $w_\alpha L_g P_h \subseteq Ann_{\Re}(K)$, for each $L = \bigoplus_{g\in G} L_g \leq_G^{id} \Re$; $P = \bigoplus_{g\in G} P_g \leq_G^{id} \Re$ and $K \leq_G^{sub} \Im$.*

Proof. (*ii*) ⇒ (*i*): Let $r_g, t_h \in h(\Re)$ and $H \leq_G^{sub} \Im$ with $r_g t_h K \subseteq H$. By part (ii), there exists $w_\alpha \in W$ such that $w_\alpha^2 r_g t_h K = w_\alpha^2 r_g K$ or $w_\alpha^2 r_g t_h K = w_\alpha^2 t_h K$ or $w_\alpha^3 r_g t_h K = 0$. Thus either $w_\alpha^3 r_g t_h K = 0$ or $w_\alpha^3 r_g K \subseteq w_\alpha^2 r_g K = w_\alpha^2 r_g t_h K \subseteq w_\alpha^2 H \subseteq H$ or $w_\alpha^3 t_h K \subseteq w_\alpha^2 t_h K = w_\alpha^2 r_g t_h K \subseteq w_\alpha^2 H \subseteq H$. Put $w'_\alpha := w_\alpha^3$, so we have either $w'_\alpha r_g t_h K = 0$ or $w'_\alpha r_g K \subseteq H$ or $w'_\alpha t_h K \subseteq H$, as required. (*i*) ⇒ (*iii*): This is clear. (*iii*) ⇒ (*ii*): By part (iii), there exists $w_\alpha \in W$. Assume there are $r_g, t_h \in h(\Re)$ such that $w_\alpha^2 r_g t_h K \neq w_\alpha^2 r_g K$ and $w_\alpha^2 r_g t_h K \neq w_\alpha^2 t_h K$. Then there exists a completely graded irreducible submodule N_1, N_2 of $\Im$ with $w_\alpha^2 r_g t_h K \subseteq N_1$, $w_\alpha^2 r_g t_h K \subseteq N_2$, $w_\alpha^2 r_g K \nsubseteq N_1$ and $w_\alpha^2 t_h K \nsubseteq N_2$ by remark 3.4. Now $(w_\alpha r_g)(w_\alpha t_h)K = w_\alpha^2 r_g t_h K \subseteq N_1 \cap N_2$ implies either $w_\alpha^2 r_g K \subseteq N_1 \cap N_2$ or $w_\alpha^2 t_h K \subseteq N_1 \cap N_2$ or $w_\alpha^3 r_g t_h K = 0$ by part (iii). If $w_\alpha^2 r_g K \subseteq N_1 \cap N_2$ or $w_\alpha^2 t_h K \subseteq N_1 \cap N_2$, then $w_\alpha^2 r_g K \subseteq N_1$ and $w_\alpha^2 t_h K \subseteq N_2$, which is a contradiction. Thus $w_\alpha^3 r_g t_h K = 0$.

(*i*) ⇒ (*iv*): By Lemma 3.3. (*iv*) ⇒ (*i*): Let $r_g, t_h \in h(\Re)$ and $H \leq_G^{sub} \Im$ with $r_g t_h K \subseteq H$. Now, let $L = Rr_g$ and $P = Rt_h$ be two graded ideals of $\Re$ generated by r_g, t_h, respectively. Then $L_g P_h K \subseteq H$. By assumption, there exists $w_\alpha \in W$ such that either $w_\alpha L_g K \subseteq H$ or $w_\alpha P_h K \subseteq H$ or $w_\alpha L_g P_h K = 0$ and so either $w_\alpha r_g K \subseteq H$ or $w_\alpha t_h K \subseteq H$ or $w_\alpha r_g t_h \in Ann_{\Re}(K)$. □

Remark 3.6. Let $W \subseteq h(\Re)$ be a *m.c.s.* of $\Re$. Clearly every *Gr-W*-second submodule of $\Im$ and every graded *Gr*-2_{st}^{abs}-second submodule K of $\Im$ with $Ann_{\Re}(K) \cap W = \emptyset$ is *Gr-W*-2^{abs}-second submodule of $\Im$. However, the converse is not generally true, as the following example demonstrates.

Example 3.7. Let $\Re = \mathbb{Z}$ and $G = \mathbb{Z}_2$, then $\Re$ is a *G*-graded ring with $\Re_0 = \mathbb{Z}$ and $\Re_1 = \{0\}$. Consider $\Im = \mathbb{Z}_4$ as a $\mathbb{Z}$-module Then $\Im$ is a *G*-graded module with $\Im_0 = \mathbb{Z}_4$ and $\Im_1 = \{\bar{0}\}$. Take $W = \mathbb{Z}\backslash 2\mathbb{Z}$. Then $\Im$ is not a *Gr-W*-second $\mathbb{Z}$-module since for each $w \in W$, $2\mathbb{Z}_4 = 2s\mathbb{Z}_4 \neq w\mathbb{Z}_4 = \mathbb{Z}_4$ and $2s\mathbb{Z}_4 \neq 0$. However, if we consider $w = 1$, and $i, j \in \mathbb{Z}$ then there are three cases to consider:

Case 1 : If $i \neq 2n$ and $j \neq 2n$ for each $n \in \mathbb{N}$, then

$$ij(1)^2\mathbb{Z}_4 = \mathbb{Z}_4 = (1)^2 i\mathbb{Z}_4 = (1)^2 j\mathbb{Z}_4$$

Case 2 : If $i = 2n_1$ and $j = 2n_2$ for some $n_1, n_2 \in \mathbb{N}$, then

$$ij(1)^3\mathbb{Z}_4 = 0.$$

Case 3 : If $i = 2n_1$ for some $n_1 \in \mathbb{N}$ and $j \neq 2n$ for each $n \in \mathbb{N}$, then

$$ij(1)^2\mathbb{Z}_4 = \overline{2}\mathbb{Z}_4 = (1)^2 i\mathbb{Z}_4.$$

So, $\mathbb{Z}_4$ is a Gr-W-2^{abs}-second $\mathbb{Z}$-module.

Lemma 3.8. *Let $W \subseteq h(\Re)$ be a m.c.s. of $\Re$ and K be a graded finitely generated submodule of $\Im$. If $W^{-1}K = 0$, then there exists an $w_\alpha \in W$ such that $w_\alpha K = 0$.*

Proof. Suppose $W^{-1}K = 0$ and K is generated by $x_1, x_2, ..., x_n \in h(K)$, then $K = \Re x_1 + \Re x_2 + ...\Re x_n$. We have for every $i = 1, 2, ..., n$, $\frac{xi}{1} = 0$ in $W^{-1}K$, which means there is $w_i \in W$ such that $w_i x_i = 0$. Let $w_\alpha = w_1 w_2 ....w_n \in W$. Then $w_\alpha x_i = 0$, for each $i = 1, 2, ..., n$ and therefore $w_\alpha K = 0$ as K is generated by $x_1, x_2, ..., x_n$. □

Let $W \subseteq h(\Im)$ be a m.c.s. of $\Re$. Then $W^* = \{x_g \in h(\Re) : \frac{xg}{1}$ is a unit of $W^{-1}\Re\}$ is *m.c.s.* of $\Re$ containing W.

Theorem 3.9. *Let $W \subseteq h(\Re)$ be a m.c.s. of $\Re$. Then*:

(a) *If K is Gr-2^{abs}_{st}-second submodule with $Ann_\Re(K) \cap W = \emptyset$, then K is Gr-W-2^{abs}-second submodule. In fact, if $W \subseteq u(\Re)$ and K is Gr-W-2^{abs}-second submodule of $\Im$, then K is a Gr-2^{abs}_{st}-second submodule of $\Im$.*

(b) *If $W_1 \subseteq W_2 \subseteq h(\Re)$ are multiplicative closed subsets of, and K is graded W_1-2-absorbing second submodule of $\Im$, then K is graded W_2-2-absorbing second submodule of $\Im$ in case of $Ann_\Re(K) \cap W_2 = \emptyset$.*

(c) *K is Gr-W-2^{abs}-second submodule of $\Im$ if and only if K is Gr-W^*-2^{abs}-second submodule of $\Im$.*

(d) *If K is a finitely generated Gr-W-2^{abs}-second submodule of $\Im$, then $W^{-1}K$ is a Gr-2^{abs}_{st}-second submodule of $W^{-1}\Im$.*

Proof. (a) and (b) are clear.

(c) Suppose K is Gr-W-2^{abs}-second submodule of $\Im$. First we want to show $Ann_\Re(K) \cap W^* = \emptyset$. To see that suppose there exists $x_\beta \in Ann_\Re(K) \cap W^*$. As $x_\beta \in W^*$, $x_\beta/1$ is a unit of $W^{-1}\Re$, so there exists $w_\gamma \in W$ and $a_i \in h(\Re)$ such that $(x_\beta/1)(a_i/w_\gamma) = 1$, hence $u_\lambda x_\beta a_i = u_\lambda w_\gamma$ for some $u_\lambda \in W$, so $u_\lambda w_\gamma = u_\lambda x_\beta a_i \in Ann_\Re(K) \cap W$, which is a contradiction. Now as $W \subseteq W^*$, by part (b), K is graded Gr-W^*-2^{abs}-second submodule of $\Im$. Conversely, assume that K is Gr-W^*-2^{abs}-second submodule of $\Im$. Let $r_g, t_h \in h(\Re)$ and H a graded submodule of $\Im$ with $r_g t_h K \subseteq H$. Since K is Gr-W^*-2^{abs}-second submodule of $\Im$, there exists $w_\alpha \in W^*$ such that $w_\alpha r_g K \subseteq H$ or $w_\alpha t_h K \subseteq H$ or $w_\alpha r_g t_h \in Ann_\Re(K)$. Since $w_\alpha \in W^*$, $w_\alpha/1$ is unit of $W^{-1}\Re$, so there exists $w_i, c_j \in W$ and $d \in h(\Re)$ such that $w_i c_j = w_i w_\alpha d$. Then $w_i c_j \in W$, note that $(w_i c_j) r_g t_h = (w_i w_\alpha d) r_g t_h = w_i d s_\alpha r_g t_h \in Ann_\Re(K)$. or $(w_i c_j) r_g K = (w_i w_\alpha d) r_g K = w_i d s_\alpha r_g K \subseteq H$ or $(w_i c_j) t_h K = (w_i w_\alpha d) t_h K = w_i d s_\alpha t_h K \subseteq H$. Therefore K is Gr-W-2^{abs}-second submodule of $\Im$.

(d) As K is a Gr-W-2^{abs}-second submodule of $\Im$, there is $w_\alpha \in W$. If $W^{-1}K = 0$, then as K is a graded finitely generated, there is a $u_\beta \in W$ such that $u_\beta K = 0$ by Lemma 3.8. Hence $Ann_\Re(K) \cap W \neq \emptyset$, a contradiction. Thus $W^{-1}K \neq 0$. Now let a_i/t_{g1} , $b_j/v_{g2} \in W^{-1}\Re$. Since K is a graded W-2-absorbing second submodule of $\Im$, we have either $a_i b_j w_\alpha^2 K = a_i w_\alpha^2 K$ or $a_i b_j w_\alpha^2 K = b_j w_\alpha^2 K$ or $a_i b_j w_\alpha^3 K = 0$. This implies that either $(a_i/t_{g1})(b_j/v_{g2})W^{-1}K = (a_i/t_{g1})W^{-1}K$ or $(a_i/t_{g1})(b_j/v_{g2})W^{-1}K = (b_j/v_{g2})W^{-1}K$ or $(a_i/t_{g1})(b_j/v_{g2})W^{-1}K = 0$, as required. □

Theorem 3.10. *Let $W \subseteq h(\mathfrak{R})$ be a m.c.s. of $\mathfrak{R}$, and $K \leq_G^{sub} \mathfrak{I}$ with $Ann_{\mathfrak{R}}(K) \cap W = \emptyset$. Then K is a Gr-W-2^{abs}-second submodule of $\mathfrak{I}$ if and only if $w_\alpha^3 K$ is Gr-2_{st}^{abs}-second submodule of $\mathfrak{I}$ for some $w_\alpha \in W$.*

Proof. Suppose that K is a *Gr-W-*2^{abs}-second submodule of $\mathfrak{I}$ and $r_g, t_h \in h(\mathfrak{R})$. Then for some $w_\alpha \in W$, we get $w_\alpha^2 r_g t_h K = w_\alpha^2 r_g K$ or $w_\alpha^2 r_g t_h K = w_\alpha^2 t_h K$ or $w_\alpha^3 r_g t_h K = 0$ by Theorem 3.5. Hence $r_g t_h w_\alpha^3 K = r_g w_\alpha^3 K$ or $r_g t_h w_\alpha^3 K = t_h w_\alpha^3 K$ or $r_g t_h w_\alpha^3 K = 0$. Since $Ann_{\mathfrak{R}}(K) \cap W = \emptyset$, then $w_\alpha^3 K \neq 0$. So $w_\alpha^3 K$ is *Gr-* 2_{st}^{abs}-second submodule of $\mathfrak{I}$, by [3, Theorem 3.2]. Conversely, let $w_\alpha^3 K$ is *Gr-* 2_{st}^{abs}-second submodule of $\mathfrak{I}$, for some $w_\alpha \in W$ and $r_g, t_h \in h(\mathfrak{R})$. Then $r_g t_h w_\alpha^3 K = r_g w_\alpha^3 K$ or $r_g t_h w_\alpha^3 K = t_h w_\alpha^3 K$ or $r_g t_h w_\alpha^3 K = 0$ by [3, Theorem 3.2]. Thus $w_\alpha^6 r_g t_h K = w_\alpha^6 r_g K$ or $w_\alpha^6 r_g t_h K = w_\alpha^6 t_h K$ or $w_\alpha^9 r_g t_h K = 0$. Put $u_\alpha :_{\mathfrak{I}} w_\alpha^3 \in W$, then $u_\alpha^2 r_g t_h K = u_\alpha^2 r_g K$ or $u_\alpha^2 r_g t_h K = u_\alpha^2 t_h K$ or $u_\alpha^3 r_g t_h K = 0$. Hence, by Theorem 3.5, K is a *Gr-W-*2^{abs}-second submodule of $\mathfrak{I}$. □

Theorem 3.11. *Let $W \subseteq h(\mathfrak{R})$ be m.c.s. of $\mathfrak{R}$. Let $N \subset K$ be two graded submodules of $\mathfrak{I}$ with $Ann_{\mathfrak{R}}(K/N) \cap W = \emptyset$ and K is Gr-W-2^{abs}-second submodule of $\mathfrak{I}$. Then K/N is Gr-W-2^{abs}-second submodule of $\mathfrak{I}/N$.*

Proof. Assume that K is a *Gr-W-*2^{abs}-second submodule of $\mathfrak{I}$. Let $r_g, t_h \in h(\mathfrak{R})$ and $H/N \leq_G^{sub} \mathfrak{I}/N$ such that $r_g t_h (K/N) \subseteq (H/N)$. Thus $r_g t_h K \subseteq H$, $H \leq_G^{sub} \mathfrak{I}$. Since K is *Gr-W-*2^{abs}-second submodule of $\mathfrak{I}$, then there is $w_\alpha \in W$ such that $w_\alpha r_g K \subseteq H$ or $w_\alpha t_h K \subseteq H$ or $w_\alpha r_g t_h K = 0$. So, $w_\alpha r_g (K/N) \subseteq (H/N)$ or $w_\alpha t_h (K/N) \subseteq (H/N)$ or $w_\alpha r_g t_h (K/N) = N$, as needed. □

Theorem 3.12. *Let $W \subseteq h(\mathfrak{R})$ be a m.c.s. of $\mathfrak{R}$ and K is a Gr-W-2^{abs}-second submodule of a graded $\mathfrak{R}$-module $\mathfrak{I}$. Then we have the following:*

(i) *$Ann_{\mathfrak{R}}(K)$ is a Gr-W-2^{abs} ideal of $\mathfrak{R}$.*

(ii) *If $H \leq_G^{sub} \mathfrak{I}$ with $(H :_{\mathfrak{R}} K) \cap W = \emptyset$, then $(H :_{\mathfrak{R}} K)$ is a Gr-W-2^{abs} ideal of $\mathfrak{R}$.*

(iii) *There exists $w_\alpha \in W$* with *$w_\alpha^n K = w_\alpha^{n+1} K$, for all $n \geq 3$.*

Proof. (i) Let $r_g, t_h, c_\lambda \in h(\mathfrak{R})$ and $r_g t_h c_\lambda \in Ann_{\mathfrak{R}}(K)$, then there is $w_\alpha \in W$ and $r_g t_h K \subseteq r_g t_h K$ implies that $r_g w_\alpha K \subseteq r_g t_h K$ or $t_h w_\alpha K \subseteq r_g t_h K$ or $w_\alpha r_g t_h K = 0$. If $w_\alpha r_g t_h K = 0$, we are done. If $r_g w_\alpha K \subseteq r_g t_h K$, then $c_\lambda r_g w_\alpha K \subseteq c_\lambda r_g t_h K = 0$, hence $c_\lambda r_g w_\alpha K = 0$. If $t_h w_\alpha K \subseteq r_g t_h K$, then $c_\lambda t_h w_\alpha K \subseteq c_\lambda r_g t_h K = 0$, so $c_\lambda t_h w_\alpha K = 0$.

(ii) Let r_g, t_h, $c_\lambda \in h(\mathfrak{R})$ and $r_g t_h c_\lambda = (r_g c_\lambda) t_h \in (H :_{\mathfrak{R}} K)$, so $(r_g c_\lambda) t_h K \subseteq H$. Then there exists $w_\alpha \in W$ with $w_\alpha r_g c_\lambda K \subseteq H$ or $w_\alpha c_\lambda t_h K \subseteq w_\alpha t_h K \subseteq H$ or $w_\alpha r_g t_h c_\lambda \in Ann_{\mathfrak{R}}(K) \subseteq (H :_{\mathfrak{R}} K)$. Thus $w_\alpha r_g c_\lambda \in (H :_{\mathfrak{R}} K)$ or $w_\alpha c_\lambda t_h \in (H :_{\mathfrak{R}} K)$ or $w_\alpha r_g t_h c_\lambda \in (H :_{\mathfrak{R}} K)$.

(iii) Since K is *Gr-W-*2^{abs}-second submodule of $\mathfrak{I}$, there exists $w_\alpha \in W$. It is enough to show that $w_\alpha^3 K = w_\alpha^4 K$, it is clear that $w_\alpha^4 K \subseteq w_\alpha^3 K$. Since K is *Gr-W-*$2^{abs}$-second submodule and $w_\alpha^2 w_\alpha^2 K = w_\alpha^4 K \subseteq w_\alpha^4 K$, either $w_\alpha^3 K \subseteq w_\alpha^4 K$ or $w_\alpha^5 K = 0$. If $w_\alpha^5 K = 0$, then $w_\alpha^5 \in Ann_{\mathfrak{R}}(K) \cap W = \emptyset$ which is a contradiction. Thus $w_\alpha^3 K \subseteq w_\alpha^4 K$, as needed. □

Theorem 3.13. *Let $W \subseteq h(\mathfrak{R})$ be a m.c.s. and K is a Gr-W-2^{abs}-second submodule of $\mathfrak{I}$. Then the following statement hold for some $w_\alpha \in W$.*

(a) *$a_\beta w_\alpha K \subseteq a_\beta d_\gamma K$ or $d_\gamma w_\alpha K \subseteq a_\beta d_\gamma K$, for all a_β, $d_\gamma \in W$.*

(b) *$(Ann_{\mathfrak{R}}(K) :_{\mathfrak{R}} a_\beta d_\gamma) \subseteq (Ann_{\mathfrak{R}}(K) :_{\mathfrak{R}} w_\alpha a_\beta)$* or *$(Ann_{\mathfrak{R}}(K) :_{\mathfrak{R}} a_\beta d_\gamma) \subseteq (Ann_{\mathfrak{R}}(K) :_{\mathfrak{R}} w_\alpha d_\gamma)$*, for all a_β, $d_\gamma \in W$.

Proof. (a) Let K be a *Gr-W-*2^{abs}-second submodule, then exists $w_\alpha \in W$. Let N be a completely graded irreducible submodule of $\mathfrak{I}$ with $a_\beta d_\gamma K \subseteq$ N, where a_β, $d_\gamma \in W$. Then $a_\beta w_\alpha K \subseteq N$ or $w_\alpha d_\gamma K \subseteq N$ or $w_\alpha a_\beta d_\gamma K = 0$. As $Ann_{\mathfrak{R}}(K) \cap W = \emptyset$, we get $w_\alpha a_\beta d_\gamma K \neq 0$. If for each completely graded submodule of $\mathfrak{I}$ we have $a_\beta w_\alpha K \subseteq N$ (resp. $w_\alpha d_\gamma K \subseteq N$), then we are done by Remark 3.4. So suppose that there are completely graded submodules N_1 and N_2 of $\mathfrak{I}$ with $a_\beta w_\alpha K \nsubseteq N_1$ and $w_\alpha d_\gamma K \nsubseteq N_2$. Since K is *Gr-W-*2^{abs}-second submodule of $\Im$, $w_\alpha d_\gamma K \subseteq N_1$ and $a_\beta w_\alpha K \subseteq N_2$. As $a_\beta d_\gamma K \subseteq N_1 \cap N_2$, we get $w_\alpha d_\gamma K \subseteq N_1 \cap N_2$ or $a_\beta w_\alpha K \subseteq N_1 \cap N_2$, which is a contradiction.

(b) Let $r_g \in (Ann_\Re(K) :_\Re a_\beta d_\gamma) \cap h(\Re)$. Then $r_g a_\beta d_\gamma \in Ann_\Re(K)$. By Theorem 3.12 (i), $Ann_\Re(K)$ is Gr-W-2^{abs} ideal of $\Re$, so $w_\alpha r_g a_\beta \in Ann_\Re(K)$ or $w_\alpha r_g d_\gamma \in Ann_\Re(K)$ or $w_\alpha a_\beta d_\gamma \in Ann_\Re(K)$, for some $w_\alpha \in W$. Since $Ann_\Re(K) \cap W = \emptyset$, then $w_\alpha a_\beta d_\gamma \notin Ann_\Re(K)$. Thus $r_g \in (Ann_\Re(K) :_\Re w_\alpha a_\beta)$ or $r_g \in (Ann_\Re(K) :_\Re w_\alpha d_\gamma)$. □

Lemma 3.14. *Let $\Im$ be gr-comultiplication $\Re$-module, $W \subseteq h(\Re)$ be a m.c.s. of $\Re$ and $K \leq_G^{sub} \Im$. If $Ann_\Re(K)$ is a Gr-W-2^{abs} ideal of $\Re$, then K a Gr-W-2^{abs}-second submodule of $\Im$.*

Proof. Let $r_g, t_h \in h(\Re)$ and $H \leq_G^{sub} \Im$ with $r_g t_h K \subseteq H$. Thus $Ann_\Re(H) r_g t_h K = 0$. Since $Ann_\Re(K)$ is a Gr-W-2^{abs} ideal of $\Re$ and $Ann_\Re(H) r_g t_h \subseteq Ann_\Re(K)$, then there is $w_\alpha \in W$ such that either $w_\alpha Ann_\Re(H) r_g K = 0$ or $w_\alpha Ann_\Re(H) t_h K = 0$ or $w_\alpha r_g t_h K = 0$. If $w_\alpha r_g t_h K = 0$, we are done. If $w_\alpha Ann_\Re(H) r_g K = 0$ or $w_\alpha Ann_\Re(H) t_h K = 0$, then $Ann_\Re(H) \subseteq Ann_\Re(w_\alpha r_g K)$ or $Ann_\Re(H) \subseteq Ann_\Re(w_\alpha t_h K)$. Since $\Im$ is gr-comultiplication module, $w_\alpha r_g K \subseteq H$ or $w_\alpha t_h K \subseteq H$. □

Example 3.15. Let $\Re = \mathbb{Z}$ and $G = \mathbb{Z}_2$. Then $\Re$ is a G-graded ring with $\Re_0 = \mathbb{Z}$ and $\Re_1 = \{0\}$. Let $\Im = \mathbb{Z}$ as a $\mathbb{Z}$-module, = is a G-graded module with $\Im_0 = \mathbb{Z}$ and $\Im_1 = \{0\}$. $\Im$ is not a gr-comultiplication module, see [15, Example 3.3]. Take the multiplicative closed set $W = \mathbb{Z}\setminus\{0\}$. The graded submodule $q\mathbb{Z}$ of $\Im$, where q is prime number, is not Gr-W-2^{abs}-second submodule of $\Im$, since take $n = 3$, $m = 2 \in \mathbb{Z}$, then for all $w \in W$ we have $w^2(3)(2)q\mathbb{Z} \neq w^2(3)q\mathbb{Z}$ and $w^2(3)(2)q\mathbb{Z} \neq w^2(2)q\mathbb{Z}$ and $w^3(3)(2)q\mathbb{Z} \neq 0$. But $Ann_\Re(q\mathbb{Z}) = 0$ is a Gr-W-2^{abs} ideal of $\mathbb{Z}$.

Theorem 3.16. *Let $\Im$ be a gr-comultiplication* $\Re$-module and $W \subseteq h(\Re)$ *be a m.c.s. of* $\Re$: *If the zero graded submodule of $\Im$ is Gr-W-2^{abs} submodule, then every $K \leq_G^{sub} \Im$ with $Ann_\Re(K) \cap W = \emptyset$ is a Gr-W-2^{abs}-second submodule of $\Im$.*

Proof. Let $\Im$ be a gr-comultiplication $\Re$-module with the zero graded submodule is Gr-W-2^{abs}-submodule and $K \leq_G^{sub} \Im$ *with* $Ann_\Re(K) \cap W = \emptyset$. We show that $Ann_\Re(K)$ is Gr-W-2^{abs} ideal of $\Re$. Let $r_g, t_h, c_\lambda \in h(\Re)$ and $r_g t_h c_\lambda = (r_g c_\lambda) t_h \in Ann_\Re(K)$. Then there exists $w_\alpha \in W$ such that $r_g c_\lambda w_\alpha K = 0$ or $c_\lambda t_h w_\alpha K \subseteq t_h w_\alpha K = 0$ or $w_\alpha r_g t_h c_\lambda \in Ann_\Re(\Im) \subseteq Ann_\Re(K)$. Thus $Ann_\Re(K)$ is Gr-W-2^{abs} ideal of $\Re$. By Lemma 3.14, we get the result. □

Definition 3.17. We say that $\Im$ satisfy the double annihilator conditions (DAC) if $P \leq_G^{id} \Re$, then $P = Ann_\Re((0 :_\Im P))$. A graded $\Re$-module $\Im$ is said to be strong gr-comultiplication module if $\Im$ is a gr-comultiplication $\Re$-module and satisfy the DAC conditions.

Theorem 3.18. *Let $\Im$ be strong gr-comultiplication $\Re$-module and $K \leq_G^{sub} \Im$ with $Ann_\Re(K) \cap W = \emptyset$, where $W \subseteq h(\Re)$ be a m.c.s. of $\Re$. Then the following are equivalent*

(a) *K is Gr-W-2^{abs}-second submodule of $\Im$.*

(b) *$Ann_\Re(K)$ is Gr-W-2^{abs} ideal of $\Re$.*

(c) *$K = (0 :_\Im I)$ for some Gr-W-2^{abs} ideal I of $\Re$ with $Ann_\Re(K) \subseteq I$.*

Proof. $(a) \Rightarrow (b)$: From Theorem 3.12.

$(b) \Rightarrow (c)$: Since $\Im$ is gr-comultiplication $\Re$-module, $K = (0 :_\Im Ann_\Re(K))$. We can see the result clearly.

(c) ⇒ (a): Since $\Im$ satisfy the double annihilator conditions (DAC), $Ann_\Re((0 :_\Im I)) = I$. By Lemma 3.16, we get the result. □

Lemma 3.19. *Let* $W \subseteq h(\Re)$ *be m.c.s. of* $\Re$ *and* K *be a Gr-W-second submodule of* $\Im$*. Then there exists* $w_\alpha \in W$ *and whenever* $r_g t_h K \subseteq H$*, where* $r_g, t_h \in h(\Re)$ and $H \leq_G^{sub} \Im$, then either $w_\alpha r_g \in Ann_\Re(K)$ or $w_\alpha t_h \in Ann_\Re(K)$ *or* $w_\alpha K \subseteq H$*.*

Proof. Let K be a Gr-W-second submodule of $\Im$ and $r_g t_h K \subseteq H$, where $r_g, t_h \in h(\Re)$ and $H \leq_G^{sub} \Im$. Then $r_g K \subseteq (H :_\Im t_h)$, since K is a Gr-W-second submodule of $\Im$, there exists $w_\alpha \in W$ such that $w_\alpha r_g \in Ann_\Re(K)$ or $w_\alpha t_h K \subseteq H$, we will show if $w_\alpha t_h K \subseteq H$; then $w_\alpha t_h \in Ann_\Re(K)$ or $w_\alpha K \subseteq H$. Assume that $t_h K \subseteq (H :_\Im w_\alpha)$, since K is a Gr-W-second submodule of $\Im$, we get either $w_\alpha t_h \in Ann_\Re(K)$ or $w_\alpha^2 K \subseteq H$. If $w_\alpha t_h \in Ann_\Re(K)$, we are done. Suppose $w_\alpha^2 K \subseteq H$, let N be a completely graded irreducible submodule of $\Im$ such that $w_\alpha^2 K \subseteq N$, then either $w_\alpha K \subseteq N$ or $w_\alpha^3 K = 0$, since $Ann_\Re(K) \cap W = \emptyset$, we get $w_\alpha^3 K \neq 0$, so $w_\alpha K \subseteq N$. By Remark 3.4, $w_\alpha K \subseteq w_\alpha^2 K$. Hence $w_\alpha K \subseteq H$. □

Theorem 3.20. *Let* $W \subseteq h(\Re)$ *be a m.c.s. of* $\Re$*. Then the sum of two Gr-W-second submodules is a Gr-W-*2^{abs}*-second submodule of* $\Im$*.*

Proof. Let K_1, K_2 be two Gr-W-second submodules of $\Im$ and let $K = K_1 + K_2$. Let $r_g t_h K \subseteq H$, where $r_g, t_h \in h(\Re)$ and H is graded submodule of $\Im$. As $r_g t_h K_1 \subseteq r_g t_h K \subseteq H$ and K_1 is Gr-W-second submodule of $\Im$, there exists $w_{\alpha 1} \in W$ such that $w_{\alpha 1} r_g \in Ann_\Re(K_1)$ or $w_{\alpha 1} t_h \in Ann_\Re(K_1)$ or $w_{\alpha 1} K_1 \subseteq H$ by Lemma 3.19. Also, K_2 is Gr-W-second submodule of $\Im$, there exists $w_{\alpha 2} \in W$ such that $w_{\alpha 2} r_g \in Ann_\Re(K_2)$ or $w_{\alpha 2} t_h \in Ann_\Re(K_2)$ or $w_{\alpha 2} K_2 \subseteq H$. Without loss of generality, we may assume $w_{\alpha 1} r_g \in Ann_\Re(K_1)$ and $w_{\alpha 2} K_2 \subseteq H$. Now, Set $w_\alpha = w_{\alpha 1} w_{\alpha 2} \in W$. Thus $w_\alpha r_g K \subseteq H$ and hence K is a Gr-W-2^{abs}-second submodule of $\Im$. □

As shown in the example below, the sum of two Gr-W-2^{abs}-second submodules is not necessarily a Gr-W-2^{abs}-second submodule.

Example 3.21. Let $\Re = \mathbb{Z}$ and $G = \mathbb{Z}_2$. Then $\Re$ is a G-graded ring with $\Re_0 = \mathbb{Z}$ and $\Re_1 = \{0\}$. Let $\Im = \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{q^k}$ as a $\mathbb{Z}$-module, where $k \in N$ and p,q are distinct prime numbers. Then $\Im$ is a G-graded module with $\Im_0 = \mathbb{Z}_{p^k} \oplus 0$ and $\Im_1 = 0 \oplus \mathbb{Z}_{q^k}$. Put $W = \{t \in \mathbb{Z} : gcd(t, pq) = 1\}$. So W is a $m.c.s.$ of $\mathbb{Z}$. We have $\mathbb{Z}_{p^k} \oplus 0$ and $0 \oplus \mathbb{Z}_{q^k}$ both are Gr-W-2^{abs}-second submodules. However $\Im$ is not a Gr-W-2^{abs}-second $\mathbb{Z}$-module, since $p^k \Im \subseteq 0 \oplus \mathbb{Z}_{q^k}$, $p^{k-1} tM \nsubseteq 0 \oplus \mathbb{Z}_{q^k}$, $pt\mathrm{M} \nsubseteq 0 \oplus \mathbb{Z}_{q^k}$, and $tp^k \Im \neq 0$ for each $t \in W$.

Lemma 3.22. *If* P *is* Gr*-*W*-*2^{abs} *ideal of* $\Re$*, then* $Gr(P)$ *is* Gr*-*W*-*2^{abs} *ideal of* $\Re$*.*

Proof. Let $r_g, t_h, c_\lambda \in h(\Re)$ and $r_g t_h c_\lambda \in Gr(P)$, so there exists $n \in \mathbb{N}$ such that $(r_g t_h c_\lambda)^n \in P$. Since P is Gr-W-2^{abs} ideal of $\Re$ and $r_g^n t_h^n c_\lambda^n \in P$, then there exists $w_\alpha \in W$ such that either $w_\alpha r_g^n t_h^n \in P$ or $w_\alpha r_g^n c_\lambda^n \in P$ or $w_\alpha t_h^n c_\lambda^n \in P$. Hence $(w_\alpha r_g t_h)^n = w_\alpha^n r_g^n t_h^n \in P$ or $(w_\alpha r_g c_\lambda)^n = w_\alpha^n r_g^n c_\lambda^n \in P$ or $(w_\alpha t_h c_\lambda)^n = w_\alpha^n t_h^n c_\lambda^n \in P$. Therefore, $w_\alpha r_g t_h \in Gr(P)$ or $w_\alpha r_g c_\lambda \in Gr(P)$ or $w_\alpha t_h c_\lambda \in Gr(P)$. Thus $Gr(P)$ is Gr-W-2^{abs} ideal of $\Re$. □

For a graded $\Re$-submodule U of $\Im$, the graded second radical of U is defined as the sum of all graded second $\Re$-submodules of $\Im$ contained in U, and its denoted by $GSec(U)$. If U does not contain any graded second $\Re$-submodule, then $GSec(U) = \{0\}$. The graded second spectrum of $\Im$ is the collection of all graded second $\Re$-submodules, and it is represented by the symbol $GSpec^s(\Im)$. On the other hand, the set of all graded prime $\Re$-submodules of $\Im$ is called the graded spectrum of $\Im$, and is denoted by $GSpec(\Im)$. The map $\phi : GSpec^s(\Im) \to GSpec(\Re/Ann_\Re(\Im))$ defined by $\phi(U) = Ann_\Re(U)/Ann_\Re(\Im)$ is called the natural map of $GSpec^s(\Im)$; see [12].

Theorem 3.23. *Let* $\Im$ *be a gr-comultiplication* $\Re$*-module and the natural map* ϕ of $GSpec^s(K)$ *is surjective, if* K *is* Gr*-*W*-*2^{abs}*-second submodule of* $\Im$*, then* $GSec(K)$ *is a* Gr*-*W*-*2^{abs}*-second submodule of* $\Im$*.*

Proof. Let K be Gr-W-2^{abs}-second of $\Im$. By Theorem 3.12 (i), $Ann_{\Re}(K)$ is Gr-W-2^{abs} ideal of $\Re$. Hence $Gr(Ann_{\Re}(K))$ is Gr-W-2^{abs} ideal of $\Re$ by Lemma 3.22. Using [12, Lemma 4.7], $Gr(Ann_{\Re}(K)) = Ann_{\Re}(GSec(K))$ so $Ann_{\Re}(GSec(K))$ is Gr-W-2^{abs} ideal of $\Re$. By Lemma 3.14, we get the result. □

Theorem 3.24. *Let $W \subseteq h(\Re)$ be a m.c.s. of $\Re$ and $\varphi : \Im \to \Im'$ be a graded monomorphism of graded $\Re$-modules. Then we have the following:*

(i) *If K is a Gr-W-2^{abs}-second submodule of $\Im$, then $\varphi(K)$ is a Gr-W-2^{abs}-second submodule of $\Im'$.*

(ii) *If K' is a Gr-W-2^{abs}-second submodule of $\Im'$ and $K' \subseteq \varphi(\Im)$, then $\varphi^{-1}(K')$ is a Gr-W-2^{abs}-second submodule of $\Im$.*

Proof. (i) $Ann_{\Re}(\varphi(K)) \cap W = \emptyset$, since $Ann_{\Re}(K) \cap W = \emptyset$ and φ graded monomorphism. Let $r_g, t_h \in h(\Re)$: Since K is a Gr-W-2^{abs}-second submodule of $\Im$, there exists $w_\alpha \in W$ such that $w_\alpha^2 r_g t_h K = w_\alpha^2 r_g K$ or $w_\alpha^2 r_g t_h K = w_\alpha^2 t_h K$ or $w_\alpha^3 r_g t_h K = 0$. Hence, $w_\alpha^2 r_g t_h \varphi(K) = w_\alpha^2 r_g \varphi(K)$ or $w_\alpha^2 r_g t_h \varphi(K) = w_\alpha^2 t_h \varphi(K)$ or $w_\alpha^3 r_g t_h \varphi(K) = 0$.

(ii) Since $Ann_{\Re}(K) \cap W = \emptyset$, then $Ann_{\Re}(f^{-1}(K)) \cap W = \emptyset$. Let $r_g, t_h \in h(\Re)$. Since K is a Gr-W-2^{abs}-second of of $\Im$, then there exists a fixed $w_\alpha \in W$ such that $w_\alpha^2 r_g t_h K = w_\alpha^2 r_g K$ or $w_\alpha^2 r_g t_h K = w_\alpha^2 t_h K$ or $w_\alpha^3 r_g t_h K = 0$. Thus $w_\alpha^2 r_g t_h f^{-1}(K) = s_\alpha^2 r_g f^{-1}(K)$ or $w_\alpha^2 r_g t_h f^{-1}(K) = w_\alpha^2 t_h f^{-1}(K)$ or $w_\alpha^3 r_g t_h f^{-1}(K) = 0$, as needed. □

Theorem 3.25. *Let $\Re = \Re_1 \times \Re_2$ be graded ring, where $\Re_1$ and $\Re_2$ be two commutative graded rings with $1 \neq 0$ and let $W_1 \subseteq (\Re_1)_e$ and $W_2 \subseteq (\Re_2)_e$ be two multiplicatively closed sets. Let $\Im = \Im_1 \times \Im_2$ be graded $\Re$-module, where $\Im_1$ is a graded $\Re_1$-module and $\Im_2$ is a graded $\Re_2$-module. Suppose that $K = K_1 \times K_2 \leq_G^{sub} \Im$. If either $Ann_{\Re 1}(K_1) \cap W_1 \neq \emptyset$ and K_2 is a graded Gr-W_2-2^{abs}-second submodule of $\Im_2$ or $Ann_{\Re 2}(K_2) \cap W_2 \neq \emptyset$ and K_1 is a Gr-W_1-2^{abs}-second submodule of $\Im_1$ or K_1 is Gr-W_1-second submodule of $\Im_1$ and K_2 is Gr-W_2-second submodule of $\Im$2, then K is Gr-W-2^{abs}-second submodule of $\Im$.*

Proof. Suppose K_1 is a Gr-W_1-2^{abs}-second submodule of $\Im_1$ and $Ann_{\Re 2}(K_2) \cap W_2 \neq \emptyset$. We will show that K is Gr-W-2^{abs}-second submodule of $\Im$. Then there exists $(w_2)_e \in Ann_{\Re 2}(K_2) \cap W_2$. Let $((r_1)_g, (r_2)_g)((t_1)_h, (t_2)_h)K_1 \times K_2 \subseteq H_1 \times H_2$, where $(r_i)_g \in (\Re_i)_g$, $(t_i)_h \in (\Re_i)_h$ and $H_i \leq_G^{sub} \Im_i$, where $i = 1, 2$. Then $(r_1)_g (t_1)_h K_1 \subseteq H_1$. Since K_1 is a Gr-W_1-2^{abs}-second submodule of $\Im_1$, there exists $(w_1)_e \in W_1$ such that $(w_1)_e (r_1)_g K_1 \subseteq H_1$ or $(w_1)_e (t_1)_h K_1 \subseteq H_1$ or $(w_1)_e (r_1)_g (t_1)_h K_1 = 0$. Put $w_e = ((w_1)_e, (w_2)_e) \in W_1 \times W_2$. Then $w_e((r_1)_g, (r_2)_g)K_1 \times K_2 \subseteq H_1 \times H_2$ or $w_e((t_1)_h, (t_2)_h)K_1 \times K_2 \subseteq H_1 \times H_2$ or $w_e((r_1)_g, (r_2)_g)((t_1)_h, (t_2)_h)K_1 \times K = 0$. Therefore, K is Gr-W-2^{abs}-second submodule of $\Im$. Similarly for if $Ann_{\Re 1}(K_1) \cap W_1 \neq \emptyset$ and K_2 is a Gr-W_2-2^{abs}-second submodule of $\Im_2$, then K is Gr-W-2^{abs}-second submodule of $\Im$. Now suppose K_1 is Gr-W_1-second submodule of $\Im_1$ and K_2 is Gr-W_2-second submodule of $\Im_2$. Let $(a_g, x_g)(b_h, y_h)K_1 \times K_2 \subseteq H_1 \times H_2$, where $a_g \in (\Re_1)_g$, $x_g \in (\Re_2)_g$, $b_h \in (\Re_1)h$, $y_h \in (\Re_2)h$, H_1 is graded submodule of $\Im_1$ and H_2 is graded submodule of $\Im_2$. Then we have $a_g b_h K_1 \subseteq H_1$ and $x_g y_h K_2 \subseteq H_2$. As K_1 is Gr-W_1-second submodule of $\Im_1$, there exists $w'_e \in W_1$ such that $w'_e a_g \in Ann_{\Re 1}(K_1)$ or $w'_e b_h \in Ann_{\Re 1}(K_1)$ or $w'_e K_1 \subseteq H_1$ by Lemma 3.19. Similarly, there exists $w''_e \in W_2$ such that $w''_e x_g \in Ann_{\Re 2}(K_2)$ or $w''_e y_h \in Ann_{\Re 2}(K_2)$ or $w''_e K_2 \subseteq H_2$ by Lemma 3.19. Without loss of generality, we have three cases:

Case 1: If $w'_e a_g \in Ann_{\Re 1}(K_1)$ and $w''_e K_2 \subseteq H_2$, then

$$(w'_e, w''_e)(a_g, x_g)K_1 \times K_2 = w'_e a_g K_1 \times w''_e x_g K_2 \subseteq 0 \times K_2 \subseteq K_1 \times K_2.$$

Case 2: If $w'_e a_g \in Ann_{\Re 1}(K_1)$ and $w''_e x_g \in Ann_{\Re 2}(K_2)$, then

$$(w'_e, w''_e)(a_g, x_g)(b_h, y_h)K_1 \times K_2 = 0$$

Case 3: If $w'_e K_1 \subseteq H_1$ and $w''_e K_2 \subseteq H_2$, then

$$(w'_e, w''_e)(b_h, y_h)K_1 \times K_2 \subseteq (w'_e, w''_e)K_1 \times K_2 \subseteq H_1 \times H_2.$$

Hence, K is Gr-W-2^{abs}-second submodule of $\Im$. □

Definition 3.26. Let $W \subseteq \Re_e$ be *m.c.s.* of $\Re$ and $K \leq_G^{sub} \Im$ with $Ann_{\Re}(K) \cap W = \emptyset$. We say that K is *e-W-2-*absorbing second (*e-W-2*^{abs}-second) submodule of $\Im$, if there exists $w_e \in W$ and whenever $r_e t_e K \subseteq H$, then $w_e r_e K \subseteq H$ or $w_e t_e K \subseteq H$ or $w_e r_e t_e K = 0$, for every r_e, $t_e \in \Re_e$ and $H \leq_G^{sub} \Im$.

Definition 3.27. Let $W \subseteq \Re_e$ be a *m.c.s.* of $\Re$ and $K \leq_G^{sub} \Im$ such that $Ann_{\Re}(K) \cap W = \emptyset$. We say that K is a *e-W*-second submodule of $\Im$, if there exists $w_e \in W$ and whenever $r_e K \subseteq H$, where $r_e \in \Re_e$ and $H \leq_G^{sub} \Im$, then $w_e K \subseteq H$ or $w_e r_e K = 0$

Theorem 3.28. *Let $\Re = \Re_1 \times \Re_2$ be G-graded ring, where $\Re_1$ and $\Re_2$ be two commutative G-graded rings and let $W_1 \subseteq (\Re_1)_e$ be m.c.s. of $\Re_1$ and $W_2 \subseteq (\Re_2)_e$ be a m.c.s. of $\Re_2$. Let $\Im = \Im_1 \times \Im_2$ be a graded $\Re$-module, where $\Im_1$ is a graded $\Re_1$-module and $\Im_2$ is a graded $\Re_2$-module. Suppose that $K = K_1 \times K_2 \leq_G^{sub} \Im$. Then the following conditions are equivalent.*

(i) *K is e-W-2^{abs}-second submodule of $\Im$.*

(ii) *Either $Ann_{\Re 1}(K_1) \cap W_1 \neq \emptyset$ and K_2 is a e-W_2-2^{abs}-second submodule of $\Im_2$ or $Ann_{\Re 2}(K_2) \cap W_2 \neq \emptyset$ and K_1 is a e-W_1-2^{abs}-second submodule of $\Im_1$ or K_1 is e-W_1-second submodule of $\Im_1$ and K_2 is e-W_2-second submodule of $\Im_2$.*

Proof. (i) $\Rightarrow$ (ii) Let $K = K_1 \times K_2$ be *e-W-2*^{abs}-second submodule of $\Im$. By Theorem 3.12, $Ann_{\Re}(K) = Ann_{\Re 1}(K_1) \cap Ann_{\Re 2}(K_2)$ is *Gr-W-2*^{ab} ideal of $\Re$. Thus either $Ann_{\Re 1}(K_1) \cap W_1 = \emptyset$ or $Ann_{\Re 2}(K_2) \cap W_2 = \emptyset$. Assume that $Ann_{\Re 1}(K_1) \cap W_1 \neq \emptyset$. We show that K_2 is a *e-*W_2-2^{abs}-second submodule of $\Im_2$. Let $r_{e2} t_{e2} K_2 \subseteq H_2$ for some r_{e2}, $t_{e2} \in (\Re_2)_e$ and $H_2 \leq_G^{sub} \Im_2$. Hence $(1, r_{e2})(1, t_{e2})K_1 \times K_2 \subseteq \Im_1 \times H_2$. Since K is *e-W-2*^{abs}-second submodule of $\Im$, there exists $w_e = (w_{e1}, w_{e2}) \in W$ such that $(w_{e1}, w_{e2})(1, r_{e2})K_1 \times K_2 \subseteq \Im_1 \times H_2$ or $(w_{e1}, w_{e2})(1, t_{e2})K_1 \times K_2 \subseteq \Im_1 \times H_2$ or $(w_{e1}, w_{e2})(1, r_{e2})(1, t_{e2})K_1 \times K_2 = 0$, it follows that either $w_{e2} r_{e2} K_2 \subseteq H_2$ or $w_{e2} t_{e2} K_2 \subseteq H_2$ or $w_{e2} r_{e2} t_{e2} K_2 = 0$. So K_2 is *e-*W_2-2^{abs}-second submodule of $\Im_2$. Similarly if $Ann_{\Re 2}(K_2) \cap W_2 \neq \emptyset$, then K_1 is a *e-*W_1-2^{abs}-second submodule of $\Im_1$. Assume that $Ann_{\Re 1}(K_1) \cap W_1 = \emptyset$ and $Ann_{\Re 2}(K_2) \cap W_2 = \emptyset$. We show that K_1 is *e-*W_1-second submodule of $\Im_1$ and K_2 is *e-*W_2-second submodule of $\Im_2$. Note that there exists $w_e = (w_{e1}, w_{e2}) \in W$ satisfying that K is *e-W-2*^{abs}-second submodule of $\Im$. Suppose that K_1 is not *e-*W_1-second submodule of $\Im_1$. So there exists $a_{e1} \in (\Re_1)_e$ and $H_1 \leq_G^{sub} \Im_1$ such that $a_{e1} K_1 \subseteq H_1$ but $w_{e1} K_1 \not\subseteq H_1$ and $w_{e1} a_{e1} K_1 \neq 0$. Moreover, $Ann_{\Re 2}(K_2) \cap W_2 = \emptyset$ so $w_{e2} k_2 \neq 0$. Thus by Remark 3.4, there exists a completely graded irreducible submodule N_2 of $\Im_2$ such that $w_{e2} K_2 \not\subseteq N_2$. Furthermore,

$$(a_{e1}, 1)(1, 0)K_1 \times K_2 \subseteq a_{e1} K_1 \times 0 \subseteq H_1 \times 0 \subseteq H_1 \times N_2.$$

Since K is *e-W-2*^{abs}-second submodule of $\Im$, either $(w_{e1}, w_{e2})(a_{e1}, 1)K_1 \times K_2 \subseteq H_1 \times N_2$ or $(w_{e1}, w_{e2})(1, 0)K_1 \times K_2 \subseteq H_1 \times N_2$ or $(w_{e1}, w_{e2})(1, 0)(a_{e1}, 1)K_1 \times K_2 = 0$. Hence, $w_{e2} K_2 \subseteq N_2$ or $w_{e1} K_1 \subseteq H_1$ or $w_{e1} a_{e1} K_1 = 0$, which them are contradictions. So K_1 is *e-*W_1-second submodule of $\Im_1$. Similarly one can see that K_2 is *e-*W_2-2^{abs}-second of $\Im_2$.

(ii) $\Rightarrow$ (i) Suppose that K_1 is a *e-*W_1-2^{abs}-second submodule of $\Im_1$ and $Ann_{\Re 2}(K_2) \cap W_2 \neq \emptyset$. We show that K is *e-W-2*^{abs}-second submodule of $\Im$. Then there exists $w''_e \in Ann_{\Re 2}(K_2) \cap W_2$. Let $(c_1, c_2)(d_1, d_2)K_1 \times K_2 \subseteq H_1 \times H_2$ for some c_1, $d_1 \in (\Re_1)_e$, c_2, $d_2 \in (\Re_2)_e$ and H_1(resp. H_2) $\leq_G^{sub} \Im_1$ (resp. $\Im_2$). Then $c_1 d_1 K_1 \subseteq H_1$. Since K_1 is a *e-*W_1-2^{abs}-second submodule of $\Im_1$, there exists $w'_e \in W_1$ such that $w'_e c_1 K_1 \subseteq H_1$ or $w'_e d_1 K_1 \subseteq H_1$ or $w'_e c_1 d_1 K_1 = 0$. Put $w_e = (w'_e, w''_e)$. Then $w_e(c_1, c_2)K_1 \times K_2 \subseteq H_1 \times H_2$ or $w_e(d_1, d_2)K_1 \times K_2 \subseteq H_1 \times H_2$ or $w_e(c_1, c_2)(d_1, d_2)K_1 \times K_2 = 0$. Thus K is *e-W-2*^{abs}-second submodule of $\Im$. Similarly if K_2 is a *e-*W_2-2^{abs}-second submodule of $\Im_2$ and $Ann_{\Re 1}(K_1) \cap W_1 \neq \emptyset$, then K is *e-W-2*^{abs}-second submodule of $\Im$. Assume that K_1 is *e-*W_1-second submodule of $\Im_1$ and K_2 is *e-*W_2-second submodule of $\Im_2$. Let a_{e1}, $b_{e1} \in (\Re_1)_e$, x_{e2}, $y_{e2} \in (\Re_2)_e$ and H_1(resp. H_2) $\leq_G^{sub} \Im_1$ (resp. $\Im_2$) such that Let $(a_{e1}, x_{e2})(b_{e1}, y_{e2})K_1 \times K_2 \subseteq H_1 \times H_2$. Then we have $a_{e1} b_{e1} K_1 \subseteq H_1$ and $x_{e2} y_{e2} K_1 \subseteq H_1$. As K_1 is *e-*W_1-second submodule of $\Im_1$, then there exists $w_{e1} \in W_1$ such that $w_{e1} a_{e1} \in Ann_{\Re 1}(K_1)$ or $w_{e1} b_{e1} \in Ann_{\Re 1}(K_1)$ or $w_{e1} K_1 \subseteq H_1$ by Lemma 3.19. Similarly, there exists $w_{e2} \in W_2$

such that $w_{e2}x_{e2} \in Ann_{\Re 2}(K_2)$ or $w_{e2}y_{e2} \in Ann_{\Re 2}(K_2)$ or $w_{e2}K_2 \subseteq H_2$. Without losing generality, we can infer $w_{e1}a_{e1} \in Ann_{\Re 1}(K_1)$ and $w_{e2}K_2 \subseteq H_2$ or $w_{e1}a_{e1} \in Ann_{\Re 1}(K_1)$ and $w_{e2}x_{e2} \in Ann_{\Re 2}(K_2)$ or $w_{e1}K_1 \subseteq H_1$ and $w_{e2}K_2 \subseteq H_2$. If $w_{e1}a_{e1} \in Ann_{\Re 1}(K_1)$ and $w_{e2}K_2 \subseteq H_2$, then

$$(w_{e1}, w_{e2})(a_{e1}, x_{e2})K_1 \times K_2 \subseteq w_{e1}a_{e1}K_1 \times w_{e2}x_{e2}K_2 \subseteq 0 \times H_2 \subseteq H_1 \times H_2.$$

If $w_{e1}a_{e1} \in Ann_{\Re 1}(K_1)$ and $w_{e2}x_{e2} \in Ann_{\Re 2}(K_2)$, then

$$(w_{e1}, w_{e2})(\mathrm{a}a_{e1}, x_{e2})(b_{e1}, y_{e2})K_1 \times K_2 = 0.$$

If $w_{e1}K_1 \subseteq H_1$ and $w_{e2}K_2 \subseteq H_2$, then

$$(w_{e1}, w_{e2})(a_{e1}, x_{e2})K_1 \times K_2 \subseteq (w_{e1}, w_{e2})K_1 \times K_2 \subseteq H_1 \times H_2.$$

Thus K is e-W-2^{abs}-second submodule of $\Im$. □

Declaration of interests statement

The authors declare no conflict of interest

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