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Generalized Caputo-Katugampola for solving fuzzy fractional Heat Equation

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Abstract

This paper explores the application of fuzzy theory to solve fractional heat equations using a novel approach, the Optimal Homotopy Asymptotic Method (OHAM). We introduce a semi-analytical method to address fuzzy fractional-order heat equations, aiming to overcome the limitations of existing approaches. Our methodology leverages the generalized Caputo-Katugampola (CK) definition with two parameters α and ρ, to define fractional derivatives. Through this research, we present a comprehensive framework for tackling this challenging problem. To illustrate the effectiveness and feasibility of our method, we provide several practical examples. The results are presented in tables and figures, and our approach is compared to the exact solutions. This study not only contributes to the field but also offers a powerful and efficient way to address fuzzy fractional heat equations with increased accuracy and reduced computational effort.

Key words and phrases: Fuzzy fractional Heat equation, Caputo-Katugampola Derivative, Fractional Derivative, Optimal Homotopy Asymptotic Method, Fuzzy Fractional Differential Equations.

Mathematics Subject Classification (2010): 26A33, 34A07, 34K28, 49Mxx

1. Introduction

In the last three decades, there has been a significant surge in interest in fractional calculus, primarily due to its established utility in various fields of physics, biology, and engineering. Because

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the fractional-order differential operators are non-local, they can more accurately explain some phenomena than integer-order differential operators. It is frequently employed to substitute a fractional derivative for the traditional time derivative in evolutionary equations [1–4].

Fuzzy differential equations (FDEs) play a pivotal role in expanding the repertoire of system models and fundamental tools for diversifying the array of machine models utilized across physics, engineering, biology, and other medical disciplines [5–7]. They find applications in addressing production problems, optimization tasks, artificial intelligence, diagnostic procedures in clinical settings, cosmological studies, and the development of specialized robotic systems, and serve as valuable tools for modeling uncertain nonlinear systems [8]. Hoa et al [9] have made major contributions to non-integer order differential equations with fuzzy solutions. They broadened the concept from classical to arbitrary order, resulting in important results in fuzzy fractional calculus. They also obtained significant results while employing the "Caputo generalized Hukuhara" fractional derivative. They looked into a few fascinating subjects concerning fractional-order fuzzy problems. Other significant investigations can be found in [10–14]. Several numerical and analytical methods have been developed for solving fuzzy fractional differential equations, such as the Power series expansion method [15], Sumudu transform [16], Adomian Decomposition Method (ADM) [17], Gegenbauer Wavelet Polynomials [18] and Variation Iteration Method (VIM) [19].

Caputo-Katugampola (CK) fractional integral and derivative [20] is a revolutionary fractional integral and derivative concept that generalizes the fractional derivatives with two parameters and retains certain essential and basic features of Caputo and Caputo-Hadamard derivatives [21]. The parameter values have a significant influence on the used derivative, presenting a helpful tool for creating fractional calculus models. In modeling and simulations, the CK derivative has two fractional parameters that provide greater generalization than other fractional formulations.

Optimal Homotopy Asymptotic Method (OHAM) has now evolved as a result of Marinca [22] and is used to solve nonlinear problems without relying on a tiny parameter. When compared to the Homotopy Perturbation Method (HPM), it was established that OHAM provides the best solution. Furthermore, unlike the Homotopy Analysis Method (HAM), OHAM does not require any initial guess or being aware of the h-curve. Furthermore, like the HAM, OHAM contains built-in convergence criteria, but with a greater degree of flexibility. OHAM is also parameter-free and provides more precision than approximate analytical approaches of the same order of approximation. The downside of OHAM is that it necessitates the solution of a set of nonlinear algebraic equations at each order of approximation, and this technique comprises multiple unknown convergence-control factors, making calculation time-consuming. Over and above that, OHAM has been carried out effectively in a variety of technological and mathematical domains [23–26].

The general heat equation can be applied to the flow or distribution of heat in a thin rod, plane, or space from a high-temperature point to a lower-temperature point during a particular time interval. The considered equation is one of the diffusion equation's special instances. Using the heat equation, one may anticipate that the heat flow will continue until the temperature of all bodies or particles remains constant [27]. The fuzzy fractional heat equation was investigated by Snehashish and other researchers [28] using the HPM. The variational iteration method (VIM) was discussed and investigated to solve fuzzy time-fractional diffusion equations which is taken in the Caputo sense [29]. Furthermore, the analysis for semi-analytical methods especially the Laplace transform along with decomposition techniques and the Adomian polynomial for Two-Dimensional Heat Equation under the Caputo–Fabrizio fractional differential operator was introduced by Sitthiwirattham et al [30]. The time-fractional heat equation has been investigated using the spectral tau technique under non-local conditions where CPs6 and their modified polynomials were used to choose suitable sets of basis functions [3]. Additionally, the one- and two-dimensional heat partial differential equations (PDEs) with generalized Lucas polynomials (GLPs) involving two parameters are utilized as basis functions by the tau and collocation methods to convert the heat equations into subject to their underlying conditions into systems of linear algebraic equations [31]. Investigating the convergence analysis of the shifted fifth-kind Chebyshev polynomials (5CPs) utilized a spectral tau solution to the heat conduction equation is introduced in [32].

This paper develops a novel efficient approach for solving FFHE in one dimension with CK derivative of two parameters using OHAM. This algorithm is based on selecting appropriate parameters that ensure the guaranteed convergence of solutions. Combining the fractional derivative form as CK definition with a powerful algorithm that regulates the convergence parameters (OHAM) will ensure an accurate solution that obeys real-world data.

An outline of this article is as follows: Section (2) goes over some basic definitions and preliminaries that will be used throughout this work. Section (3) proposes the defuzzification of a fuzzy fractional heat equation. Section (4) shows the general form of fuzzy fractional OHAM. In Section (5), we use OHAM's convergent analysis and provide a structural formula for the fuzzy fractional heat equation. In Section (6), we will illustrate and analyze the capabilities of the suggested fuzzy fractional OHAM in addressing a test scenario through our findings. Lastly, some concluding remarks for this work are presented in the final section.

2. Preliminary

Some essential definitions and basics of fuzzy and fractional calculus with Caputo-Katugampola fractional derivatives are introduced in this section.

2.1. Fuzzy concept

Zadeh created a fuzzy set theory [33] in 1965. It is regarded as carrying crisp or (classical) set theory [34]. The crisp sets notion classifies the membership of items on a set's topic in binary terms, with a detail either belonging to or not belonging to the set. Fuzzy set theory is defined by a membership function with a value in the interval [0, 1] and is defined as an extension of the classical set theory.

Definition 2.1: [35] *Let $\widetilde{A}: X \to [0,1]$ be a fuzzy set. The ζ-level (ζ-cut) representation of a fuzzy set $\widetilde{A}$ is defined as:*

$$[\widetilde{A}]_\zeta = \{s \in X \,|\, \delta\widetilde{A}(s) > \zeta\}, \quad \zeta \in [0,1]$$

Definition 2.2: [36] *A triangular fuzzy number is a fuzzy number defined as triple numbers $\alpha_1 < \alpha_2 < \alpha_3$ with the base on the interval $[\alpha_1, \alpha_3]$ and at $x = \alpha_2$ as a peak point and membership function is as the following:*

$$\delta(s;\alpha_1,\alpha_2,\alpha_3) = \begin{cases} 0, & x < \alpha_1; \\ \dfrac{s-\alpha_1}{\alpha_2-\alpha_1}, & \alpha_1 \le x \le \alpha_2; \\ \dfrac{\alpha_3 - s}{\alpha_3-\alpha_2}, & \alpha_2 \le x \le \alpha_3; \\ 0, & x > \alpha_3. \end{cases}$$

such that the ζ-level as follows:

$$\left[\delta(s)\right]_\zeta = \left[\alpha_1 + \zeta(\alpha_2-\alpha_1), \alpha_3 - \zeta(\alpha_3-\alpha_2)\right], \quad \zeta \in [0,1]$$

Definition 2.3: [37] *Let $\widetilde{S}$ be the set of all normal upper semi-continuous convex fuzzy numbers with ζ-level bounded intervals that satisfy the following condition:*

$$[\delta(s)]_\zeta = \{s \in \mathbb{R} : \delta \ge \zeta\}, \quad \zeta \in [0,1].$$

An arbitrary fuzzy number is represented by an ordered pair of membership functions $\left[\delta(s)\right]_\zeta = [\underline{\delta}(s), \overline{\delta}(s)]_\zeta$ for all which is satisfying

1. $\delta(s)$ is normal: there exists $s_0 \in \mathbb{R}$ such that $\delta(s_0) = 1$.
2. $\delta(s)$ is convex: $\forall s,t \in \mathbb{R}$ and $\lambda \in [0,1]$, it holds that:

$$\delta(\lambda s + (1-\lambda)t) \geq \min\{(\delta(s), \delta(t)\}$$

3. δ is upper semi continues: for any $s_0 \in \mathbb{R}$, it satisfied that $\delta(s_0) \geq \lim_{s \to s_0^{\pm}} \delta(s)$.
4. $\overline{\{s \in \mathbb{R} : \delta \geq \zeta\}}$ is compact subset of $\mathbb{R}$.

In the parametric form, which is represented by an ordered pair of functions $[\delta]_\zeta = [\underline{\delta}(s), \overline{\delta}(s)]_\zeta = [\underline{\delta}(s;\zeta), \overline{\delta}(s;\zeta)]$, $\zeta \in [0,1]$, that hold the below conditions:

1. $\underline{\delta}(s;\zeta)$ is a bounded left continuous non-decreasing in $[0,1]$.
2. $\overline{\delta}(s;\zeta)$ is a bounded left continuous non-increasing in $[0,1]$.
3. $\underline{\delta}(s;\zeta) \leq \overline{\delta}(s;\zeta)$.

Definition 2.4: [38] *Let $\tilde{h} : M \to \widetilde{S}$ be a map, so, for interval $M \subseteq \widetilde{S}$ denote a fuzzy function with crisp variable, and we define ζ-level set as*

$$[\tilde{h}(s)]_\zeta = [\underline{h}(s;\zeta), \overline{h}(s;\zeta)] \quad s \in M, \zeta \in [0,1],$$

where $\widetilde{S}$ sets all upper semi-continuous normal convex fuzzy numbers. That is, the fuzzifying function is a mapping from a domain to a set of fuzzy ranges. In a mathematical sense, the fuzzifying function and the fuzzy relation coincide.

Definition 2.5: [39] *Given a function $g : S \to T$, where $S = S_1 \times S_2 \times \ldots \times S_n$ and let $\widetilde{A} = \widetilde{A}_1 \times \widetilde{A}_2 \times \ldots \times \widetilde{A}_n$, where $\widetilde{A}_i, i = 1,2,\ldots,n$, be n-fuzzy subset in S and $t = g(s_1, s_2, \ldots, s_n)$ in T. Then, the extension principle allows defining a fuzzy subset $\widetilde{B} = g(\widetilde{A})$ in T by:*

$$\widetilde{B} = \{(t, \delta_{\widetilde{B}}(t)) : t = g(s_1, s_2, \ldots, s_n), s_1, s_2, \ldots, s_n \in S\}.$$

such that,

$$\delta_{\widetilde{B}}(t) = \begin{cases} \sup\limits_{s_1, s_2, \ldots, s_n \in g^{-1}(t)} \min\{\delta_{\widetilde{A}_1}(s_1), \delta_{\widetilde{A}_2}(s_2), \ldots, \delta_{\widetilde{A}_n}(s_n)\}, & if \quad g^{-1}(t) \neq \phi, \\ 0, & Otherwise. \end{cases}$$

the extension principle can be written for $n = 1$ as

$$\widetilde{B} = \{(t, \delta_{\widetilde{B}}(t)) : t = g(s), s \in S\}.$$

such that,

$$\delta_{\widetilde{B}}(t) = \begin{cases} \sup\limits_{s \in g^{-1}(t)} \{\delta_{\widetilde{A}}(s)\} & if, \quad g^{-1}(t) \neq \phi, \\ 0, & Otherwise. \end{cases}$$

For $s,t \in \widetilde{S}$, and $\lambda \in \mathbb{R}$, the sum $s+t$ is $[s+t]_\zeta = [s]_\zeta + [t]_\zeta$ and the product $\lambda.s$ is $[\lambda.s]_\zeta = \lambda[s]_\zeta$, and the diameter of the ζ-level set of s as $diam[s]_\zeta = [\underline{s}(\zeta) - \overline{s}(\zeta)]$.

Definition 2.6: [40] *Let $s,t \in \widetilde{S}$. If there is $r \in \widetilde{S}$: $s = t + r$, then r is said to be the Hukuhara difference of s and t and it is denoted by $s \ominus t$.*

Definition 2.7: [40] *Let s,t be two fuzzy numbers then the distance $D[s,t]$ (Hausdorff distance) is defined as*

$$D[s,t] = \sup_{0\le\zeta\le1} \max\{|\underline{s}(\zeta) - \underline{t}(\zeta)|, |\overline{s}(\zeta) - \overline{t}(\zeta)|\}.$$

Definition 2.8: [41] *Let $g : I \to \widetilde{E}$ and $s_0 \in I$. Then the fuzzy function g is said to be Hukuhara differentiable (H-differentiable) at s_0, if there is $g'(s_0) \in \widetilde{E}$, and for $h > 0$, there are $g(s_0 + h) \ominus g(s_0)$ and $g(s_0) \ominus g(s_0 - h)$ such that*

$$g'(s_0) = lim_{h\to0} \frac{g(s_0+h) \ominus g(s_0)}{h} = lim_{h\to0} \frac{g(s_0) \ominus g(s_0-h)}{h}.$$

Definition 2.9: [42] *Let $g : I \to \widetilde{E}$ for $s \in I \subseteq \mathbb{R}$. The n^{th} order Hukuhara differentiable functions at t*

$$[\widetilde{g}(t)]_\zeta = [\underline{g}(t;\zeta), \overline{g}(t;\zeta)], \forall\zeta \in [0,1].$$

The functions $\underline{g}(t;\zeta), \overline{g}(t;\zeta)$ are both n^{th} order Hukuhara differentiable functions and

$$[\widetilde{g}^{(n)}(t)]_\zeta = [\underline{g}^{(n)}(t;\zeta), \overline{g}^{(n)}(t;\zeta)], \forall\zeta \in [0,1].$$

2.2. Caputo-Katugampola fractional derivative

The Caputo-Katugampola derivative is a novel fractional operator that generalizes the idea of Caputo and Caputo-Hadamard fractional derivatives.

Definition 2.10: [21] *Both left and right generalized fractional integrals of the function f, called the Katugampola fractional integral are respectively given by:*

$$I_{a+}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x s^{\rho-1}(x^\rho - s^\rho)^{\alpha-1} f(s)ds,$$

and

$$I_{b-}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b s^{\rho-1}(x^\rho - s^\rho)^{\alpha-1} f(s)ds,$$

where $0 < a < b < \infty$, $f : [a,b] \to \mathbb{R}$ is an integrable function, and $\alpha > 0$ and $\rho > 0$ two fixed real numbers.

The following properties of the fractional integral operator $I^{\alpha,\rho}$ where $\alpha \ge 0, \rho > 0$, and for constant $c \in \mathbb{R}$, holds:

1. $I^{\alpha,\rho} I^{\beta,\rho} f(x) = I^{\alpha+\beta,\rho} f(x) = I^{\beta+\alpha,\rho} f(x) = I^{\beta,\rho} I^{\alpha,\rho} f(x)$,
2. $I^{\alpha,\rho}(cf(x)) = cI^{\alpha,\rho}(f(x))$,
3. $I^{\alpha,\rho}(x^n) = \dfrac{\rho^{-\alpha}\Gamma\left(\dfrac{n}{\rho}+1\right)}{\Gamma\left(\alpha+\dfrac{n}{\rho}+1\right)} x^{n+\alpha\rho}$.

The first two properties are found in [43]. and the Proof of property (3) as follow.

Proof. By Definition (2.10) and set $a = 0$ we have

$$I^{\alpha,\rho}(x^n) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^x s^{\rho-1}(x^\rho - s^\rho)^{\alpha-1}(s^n)ds.$$

Using the Beta function $\mathcal{B}(.,.)$, and change variables $u = \dfrac{s^\rho}{x^\rho}$, we obtain

$$\begin{aligned}
I^{\alpha,\rho}(x^n) &= \frac{\rho^{-\alpha}}{\Gamma(\alpha)} x^{\alpha\rho+n} \int_0^1 (1-u)^{\alpha-1} u^{\frac{n}{\rho}} du. \\
&= \frac{\rho^{-\alpha}}{\Gamma(\alpha)} x^{\alpha\rho+n} \mathcal{B}\left(\alpha, \frac{n}{\rho}+1\right) \\
&= \frac{\rho^{-\alpha}}{\Gamma(\alpha)} x^{\alpha\rho+n} \frac{\Gamma(\alpha)\Gamma\left(\frac{n}{\rho}+1\right)}{\Gamma\left(\alpha+\frac{n}{\rho}+1\right)} \\
&= \frac{\rho^{-\alpha}\Gamma\left(\frac{n}{\rho}+1\right)}{\Gamma\left(\alpha+\frac{n}{\rho}+1\right)} x^{n+\alpha\rho}.
\end{aligned}$$

Definition 2.11: [21] *Let* $0 < a < b < \infty$, $f : [a,b] \to \mathbb{R}$ *is an integrable function, and for* $m-1 < \alpha \le m$, $m \in \mathbb{N}$ *and* $\rho > 0$. *Both left and right Caputo-Katugampola fractional derivatives of order* α, ρ *are respectively defined by:*

$${}^{C}D_{a+}^{\alpha,\rho} f(x) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^x \frac{s^{(\rho-1)(1-m)}}{(x^\rho - s^\rho)^{\alpha-m+1}} f^{(m)}(s)ds,$$

and

$${}^{C}D_{b-}^{\alpha,\rho} f(x) = \frac{(-1)^m \rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_x^b \frac{s^{(\rho-1)(1-m)}}{(x^\rho - s^\rho)^{\alpha-m+1}} f^{(m)}(s)ds.$$

The following properties of CK fractional derivative can be held for constant $c \in \mathbb{R}$, $a < x \le b$ where $a \ge 0, \rho > 0, m-1 < \alpha \le m$ and $f \in C^j[a,b]$, first three properties can refer to [21, 44]:

1. ${}^{C}D^{\alpha,\rho}(c) = 0$,
2. ${}^{C}D_{a+}^{\alpha,\rho} I_{a+}^{\alpha,\rho} f(x) = f(x)$,
3. $I_{a+}^{\alpha,\rho}\, {}^{C}D_{a+}^{\alpha,\rho} f(x) = f(x) - \sum_{j=0}^{m-1} \dfrac{1}{\rho^j (j)!} (x^\rho - a^\rho)^j \left[\left(x^{1-\rho}\dfrac{d}{dx}\right)^j f(x)\right]|_{x=a}$,
4. ${}^{C}D^{\alpha,\rho}(x^n) = \dfrac{\Gamma(n+1)\Gamma\left(\frac{n}{\rho}-m+1\right)}{\Gamma(n-m+1)\Gamma\left(\frac{n}{\rho}-\alpha+1\right)} \rho^{\alpha-m} x^{n-\alpha\rho}$.

We going to prove property (4).

Proof. By Definition (2.11) and set $a = 0$ we have

$$ {}^{C}D^{\alpha,\rho}(x^n) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_0^x \frac{s^{(\rho-1)(1-m)}}{(x^\rho - s^\rho)^{\alpha-m+1}}\left(\frac{d^m}{dx^m}x^n\right)ds. $$

Using the Beta function $\mathcal{B}(.,.)$, and change variables $u = \frac{s^\rho}{x^\rho}$, we obtain

$$ \begin{aligned} {}^{C}D^{\alpha,\rho}(x^n) &= \frac{\Gamma(n+1)\rho^{\alpha-m}}{\Gamma(n-m+1)\Gamma(m-\alpha)}x^{n-\alpha\rho}\int_0^1 (1-u)^{m-\alpha-1}u^{\frac{n}{\rho}-m}du \\ &= \frac{\Gamma(n+1)\rho^{\alpha-m}}{\Gamma(n-m+1)\Gamma(m-\alpha)}x^{n-\alpha\rho}\mathcal{B}\left(m-\alpha,\frac{n}{\rho}-m+1\right) \\ &= \frac{\Gamma(n+1)\rho^{\alpha-m}}{\Gamma(n-m+1)\Gamma(m-\alpha)}x^{n-\alpha\rho}\frac{\Gamma(m-\alpha)\Gamma\left(\frac{n}{\rho}-m+1\right)}{\Gamma\left(\frac{n}{\rho}-\alpha+1\right)} \\ &= \frac{\Gamma(n+1)\Gamma\left(\frac{n}{\rho}-m+1\right)}{\Gamma(n-m+1)\Gamma\left(\frac{n}{\rho}-\alpha+1\right)}\rho^{\alpha-m}x^{n-\alpha\rho}. \end{aligned} $$

Remark 2.12: [45]

- When $\rho \to 1$, the Caputo-Katugampola fractional derivative is simplified to the standard Caputo.
- When $\rho \to 0^+$, the Caputo-Katugampola fractional derivative is simplified to the Caputo-Hadamard derivatives.

3. Fuzzy Fractional Heat Equation

Let's examine the equation governing fractional heat transfer in dimensional instances which may be written inside the forms:

$$ D_t^{\alpha,\rho}\widetilde{V}(s,t) = \widetilde{A}(s)\frac{\partial^2 \widetilde{V}(s,t)}{\partial s^2} + \widetilde{B}(s), 0 < s < l, t > 0, \tag{1} $$

referring to the initial/boundary conditions:

$$ \widetilde{V}(s,0) = \tilde{f}(s), \tilde{v}(0,t) = \widetilde{g}(t), \widetilde{V}(l,t) = \tilde{z}(t), $$

where $\widetilde{V}(s,t)$ is the fuzzy function of t and s as the crisp variable. Furthermore, $\frac{\partial^2 \widetilde{V}(s,t)}{\partial s^2}$ is the fuzzy second partial derivative with s, here, $D^{\alpha,\rho}\widetilde{V}(s,t)$ fuzzy time fractional H-derivative of order ν,ξ,σ, here $\widetilde{A}$ and $\widetilde{B}$ are given continuous positive and real-valued functions, respectively. $\widetilde{V}(s,0)$ is the fuzzy initial condition where $\tilde{f}(s)$ is fuzzy number and $\widetilde{V}(0,t)$ as well as $\widetilde{V}(l,t)$ is fuzzy boundary conditions with $\widetilde{g}$, $\tilde{z}$ being fuzzy convex numbers. Now the defuzzification of Eq. (1) with $\zeta \in [0,1]$ is as follows [46]:

$$ [\widetilde{V}(s,t)]_\zeta = [\underline{V}(s,t;\zeta), \overline{V}(S,t;\zeta)]. \tag{2} $$

$$\left[D_t^{\alpha,\rho}\widetilde{V}(s,t)\right]_\zeta=\left[D^{\alpha,\rho}\underline{V}(s,t;\zeta),D^{\alpha,\rho}\overline{V}(s,t;\zeta)\right]. \tag{3}$$

$$\left[\frac{\partial^2\widetilde{V}(s,t)}{\partial s^2}\right]_\zeta=\left[\frac{\partial^2\underline{V}(s,t;\zeta)}{\partial s^2},\frac{\partial^2\overline{V}(s,t;\zeta)}{\partial s^2}\right]. \tag{4}$$

$$\left[\widetilde{A}\right]_\zeta=\left[\underline{A}(s,\zeta),\overline{A}(s,\zeta)\right]. \tag{5}$$

$$\left[\widetilde{B}\right]_\zeta=\left[\underline{B}(s,\zeta),\overline{B}(s,\zeta)\right]. \tag{6}$$

$$\left[\widetilde{V}(s,0)\right]_\zeta=\left[\underline{V}(s,0;\zeta),\overline{V}(s,0;\zeta)\right]. \tag{7}$$

$$\left[\widetilde{V}(0,t)\right]_\zeta=\left[\underline{V}(0,t;\zeta),\overline{V}(0,t;\zeta)\right]. \tag{8}$$

$$\left[\widetilde{V}(l,t)\right]_\zeta=\left[\underline{V}(l,t;\zeta),\overline{V}(l,t;\zeta)\right]. \tag{9}$$

$$\left[\tilde{f}(s)\right]_\zeta=\left[\underline{f}(s;\zeta),\overline{f}(s;\zeta)\right]. \tag{10}$$

$$\begin{cases}\left[\tilde{g}\right]_\zeta=\left[\underline{g}(\zeta),\overline{g}(\zeta)\right].\\ \left[\tilde{z}\right]_\zeta=\left[\underline{z}(\zeta),\overline{z}(\zeta)\right].\end{cases} \tag{11}$$

where,

$$\begin{cases}\left[\tilde{f}(s)\right]_\zeta=\left[\underline{\phi}_1(\zeta)\psi(s),\overline{\phi}_1(\zeta)\psi(s)\right].\\ \left[\widetilde{A}(s)\right]_\zeta=\left[\underline{\phi}_2(\zeta)\psi(s),\overline{\phi}_2(\zeta)\psi(s)\right].\\ \left[\widetilde{B}(s)\right]_\zeta=\left[\underline{\phi}_3(\zeta)\psi(s),\overline{\phi}_3(\zeta)\psi(s)\right].\end{cases} \tag{12}$$

such that $\underline{\phi}_i(\zeta)$ and $\overline{\phi}_i(\zeta)$, $i=1,2,3$ are lower and upper convex fuzzy numbers and $\psi(s)$ is a crisp function of the crisp variable s. Now, by the fuzzy extension principle, the membership function can be defined as follows

$$\begin{cases}\underline{V}(s,t;\zeta)=\min\{\widetilde{V}(s,\tilde{\delta}(\zeta)):\tilde{\delta}(\zeta)\in\widetilde{V}(s,t;\zeta)\},\\ \overline{V}(s,t;\zeta)=\max\{\widetilde{V}(s,\tilde{\delta}(\zeta)):\tilde{\delta}(\zeta)\in\widetilde{V}(s,t;\zeta)\}.\end{cases} \tag{13}$$

Now, by fuzzification of Eq. (1) and defuzzification of Eqs. (2-13), we can rewrite the Eq. (1) in the following formula. The lower term of Eq. (1).

$$\begin{cases}D_t^{\alpha,\rho}\underline{V}(s,t;\zeta)=\underline{\phi}_2(\zeta)\psi(s)\dfrac{\partial^2\underline{V}(s,t;\zeta)}{\partial s^2}+\underline{\phi}_3(\zeta)\psi(s),\\ \underline{V}(s,0;\zeta)=\underline{\phi}_1(\zeta)\psi(s),\\ \underline{V}(0,t;\zeta)=\underline{g}(\zeta),\underline{V}(l,t;\zeta)=\underline{z}(\zeta).\end{cases} \tag{14}$$

The upper term is

$$\begin{cases} D_t^{\alpha,\rho}\overline{V}(s,t;\zeta) = \overline{\phi}_2(\zeta)\psi(s)\dfrac{\partial^2\overline{V}(s,t;\zeta)}{\partial s^2} + \overline{\phi}_3(\zeta)\psi(s), \\ \overline{V}(s,0;\zeta) = \overline{\phi}_1(\zeta)\psi(s), \\ \overline{V}(0,t;\zeta) = \overline{g}(\zeta), \overline{V}(l,t;\zeta) = \overline{z}(\zeta). \end{cases} \tag{15}$$

4. General OHAM Technique for Fuzzy Fractional Heat Equation

OHAM is an amendment of the HAM that is primarily based on minimizing the residual error. In OHAM, the management and adjustment of the convergence vicinity are supplied conveniently. The production of fractional OHAM in a crisp environment turned into formulated in [26, 47]. To solve the fuzzy fractional heat equation, there is a need to fuzzify and then defuzzify OHAM. Consider the fuzzy differential equation for all $\zeta \in [0,1]$ as follows:

$$\begin{cases} D_t^{\alpha,\rho}\widetilde{V}(s,t;\zeta) = \widetilde{\mathcal{A}}_\eta\left(s,t,\widetilde{V}(s,t;\zeta);\zeta\right) + \tilde{f}(s,t;\zeta), 0 < s < l, \\ B\left(\widetilde{V}, \dfrac{\partial^{\nu,\xi,\sigma}\widetilde{V}}{\partial t^{\nu,\xi,\sigma}}\right) = 0, \end{cases} \tag{16}$$

where $\widetilde{V}(s,t;\zeta)$ is the fuzzy fractional function of the crisp variables t and s and $D_t^{\alpha,\rho}\widetilde{V}(s,t;\zeta)$ is the linear operator of Eq.(16) and $\widetilde{\mathcal{A}}_\eta = [\underline{\mathcal{A}}_\eta, \overline{\mathcal{A}}_\eta]$ is the fuzzy fractional function of fuzzy variable $\widetilde{V}$. Now, we can formulate fuzzy fractional optimal homotopy $\widetilde{\Phi}(s,t;\zeta;p) : [0,l] \times [0,1] \to \mathbb{R}$ which satisfy the following homotopy:

$$\begin{aligned} \widetilde{H}(\Phi(s,t;\zeta;p),p) = (1-p)&\left(\widetilde{\mathcal{L}}(\widetilde{\Phi}(s,t;\zeta;p)) - \tilde{f}(s,t;\zeta)\right) \\ &-\widetilde{\mathcal{H}}(\zeta;p)(\widetilde{\mathcal{L}}(\widetilde{\Phi}(s,t;\zeta;p)) \\ &-\widetilde{\mathcal{A}}_\eta((\widetilde{\Phi}(s,t;\zeta;p),p)) - \tilde{f}(s,t;\zeta)), \end{aligned} \tag{17}$$

with $0 \le p \le 1$ is an embedding parameter, $\widetilde{\mathcal{H}}(\zeta;p)$ is a non zero auxiliary fuzzy function for $p \neq 0$ and $\widetilde{\mathcal{H}}(\zeta;0) = 0$. When $p = 0$, $\widetilde{\Phi}(s,t;\zeta;0) = \widetilde{V}_0(s,t;\zeta)$ and when $p = 1$, $\widetilde{\Phi}(s,t;\zeta;1) = \widetilde{V}(s,t;\zeta)$ which is the exact solution. Therefore, the approximate solution $\widetilde{\Phi}(s,t;\zeta;p)$ will differ from the initial guess to the exact solution when p deforms from 0 to 1. The following form is used to pick the auxiliary convergence control function $\widetilde{\mathcal{H}}(\zeta;p)$ and in this situation:

$$\widetilde{\mathcal{H}}(\zeta;p) = \sum_{i=1}^{k}\widetilde{K}_i(\zeta)p^i, \tag{18}$$

where, $\widetilde{K}_i(\zeta) = \left[\underline{K}_i(\zeta), \overline{K}_i(\zeta)\right]$ the auxiliary convergence constants to be determined in each ζ-level set. Employ Taylor's series to get an approximate solution $\widetilde{\Phi}(s,t;r;\widetilde{K}_i(\zeta))$ for $0 \le \zeta \le 1$ as follows:

$$\widetilde{\Phi}(s,t;\zeta;\widetilde{K}_i(\zeta)) = \widetilde{V}_0(s,t;\zeta) + \sum_{i=1}^{\infty}\widetilde{V}_i(s,t;r;\widetilde{K}_i(\zeta))p^i. \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (??), we construct the general form of fuzzy fractional OHAM to explain the heat equation by equating the coefficient of identical powers of p. (1)

$$
\begin{aligned}
&(1-p)\left(D_t^{\alpha,\rho}\left[\widetilde{V}_0(s,t;\zeta)+\sum_{i=1}^{k}\widetilde{V}_i(s,t;\widetilde{K}_i(\zeta);\zeta)p^i\right]-\tilde{f}(s,t;\zeta)\right)\\
&=\sum_{i=1}^{k}\widetilde{K}_i(\zeta)p^i(D^{\alpha,\rho}\left[\widetilde{V}_0(s,t;\zeta)+\sum_{i=1}^{k}\widetilde{V}_i(s,t;\widetilde{K}_i(\zeta);\zeta)p^i\right]-\widetilde{\mathcal{A}}_\eta(s,t,\widetilde{V}(s,t;\zeta);\zeta)-\tilde{f}(s,t;\zeta)).
\end{aligned}
\tag{20}
$$

Afterward, equating each power of p coefficient to zero in the homotopy form of Eq. (16). This method generates a system of linear equations based on $\widetilde{V}_0(s,t;\zeta),\widetilde{V}_1(s,t;\zeta),\ldots,$ so, the approximate series solution of k^{th} order about p using the following form:

$$
\widetilde{V}(s,t,\widetilde{K}_j(\zeta);\zeta)=\widetilde{V}_0(s,t;\zeta)+\sum_{i=1}^{k}\widetilde{V}_i(s,t,\widetilde{K}_i(\zeta);\zeta). \tag{21}
$$

Now, by calculating parameters $\widetilde{K}_1(\zeta),\widetilde{K}_2(\zeta),\ldots,\widetilde{K}_k(\zeta)$ to get the best approximation solution of Eq.(1) and which should be identified for each ζ −level. Various techniques, including the least square method, the Galerkin method, the collocation method, the Ritz method, the Kantorovich method, and others, can be used to best identify these parameters. In this work, we employ the least-squares method to minimize the square residual error. Consider

$$
\widetilde{J}(s,t,\widetilde{K}_i(\zeta))=\iint_0^l\widetilde{E}^2(s,t,\widetilde{K}_i(\zeta))dxdt. \tag{22}
$$

Here, $\widetilde{E}$ is the residual error, where 0 and l are values that depend on the boundaries of the problem. The auxiliary convergence-control parameters $\widetilde{K}_i(\zeta)$ can be calculated as follows:

$$
\frac{\partial\widetilde{J}}{\partial\widetilde{K}_1}=\frac{\partial\widetilde{J}}{\partial\widetilde{K}_2}=\ldots=\frac{\partial\widetilde{J}}{\partial\widetilde{K}_k}=0. \tag{23}
$$

5. Convergence Analysis of OHAM for Fuzzy Fractional Heat Equation based on ML Operator with Three Parameters

The general form of OHAM for general fuzzy fractional differential equations and fuzzy set theory properties are merged with fractional calculus. This version has been improved to produce an approximation of the solution that was introduced in [48]. In this part, we introduce the convergence of OHAM, based on Sec. (3) and Sec. (4). Now, by rewriting Eq. (16) in the following lower and upper bound, respectively:

$$
\begin{cases}
\underline{\mathcal{L}}(\underline{V}(s,t;\zeta))-\underline{\mathcal{A}}_\eta((s,t,\widetilde{V}(s,t;\zeta))-\underline{f}(s,t;\zeta)=0,\\
\left(\underline{V}(s,t;\zeta),\dfrac{\partial\underline{V}}{\partial s}\right)=0,
\end{cases}
\tag{24}
$$

$$
\begin{cases}
\overline{\mathcal{L}}(\overline{V}(s,t;\zeta))-\overline{\mathcal{A}}_\eta(s,\widetilde{V}(s,t;\zeta))-\overline{f}(s,t;\zeta)=0,\\
\left(\overline{V}(s,t;\zeta),\dfrac{\partial\overline{V}}{\partial s}\right)=0.
\end{cases}
\tag{25}
$$

According to the defuzzification of Eq. (1). The formulation for the lower and upper bound of Eq. (17) fuzzy fractional OHAM

$$\begin{cases} (1-p)[\underline{\mathcal{L}}(\underline{\Phi}(s;p;\zeta)) - \underline{f}(s,t;\zeta)] = \underline{\mathcal{H}}(p;\zeta)[\underline{\mathcal{L}}(\underline{\Phi}(s,t;p;\zeta)) \\ -\underline{\mathcal{A}}_{\eta}((s,t,\widetilde{V}(s,t;\zeta)) - \underline{f}(s,t;\zeta)], \\ \left(\underline{\Phi}(s,t;p;\zeta), \dfrac{\partial \underline{\Phi}(s,t;p;\zeta)}{\partial s}\right) = 0, \end{cases} \tag{26}$$

$$\begin{cases} (1-p)[\overline{\mathcal{L}}(\overline{\Phi}(s,t;p;\zeta)) - \overline{f}(s,t;\zeta)] = \overline{\mathcal{H}}(p;\zeta)[\overline{\mathcal{L}}(\overline{\Phi}(s,t;p;\zeta)) \\ -\overline{\mathcal{A}}_{\eta}((s,t,\widetilde{V}(s,t;\zeta)) - \overline{f}(s,t;\zeta)], \\ \left(\overline{\Phi}(s,t;p;\zeta), \dfrac{\partial \overline{\Phi}(s,t;p;\zeta)}{\partial s}\right) = 0, \end{cases} \tag{27}$$

where the lower and upper fuzzy fractional linear operators are the lower and upper auxiliary fuzzy function and $[\underline{\Phi}(s,t;p;\zeta), \overline{\Phi}(s,t;p;\zeta)]$ the lower and upper unknown fuzzy function respectively. Obviously, when $p=0$ and $p=1$ respectively we have:

$$\underline{\Phi}(s,t;0;\zeta) = \underline{V}_0(s,t;\zeta), \qquad \underline{\Phi}(s,t;1;\zeta) = \underline{V}(s,t;\zeta). \tag{28}$$

$$\overline{\Phi}(s,t;0;\zeta) = \overline{V}_0(s,t;\zeta), \qquad \overline{\Phi}(s,t;1;\zeta) = \overline{V}(s,t;\zeta). \tag{29}$$

Therefore, when p increase from 0 to 1, the solution $\widetilde{\Phi}(\zeta,t;p;\zeta)$ vary from $\widetilde{V}_0(s,t;\zeta)$ to the exact solution. Now when $p=0$, the lower and upper bounds of zeroth-order:

$$\begin{cases} \underline{\mathcal{L}}(\underline{\Phi}(s,t;0;\zeta)) = \underline{f}(s,t;\zeta), & \left(\underline{V}(s,t;\zeta), \dfrac{\partial \underline{V}(s,t;\zeta)}{\partial s}\right) = 0, \\ \overline{\mathcal{L}}(\overline{\Phi}(s,t;0;\zeta)) = \overline{f}(s,t;\zeta), & \left(\overline{V}(s,t;\zeta), \dfrac{\partial \overline{V}(s,t;\zeta)}{\partial s}\right) = 0, \end{cases} \tag{30}$$

Now, the auxiliary function $\widetilde{\mathcal{H}}(p;r)$ for Eq. (26) and (27) is:

$$\begin{cases} \underline{\mathcal{H}}(p;\zeta) = \sum_{j=1}^{\infty} \underline{K}_j(\zeta) p^j = \underline{K}_1(\zeta)p^1 + \underline{K}_2(\zeta)p^2 + \ldots \\ \overline{\mathcal{H}}(p;\zeta) = \sum_{j=1}^{\infty} \overline{K}_j(\zeta) p^j = \overline{K}_1(\zeta)p^1 + \overline{K}_2(\zeta)p^2 + \ldots \end{cases} \tag{31}$$

where, $\widetilde{K}_1(\zeta) = [\underline{K}_1(\zeta), \overline{K}_1(\zeta)], \widetilde{K}_2(\zeta) = [\underline{K}_2(\zeta), \overline{K}_2(\zeta)], \ldots,$ is the auxiliary convergence constants. Expanding the solution $\widetilde{\Phi}(s,t;p;\zeta)$ about p by Taylor's series get the series approximate solution via fuzzy fractional OHAM

$$\begin{cases} \underline{\Phi}(x,t;\underline{K}_j(\zeta);\zeta) = \underline{V}_0(s,t;\zeta) + \sum_{j=1}^{\infty} \underline{V}_j(x,t;\underline{K}_j(\zeta);\zeta)p^j. \\ \overline{\Phi}(x;t;\overline{K}_j(\zeta);\zeta) = \overline{V}_0(s,t;\zeta) + \sum_{j=1}^{\infty} \overline{V}_j(x,t;\overline{K}_j(\zeta);\zeta)p^j. \end{cases} \tag{32}$$

Substituting Eq. (31) and (32) in Eq. (26) and (27) and then collecting the coefficient of like powers of p to find the lower and upper bound. This procedure give us a system of linear equations, For the zeroth-order given in (9) and for the first order

$$\begin{cases} \underline{\mathcal{L}}(\underline{V}_1(s,t;\zeta)) - \underline{\mathcal{L}}(\underline{V}_0(s,t;\zeta)) + \underline{f}(x,t) = \underline{K}_1(\zeta)(\underline{\mathcal{L}}(\underline{V}_0(s,t;\zeta)) \\ -\underline{\mathcal{A}}_\eta(\underline{V}_0(s,t;\zeta)) - \underline{f}(s,t;\zeta)), \\ \overline{\mathcal{L}}(\overline{V}_1(s,t;\zeta)) - \overline{\mathcal{L}}(\overline{V}_0(s,t;\zeta)) + \overline{f}(x,t) = \overline{K}_1(\zeta)(\overline{\mathcal{L}}(\overline{V}_0(s,t;\zeta)) \\ -\overline{\mathcal{A}}_\eta(\overline{V}_0(s,t;\zeta)) - \overline{f}(s,t;\zeta)), \\ \left(\widetilde{V}_1(s,t;\zeta), \dfrac{\partial \widetilde{V}_1(s,t;\zeta)}{\partial s}\right) = 0. \end{cases} \tag{33}$$

The problem of second-order

$$\begin{cases} \underline{\mathcal{L}}(\underline{V}_2(s,t;\zeta)) - \underline{\mathcal{L}}(\underline{V}_1(s,t;\zeta)) = \underline{K}_2(\zeta)\underline{\mathcal{A}}_\eta(\underline{V}_0(s,t;\zeta)) \\ +\underline{K}_1(\zeta)[\underline{\mathcal{L}}(\underline{V}_1(s,t;\zeta)) - \underline{\mathcal{A}}_\eta(\underline{V}_0(s,t;\zeta), \underline{V}_1(s,t;\zeta)) \\ -\underline{f}(s,t;\zeta)], \\ \overline{\mathcal{L}}(\overline{V}_2(s,t;\zeta)) - \overline{\mathcal{L}}(\overline{V}_1(s,t;\zeta)) = \overline{K}_2(\zeta)\overline{\mathcal{A}}_\eta(\overline{V}_0(s,t;\zeta)) \\ +\overline{K}_1(\zeta)[\overline{\mathcal{L}}(\overline{V}_1(s,t;\zeta)) - \overline{\mathcal{A}}_\eta(\overline{V}_0(s,t;\zeta), \overline{V}_1(s,t;\zeta)) \\ -\overline{f}(s,t;\zeta)], \\ \left(\widetilde{V}_2(s,t;\zeta), \dfrac{\partial \widetilde{V}_2(s,t;\zeta)}{\partial s}\right) = 0, \end{cases} \tag{34}$$

the general form of the governing problem via OHAM of k^{th} order

$$\begin{cases} \underline{\mathcal{L}}(\underline{V}_k(s,t;\zeta)) - \underline{\mathcal{L}}(\underline{V}_{k-1}(s,t;\zeta)) = \underline{K}_k(\zeta)\underline{\mathcal{A}}_\eta(\underline{V}_0(s,t;\zeta)) \\ -\sum_{i=1}^{k-1}\underline{K}_i(\zeta)[\underline{\mathcal{L}}(\underline{V}_{k-i}(s,t;\zeta)) - \underline{\mathcal{A}}_\eta(\underline{V}_0(s,t;\zeta),\ldots,\underline{V}_k(s,t;\zeta)) \\ -\underline{f}(s,t;\zeta)], \\ \overline{\mathcal{L}}(\overline{V}_k(s,t;\zeta)) - \overline{\mathcal{L}}(\overline{V}_{k-1}(s,t;\zeta)) = \overline{K}_k(\zeta)\overline{\mathcal{A}}_\eta(\overline{V}_0(s,t;\zeta)) \\ -\sum_{i=1}^{k-1}\overline{K}_i(\zeta)[\overline{\mathcal{L}}(\overline{V}_{k-i}(s,t;\zeta)) - \overline{\mathcal{A}}_\eta(\overline{V}_0(s,t;\zeta),\ldots,\underline{V}_k(s,t;\zeta)) \\ -\overline{f}(s,t;\zeta)], \\ \left(\widetilde{V}_k(s,t;\zeta), \dfrac{\partial \widetilde{V}_k(s,t;\zeta)}{\partial s}\right) = 0, \end{cases} \tag{35}$$

Depending on parameter $K_1(\zeta), K_2(\zeta), \ldots, K_k(\zeta)$, and at $p = 1$ we have:

$$\begin{cases} \underline{V}(s,t,\underline{K}_1(\zeta),\ldots;\zeta) = \underline{V}_0(s,t;\zeta) + \sum_{i=1}^{\infty}\underline{V}_i(s,t,\underline{K}_1(\zeta),\ldots;\zeta). \\ \overline{V}(x,t,\overline{K}_1(\zeta),\ldots;\zeta) = \overline{V}_0(s,t;\zeta) + \sum_{i=1}^{\infty}\overline{V}_i(s,t,\overline{K}_1(\zeta),\ldots;\zeta). \end{cases} \tag{36}$$

Approximating the series solution for k term as:

$$\begin{cases} \underline{V}_*(s,t,\underline{K}_1(\zeta),\ldots,\underline{K}_k(\zeta);\zeta) = \underline{V}_0(s,t;\zeta) \\ +\sum_{i=1}^{k}\underline{V}_i(s,t,\underline{K}_1(\zeta),\ldots,\underline{K}_i(\zeta);\zeta). \\ \overline{V}_*(s,t,\overline{K}_1(\zeta),\ldots,\overline{K}_k(\zeta);\zeta) = \overline{V}_0(s,t;\zeta) \\ +\sum_{i=1}^{k}\overline{V}_i(s,t,\overline{K}_1(\zeta),\ldots,\overline{K}_i(\zeta);\zeta). \end{cases} \tag{37}$$

We can obtain of the residual error $\widetilde{E} = [\underline{E}, \overline{E}]$, by substituting Eq. (37) in Eq. (24) and (25)

$$\begin{aligned} \underline{E}(s,t,\underline{K}_1(\zeta),\ldots,\underline{K}_i(\zeta);\zeta) = {} & \underline{\mathcal{L}}(\underline{V}_*(s,t,\underline{K}_1(\zeta),\ldots,\underline{K}_k(\zeta);\zeta)) \\ & -\underline{\mathcal{A}}_\eta(\underline{V}_*(s,\underline{K}_1(\zeta),\ldots,\underline{K}_k(\zeta);\zeta)) \\ & -\underline{f}(s,t;\zeta). \end{aligned} \tag{38}$$

$$\begin{aligned} \overline{E}(s,t,\overline{K}_1(\zeta),\ldots,\overline{K}_i(\zeta);\zeta) = {} & \overline{\mathcal{L}}(\overline{V}_*(s,t,\overline{K}_1(\zeta),\ldots,\overline{K}_k(\zeta);\zeta)) \\ & -\overline{\mathcal{A}}_\eta(\overline{V}_*(s,t,\overline{K}_1(\zeta),\ldots,\overline{K}_k(\zeta);\zeta)) \\ & -\overline{f}(s,t;\zeta). \end{aligned} \tag{39}$$

In the case of $\widetilde{E} = 0$ where $\widetilde{E} = [\underline{E}, \overline{E}]$, then $\widetilde{V}_*$ yields the exact solution. To determine the auxiliary constants $\widetilde{K}_1(\zeta), \widetilde{K}_2(\zeta), \ldots, \widetilde{K}_k(\zeta)$, we apply the least squares method

$$\begin{cases} \underline{J}(s,t,\underline{K}_1(\zeta),\underline{K}_2(\zeta),\ldots,\underline{K}_k(\zeta);\zeta) \\ = \iint_0^l \underline{E}^2(s,t,\underline{K}_1(\zeta),\underline{K}_2(\zeta),\ldots,\underline{K}_k(\zeta);\zeta)dsdt. \\ \overline{J}(s,t,\overline{K}_1(\zeta),\overline{K}_2(\zeta),\ldots,\overline{K}_k(\zeta);\zeta) \\ = \iint_0^l \overline{E}^2(s,t,\overline{K}_1(\zeta),\overline{K}_2(\zeta),\ldots,\overline{K}_k(\zeta);\zeta)dsdt. \end{cases} \tag{40}$$

Here, $\widetilde{J} = [\underline{J}, \overline{J}]$ and the optimal values for $\widetilde{K}_1(\zeta), \widetilde{K}_2(\zeta), \ldots, \widetilde{K}_K(\zeta)$, can be determined as follows:

$$\frac{\partial \widetilde{J}}{\partial \widetilde{K}_1} = \frac{\partial \widetilde{J}}{\partial \widetilde{K}_2} = \ldots = \frac{\partial \widetilde{J}}{\partial \widetilde{K}_k} = 0. \tag{41}$$

6. Numerical Experiment

In this part, the OHAM is used to get the analytical solution of the fuzzy fractional heat equation with a fuzzy initial condition. Consider the fuzzy fractional heat equation

Example 6.1: *Consider the fuzzy fractional heat equation*

$$\begin{aligned} & D_t^{\alpha,\rho}\widetilde{V}(s,t) = \frac{1}{2}s^2\frac{\partial^2 \widetilde{V}(s,t)}{\partial s^2}, 0 < s < 1, t > 0, \\ & \widetilde{V}(s,0) = \tilde{\phi}_1 s^2, \widetilde{V}(0,t) = \widetilde{V}(1,t) = 0, \end{aligned} \tag{42}$$

where, $\tilde{\phi}_1(\zeta) = [0.01\zeta - 0.01, 0.01 - 0.01\zeta]$, $\zeta \in [0,1]$. By [29], the exact solution of Eq.(42) when $\alpha, \rho \to 1$:

$$\widetilde{V}(s,t;\zeta) = \tilde{\phi}_1(\zeta)s^2e^t, \tag{43}$$

we can get the third order of the homotopy series by

$$(1-p)[D_t^{\alpha,\rho}\widetilde{V}(s,t;p;\zeta)] = \widetilde{\mathcal{H}}(\zeta)[D_t^{\alpha,\rho}\widetilde{V}(s,t;p;\zeta) - \frac{1}{2}s^2\frac{\partial^2\widetilde{V}(s,t;p;\zeta)}{\partial s^2}] \tag{44}$$

where

$$\widetilde{V}(s,t;p;\zeta) = \widetilde{V}_0(s,t;\zeta) + \sum_{i=1}^{3}\widetilde{V}_i(s,t;\widetilde{K}_i(\zeta);\zeta)p^i, \tag{45}$$

$$\widetilde{\mathcal{H}}(q;\zeta) = \sum_{i=1}^{k}\widetilde{K}_i(\zeta)p^i. \tag{46}$$

Now, substituting (45) and (46) into (44), and equating the coefficient of the same powers of p and using the initial condition in (44), we have the following equations:

Zeroth order, p^0:

$$D_t^{\alpha,\rho}\widetilde{V}_0(s,t;\zeta) = 0, \tag{47}$$

hence, by applying the left fractional integrals $(I^{\alpha,\rho})$ for Eq. (47) we have:

$$\widetilde{V}_0(s,t;\zeta) = \tilde{\phi}_1 s^2. \tag{48}$$

First order, p^1:

$$\begin{aligned} D_t^{\alpha,\rho}\widetilde{V}_1\left(s,t,\widetilde{K}_1(\zeta);\zeta\right) &= \left(1+\widetilde{K}_1(\zeta)\right)D_t^{\alpha,\rho}\widetilde{V}_0(s,t;\zeta) \\ &\quad -\frac{1}{2}s^2\widetilde{K}_1(\zeta)\frac{\partial^2\widetilde{V}_0(s,t;\zeta)}{\partial s^2}, \end{aligned} \tag{49}$$

hence, by applying the left fractional integrals $(I^{\alpha,\rho})$ for Eq. (49) we have:

$$\begin{cases} \widetilde{V}_1(s,t;\zeta) = \left(1+\widetilde{K}_1(\zeta)\right)\widetilde{V}_0(s,t;\zeta) - \frac{1}{2}s^2 I^{\alpha,\rho}\left(\widetilde{K}_1(\zeta)\frac{\partial^2\widetilde{V}_0(s,t;\zeta)}{\partial s^2}\right), \\ \widetilde{V}_1(s,0,\zeta) = 0. \end{cases} \tag{50}$$

The general equations for order $k \geq 2$ of (42) constructed as follows:

$$\begin{cases} \widetilde{V}_k\left(s,t;\zeta\right) = \widetilde{V}_{k-1}(s,t;\zeta) + \sum_{m=1}^{k}\widetilde{K}_m(\zeta)\widetilde{V}_{k-m}(s,t;\zeta) \\ -\frac{1}{2}s^2 I^{\alpha,\rho}\left(\sum_{m=1}^{k}\widetilde{K}_m(\zeta)\widetilde{V}_{k-m}(s,t;\zeta)\right), \\ \widetilde{V}_k(s,0,\zeta) = 0. \end{cases} \tag{51}$$

Now, for k^{th}-order fuzzy fractional OHAM, we can construct the approximation solution as follows:

$$\widetilde{V}_*\left(s,t,\widetilde{K}_1(\zeta),\ldots,\widetilde{K}_k(\zeta);\zeta\right) = \widetilde{V}_0(s,t;\zeta) + \sum_{i=1}^{k}\widetilde{V}_i(s,t,\widetilde{K}_i(\zeta);\zeta). \tag{52}$$

By substituting Eq.(52) into Eq.(42) we can obtain the residual error as follows:

$$\widetilde{E}(s,t;\zeta)=D_t^{\alpha,\rho}\widetilde{V}_*(s,t;\zeta)-\frac{1}{2}s^2\frac{\partial^2\widetilde{V}_*(s,t;\zeta)}{\partial s^2}. \tag{53}$$

Tables (1) and (2) provide the comparison for the upper and lower solution and accuracy for fuzzy fraction heat equation (42) between the tenth-order OHAM method and eighteenth-order term of the Variational iteration method with different values ζ and $\alpha=0.5$, $\rho=1$, $x=0.8$ and $s=0.05$.

Figure (1) shows the upper and lower three-dimensional fuzzy fractional solution for the heat equation (42), and Figures (2) and (3) are the residual error for lower and upper fuzzy fractional OHAM, respectively, at $\alpha=0.5,\rho=1,t=0.05,s\in[0,0.8]$ and $\zeta\in[0,1]$.

Table 1: Comparison between OHAM and Variational Iteration method (VIM) [29] for lower solution and accuracy for, at $\alpha=0.5,\rho=1$ and $t=0.05$, $s=0.8$.

ζ	$\underline{V}(s,t;\zeta)_{OHAM}$	$\underline{E}(s,t;\zeta)_{OHAM}$	$\underline{V}(s,t;\zeta)_{VIM}$	$\underline{E}(s,t;\zeta)_{VIM}$
0	–0.00839894	-2.91032×10^{-6}	–0.00839973	0.00000187
.2	–0.00671915	-2.32826×10^{-6}	–0.006719784	0.000001496
.4	–0.00503936	-1.74619×10^{-6}	–0.005039838	0.000001122
.6	–0.00335958	-1.16413×10^{-6}	–0.003359892	0.000000748
.8	–0.00167979	-5.82065×10^{-6}	–0.001679946	0.000000374
	0	0	0	0

Table 2: Comparison between OHAM and Variational Iteration method (VIM) [29] for upper solution and accuracy for, at $\alpha=0.5,\rho=1$ and $t=0.05$, $s=0.8$.

ζ	$\overline{V}(s,t;\zeta)_{OHAM}$	$\overline{E}(s,t;\zeta)_{OHAM}$	$\overline{V}(s,t;\zeta)_{VIM}$	$\overline{E}(s,t;\zeta)_{VIM}$
0	0.00839823	6.9994×10^{-7}	0.00839973	0.00000187
.2	0.00671859	5.59952×10^{-7}	0.006719784	0.000001496
.4	0.00503894	4.19964×10^{-7}	0.005039838	0.000001122
.6	0.00335929	2.79976×10^{-7}	0.003359892	0.000000748
.8	0.00167965	1.39988×10^{-7}	0.001679946	0.000000374
	0	0	0	0

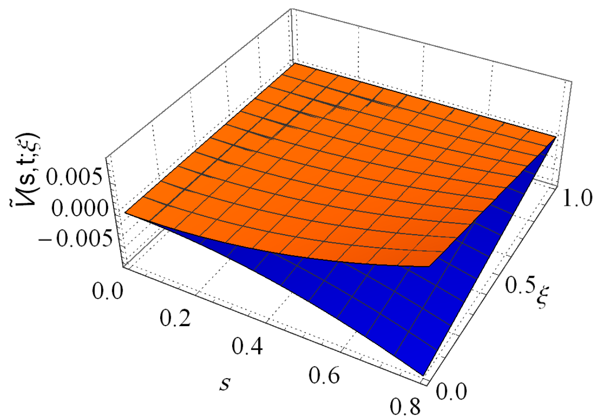

Figure 1: Upper and lower fuzzy fractional OHAM solution for (42), at $\alpha=0.5,\rho=1,t=0.05,s\in[0,0.8]$ and $\zeta\in[0,1]$

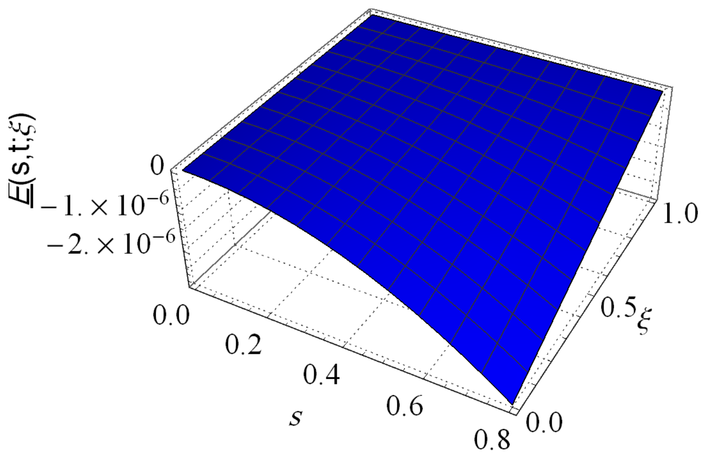

Figure 2: Residual error for lower fuzzy fractional OHAM for (42), at $\alpha = 0.5, \rho = 1, t = 0.05, s \in [0, 0.8]$ and $\zeta \in [0,1]$

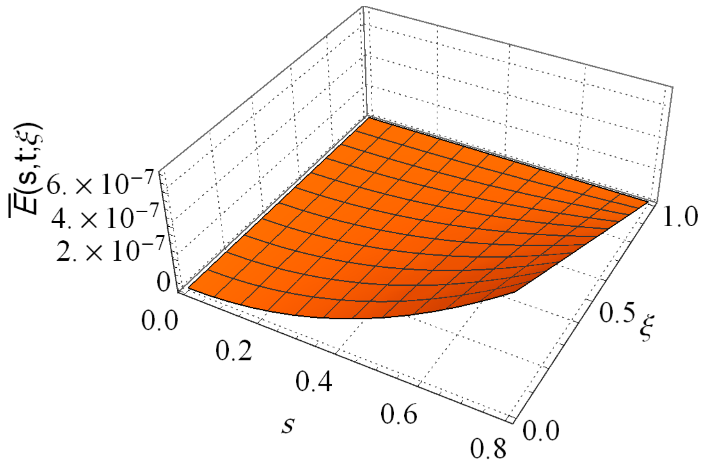

Figure 3: Residual error for upper fuzzy fractional OHAM for (42), at $\alpha = 0.5, \rho = 1, t = 0.05, s \in [0, 0.8]$ and $\zeta \in [0,1]$

Table 3: Five-order OHAM lower and upper solution and accuracy for Eq. (42) at $\alpha = 0.8, \rho = 0.8$ and $t = s = 0.5$.

ζ	$\underline{V}(s,t;\zeta)$	$\underline{E}(s,t;\zeta)$	$\overline{V}(s,t;\zeta)$	$\overline{E}(s,t;\zeta)$
0	–0.00611042	3.78087×10^{-7}	0.00611042	-3.78085×10^{-7}
.2	–0.00488833	3.02469×10^{-7}	0.00488833	-3.02468×10^{-7}
.4	–0.00366625	2.26852×10^{-7}	0.00366625	-2.26851×10^{-7}
.6	–0.00244417	1.51235×10^{-7}	0.00244417	-1.51234×10^{-7}
.8	–0.00122208	7.56174×10^{-8}	0.00122208	-7.5617×10^{-8}
	0	0	0	0

For Eq. (42), at $\alpha = 0.8, \rho = 0.8, t = 0.5, s \in [0, 0.5]$ and $\zeta = 0$. Figure (4) provides the triangular form fuzzy fractional five-order OHAM solution. Figure (5) shows the upper and lower solution. Figures (6) and (7) show the lower and upper residual error respectively.

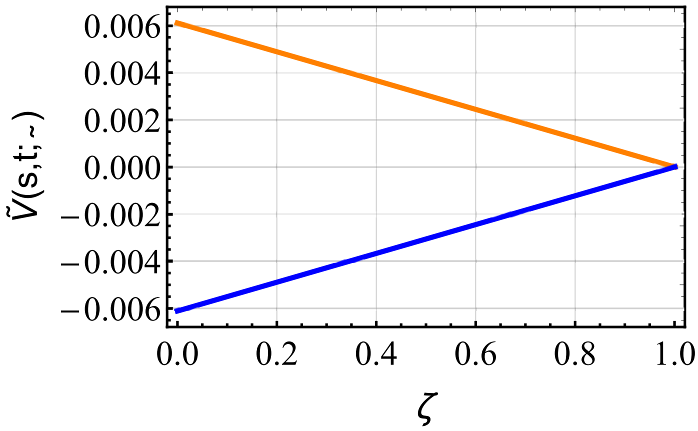

Figure 4: Triangular form fuzzy fractional five-order OHAM solution for Eq. (42), at $\alpha = 0.8, \rho = 0.8, t = 0.5, s \in [0, 0.5]$ and $\zeta = 0$

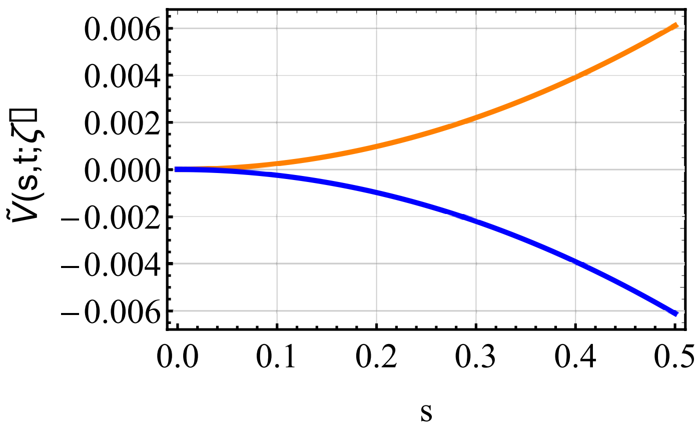

Figure 5: Upper and lower fuzzy fractional five-order OHAM solution for Eq. (42), at $\alpha = 0.8, \rho = 0.8, t = 0.5, s \in [0, 0.5]$ and $\zeta = 0$

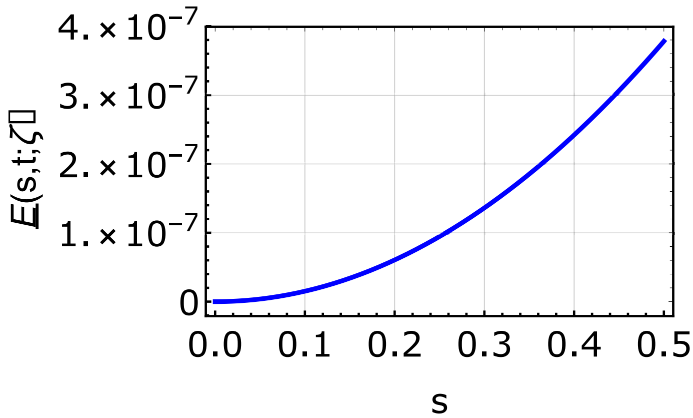

Figure 6: The residual error for lower fuzzy fractional five-order OHAM solution for Eq. (42), at $\alpha = 0.8, \rho = 0.8, t = 0.5, s \in [0, 0.5]$ and $\zeta = 0$

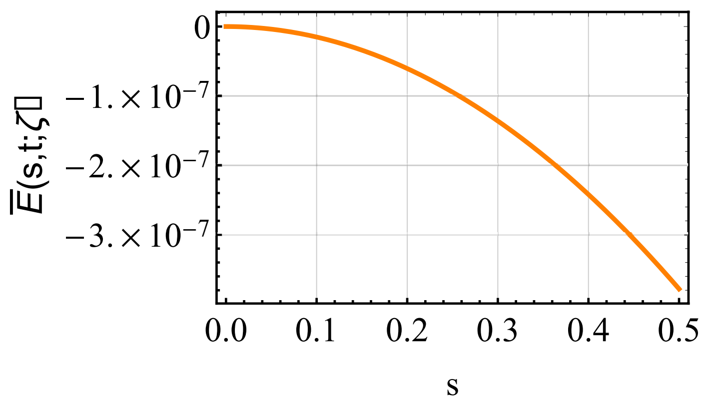

Figure 7: The residual error for upper fuzzy fractional five-order OHAM solution for Eq. (42), at $\alpha = 0.8, \rho = 0.8, t = 0.5, s \in [0, 0.5]$ and $\zeta = 0$

Conclusion

The fundamental goal of this research is to find an approximate analytical solution to the fuzzy heat equation. The fuzzy fractional heat equation, which is based on generalized Caputo-Katugampola with two parameters, was solved using the OHAM. This study demonstrates that the recommended technique for regulating solution convergence is simple, adaptable, and practical. It also performs well in terms of accuracy as the order of approximations increases. Numerical examples, such as a linear fuzzy heat equation with a fuzzy initial condition, demonstrate the method's potential through the numerical and graphical results with different orders of fractional derivatives that fulfill the criteria of the fuzzy number in the shape of a triangular fuzzy number.

References

[1] Ahmed Gamal Atta, Waleed Mohamed Abd-Elhameed, Galal Mahrous Moatimid, and Youssri Hassan Youssri. Modal shifted fifth-kind chebyshev tau integral approach for solving heat conduction equation. *Fractal and Fractional*, 6(11):619, 2022.

[2] Minh Duc Tran, Vu Ho, and Hoa Ngo Van. On the stability of fractional differential equations involving generalized caputo fractional derivative. *Mathematical Problems in Engineering*, 2020, 2020.

[3] Mohammud Foondun, Wei Liu, and McSylvester Omaba. Moment bounds for a class of fractional stochastic heat equations. *The Annals of Probability*, pages 2131–2153, 2017.

[4] Didier Dubois and Henri Prade. Towards fuzzy differential calculus part 3: Differentiation. *Fuzzy sets and systems*, 8(3):225–233, 1982.

[5] Wen Yu and Raheleh Jafari. *Modeling and Control of Uncertain Nonlinear Systems with Fuzzy Equations and Z-Number*. John Wiley & Sons, 2019.

[6] Udita N Katugampola. New approach to a generalized fractional integral. *Applied Mathematics and Computation*, 218(3):860–865, 2011.

[7] AK Alomari, Vedat Suat Erturk, Shaher Momani, and Ahmed Alsaedi. An approximate solution method for the fractional version of a singular bvp occurring in the electrohydrodynamic flow in a circular cylindrical conduit. *The European Physical Journal Plus*, 134(4):1–11, 2019.

[8] Ricardo Almeida, Agnieszka B Malinowska, and Tatiana Odzijewicz. Fractional differential equations with dependence on the caputo–katugampola derivative. *Journal of Computational and Nonlinear Dynamics*, 11(6), 2016.

[9] AA Alderremy, Khaled M Saad, Praveen Agarwal, Shaban Aly, and Shilpi Jain. Certain new models of the multi space-fractional gardner equation. *Physica A: Statistical Mechanics and Its Applications*, 545:123806, 2020.

[10] Y Chalco-Cano, H Roman-Flores, and MA Rojas-Medar. Fuzzy differential equations with generalized derivative. In *NAFIPS 2008-2008 Annual Meeting of the North American Fuzzy Information Processing Society*, pages 1–5. IEEE, 2008.

[11] Giangiacomo Gerla and Luisa Scarpati. Extension principles for fuzzy set theory. *Information Sciences*, 106(1-2):49–69, 1998.

[12] Ravi P Agarwal, Dumitru Baleanu, Juan J Nieto, Delfim FM Torres, and Yong Zhou. A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *Journal of Computational and Applied Mathematics*, 339:3–29, 2018.

[13] Esraa Magdy Abdelghany, Waleed Mohamed Abd-Elhameed, Galal Mahrous Moatimid, Youssri Hassan Youssri, and Ahmed Gamal Atta. A tau approach for solving time-fractional heat equation based on the shifted sixth-kind chebyshev polynomials. *Symmetry*, 15(3):594, 2023.

[14] Snehashish Chakraverty, Smita Tapaswini, and Diptiranjan Behera. *Fuzzy arbitrary order system: fuzzy fractional differential equations and applications*. John Wiley & Sons, 2016.

[15] Dulfikar Jawad Hashim, Ali Fareed Jameel, Teh Yuan Ying, AK Alomari, and NR Anakira. Optimal homotopy asymptotic method for solving several models of first order fuzzy fractional ivps. *Alexandria Engineering Journal*, 61(6):4931–4943, 2022.

[16] Zaid Odibat and Dumitru Baleanu. Numerical simulation of initial value problems with generalized caputo-type fractional derivatives. *Applied Numerical Mathematics*, 156:94–105, 2020.

[17] Vasile Marinca and Nicolae Herisanu. The optimal homotopy asymptotic method for solving blasius equation. *Applied Mathematics and Computation*, 231:134–139, 2014.

[18] Vangipuram Lakshmikantham and Ram N Mohapatra. *Theory of fuzzy differential equations and inclusions*. CRC press, 2004.

[19] Jiraporn Reunsumrit, Muhammad Sher, Kamal Shah, Nasser Aedh Alreshidi, and Meshal Shutaywi. On fuzzy partial fractional order equations under fuzzified conditions. *Fractals*, 30(01):2240025, 2022.

[20] Ebrahim Esmailzadeh, Davood Younesian, and Hassan Askari. Integral based methods. In *Analytical Methods in Nonlinear Oscillations*, pages 197–247. Springer, 2019.

[21] Thanin Sitthiwirattham, Muhammad Arfan, Kamal Shah, Anwar Zeb, Salih Djilali, and Saowaluck Chasreechai. Semi-analytical solutions for fuzzy caputo–fabrizio fractional-order two-dimensional heat equation. *Fractal and Fractional*, 5(4):139, 2021.

[22] NA Abdul Rahman and Muhammad Zaini Ahmad. Solving fuzzy fractional differential equations using fuzzy sumudu transform. *Journal of Nonlinear Sciences and Applications*, 10(1):2620–2632, 2017.

[23] S Siah Mansouri and N Ahmady. A numerical method for solving nth-order fuzzy differential equation by using characterization theorem. *Communication in Numerical Analysis*, 2012:12, 2012.

[24] Zahra Alijani, Dumitru Baleanu, Babak Shiri, and Guo-Cheng Wu. Spline collocation methods for systems of fuzzy fractional differential equations. *Chaos, Solitons & Fractals*, 131:109510, 2020.

[25] Hamzeh Zureigat, Ahmad Izani Ismail, and Saratha Sathasivam. Numerical solutions of fuzzy fractional diffusion equations by an implicit finite difference scheme. *Neural Computing and Applications*, 31(8):4085–4094, 2019.

[26] Xiaobin Guo and Dequan Shang. Approximate solution of th-order fuzzy linear differential equations. *Mathematical Problems in Engineering*, 2013, 2013.

[27] Fazle Mabood, Waqar A Khan, and Ahmad Izani Ismail. Optimal homotopy asymptotic method for flow and heat transfer of a viscoelastic fluid in an axisymmetric channel with a porous wall. *PLoS One*, 8(12):e83581, 2013.

[28] RK Thambynayagam. The diffusion handbook: applied solutions for engineers. *(No Title)*, 2011.

[29] N Herisanu, V Marinca, and Gh Madescu. An analytical approach to non-linear dynamical model of a permanent magnet synchronous generator. *Wind Energy*, 18(9):1657–1670, 2015.

[30] Ngo Van Hoa, Ho Vu, and Tran Minh Duc. Fuzzy fractional differential equations under caputo–katugampola fractional derivative approach. *Fuzzy Sets and Systems*, 375:70–99, 2019.

[31] Luciana T Gomes and Laécio C Barros. A note on the generalized difference and the generalized differentiability. *Fuzzy Sets and Systems*, 280:142–145, 2015.

[32] AF Jameel, Nidal Anakira, AK Alomari, I Hashim, and Shaher Momani. A new approximation method for solving fuzzy heat equations. *Journal of Computational and Theoretical Nanoscience*, 13(11):7825–7832, 2016.

[33] Lotfi Asker Zadeh. The concept of a linguistic variable and its application to approximate reasoningâ”i. *Information sciences*, 8(3):199–249, 1975.

[34] Ekhtiar Khodadadi, Mesut Karabacak, and Ercan Çelik. Solving fuzzy fractional riccati differential equations by the variational iteration method. *International Journal of Engineering and Applied Sciences*, 2(11):35–40, 2015.

[35] Meriem Araour and Abdelaziz Mennouni. A new procedures for solving two classes of fuzzy singular integro-differential equations: Airfoil collocation methods. *International Journal of Applied and Computational Mathematics*, 8(1):1–23, 2022.

[36] AK Alomari. Homotopy-sumudu transforms for solving system of fractional partial differential equations. *Advances in Difference Equations*, 2020(1):1–16, 2020.

[37] N Ratib Anakira, AK Alomari, and Ishak Hashim. Numerical scheme for solving singular two-point boundary value problems. *Journal of Applied Mathematics*, 2013, 2013.

[38] M Shahidi and A Khastan. Solving fuzzy fractional differential equations by power series expansion method. In *2018 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS)*, pages 37–39. IEEE, 2018.

[39] S Siah Mansouri and N Ahmady. A numerical method for solving nth-order fuzzy differential equation by using characterization theorem. *Communication in Numerical Analysis*, 2012:12, 2012.

[40] Saurabh Kumar and Vikas Gupta. An application of variational iteration method for solving fuzzy time-fractional diffusion equations. *Neural Computing and Applications*, 33:17659–17668, 2021.

[41] YH Youssri, WM Abd-Elhameed, SM Sayed, et al. Generalized lucas tau method for the numerical treatment of the one and two-dimensional partial differential heat equation. *Journal of Function Spaces*, 2022, 2022.

[42] Ali Ahmadian, Soheil Salahshour, Dumitru Baleanu, H Amirkhani, and Robiah Yunus. Tau method for the numerical solution of a fuzzy fractional kinetic model and its application to the oil palm frond as a promising source of xylose. *Journal of Computational Physics*, 294:562–584, 2015.

[43] Vasile Marinca, Nicolae Herisanu, and Iacob Nemes. Optimal homotopy asymptotic method with application to thin film flow. *Open Physics*, 6(3):648–653, 2008.

[44] Ali Ahmadian, Mohamed Suleiman, Soheil Salahshour, and Dumitru Baleanu. A jacobi operational matrix for solving a fuzzy linear fractional differential equation. *Advances in Difference Equations*, 2013(1):1–29, 2013.

[45] Mcsylvester Ejighikeme Omaba. On a mild solution to hilfer time-fractional stochastic differetial equation. *J. Fract. Calc. Appl*, 12:1–10, 2021.

[46] Daniela S Oliveira and Edmundo Capelas de Oliveira. On a caputo-type fractional derivative. *Advances in Pure and Applied Mathematics*, 10(2):81–91, 2019.

[47] Lotfi Asker Zadeh, George J Klir, and Bo Yuan. *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers*, volume 6. World Scientific, 1996.

[48] Manal Alqhtani, Mohamed M Khader, and Khaled Mohammed Saad. Numerical simulation for a high-dimensional chaotic lorenz system based on gegenbauer wavelet polynomials. *Mathematics*, 11(2):472, 2023.