Results in Nonlinear Analysis 6 (2023) No. 4, 13–29
https://doi.org/10.31838/rna/2023.06.04.003
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

# Fixed point approach for nonlinear $\psi$-caputo fractional differential hybrid coupled system with periodic boundary conditions

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**Abstract**

This article addresses the existence, uniqueness, and Ulam-Hyers stability of a class of nonlinear $\psi$-Caputo fractional differential hybrid coupled systems with periodic boundary conditions. Our approach is based on two key fixed point theorems: Banach's contraction principle and Scheafer's fixed point theorem. We provide a thorough discussion of the theoretical results and demonstrate their practical utility with a concrete example.

*Key words and phrases:* Existence, $\psi$-Caputo fractional derivative, coupled systems, boundary conditions, periodic conditions, uniqueness.

*Mathematics Subject Classification (2010):* 34A08, 26A33

## 1. Introduction

Fractional calculus is a branch of classical mathematics concerned with the generalization of integer order differentiation and integration of a function to non-integer order; it is a strong and expanding

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*Received July 21, 2023; Accepted August 11, 2023; Online September 16, 2023*

topic in theory as well as in its found several applications in science and engineering during the last few decades. See [1, 3–6, 11, 21, 41] and the references therein for some important results in the theory of fractional calculus and fractional differential equations. The authors of [7, 12, 14–16, 18, 31, 32, 36, 43] investigated the existence, uniqueness and stability results for several problems of differential and integral equations and inclusions.

The literature contains several definitions of fractional operators, and determining the importance of one over the other can be confusing. To address this, we can consider general operators that encompass various special kernels and differential operators. By selecting specific kernels and differential operators, we can obtain classical fractional integrals and derivatives. For instance, the $\psi$-Caputo fractional operator [10] can be reduced to the Riemann–Liouville or Hadamard fractional derivative by changing the kernel function $\psi$. Despite the unknown kernel involving $\psi$, we can deduce properties of the fractional operator. As a result, the $\psi$-Caputo fractional operator is considered a generalizing operator. Additionally, the $\psi$-Caputo fractional derivative maintains the physical interpretation of the Caputo derivative, making it advantageous for applications in physics and engineering. For some recent developments on the $\psi$-Caputo fractional operator, see the paper [2, 11, 40]

The fractional derivative of an unknown function hybrid with nonlinearity is used in hybrid differential equations. This class of equations derives from several fields of practical mathematics and physics, such as the deflection of a curved beam with a constant or variable cross-section, a three-layer beam, electromagnetic waves, or gravity-driven flows, neural networks and so forth. For more details on the subject, we recommend these publications [17, 23, 25–27, 33–35, 38, 39] to the readers.

Periodic boundary conditions are widely used in physics, engineering, and biological models to effectively model large systems by modeling a smaller unit cell that replicates regularly in space [24]. This technique allows for the study of large systems' behavior under various circumstances and over time without needing excessive computational resources. Periodic boundary conditions have been used in biological applications to investigate the behavior and interactions of membranes, proteins, and DNA with other molecules. Researchers have been able to understand the structure, dynamics, and interactions of Periodic boundary conditions with different solutes, such as ions and drugs, by using them in models of lipid bilayers. Similarly, Periodic boundary conditions have aided in the research of protein folding and stability, as well as the behavior of DNA under different conditions. See [37] for more information.

While solving differential equations precisely is difficult or impossible in several situations, along with nonlinear analysis and optimization, we investigate approximate solutions. It is important to stress that only stable estimates are acceptable. As a result, numerous methodologies for stability analysis are employed such as Lyapunov and exponential stability. Ulam, a mathematician, first raised the stability issue in functional equations in a 1940 lecture at Wisconsin University. S.M. Ulam posed the question, "Under what conditions does an additive mapping exist near an approximately additive mapping?" [42]. The succeeding year, Hyers addressed Ulam's issue for additive functions defined on Banach spaces in [19]. Rassias [28] showed the presence of unique linear mappings close to approximation additive mappings in 1978, generalizing Hyers' results. In comparison to Lyapunov and exponential stability analysis, Ulam-Hyers stability analysis focuses on the behavior of a function under perturbations, rather than the stability of a dynamical system or equilibrium point. The authors of [8, 9, 36] investigated the Ulam stabilities of fractional differential problems with different conditions. Significant attention has been paid to the research of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all types of functional equations, as evidenced by the book of Abbas *et al.* [1], and the work of Luo *et al.* [22] and Rus [29], which explored the Ulam-Hyers stability for operatorial equations.

In [40], Suwan *et al.* discussed two nonlinear fractional differential hybrid systems subjected to periodic boundary conditions. The first fractional nonlinear system is given by

$$\begin{cases} {}^c\mathbb{D}^{\alpha,\psi}_{a^+}(\varpi(t)\mathrm{g}_1(t,\varpi(t))) = \mathrm{g}_2(t,\varpi(t)), \alpha \in (0,1), \\ \varpi(a) = \varpi(b), \end{cases}$$

and the second system has the following form

$$\begin{cases} {}^c\mathbb{D}^{\alpha,\psi}_{a^+}(\varpi(t)\mathrm{g}_1(t,\varpi(t))) = \mathrm{g}_2(t,\varpi(t)), \alpha \in (1,2), \\ \varpi(a) = \varpi(b), \varpi'(a) = \varpi'(b), \end{cases}$$

where $t \in \mho := [a,b], {}^c\mathbb{D}^{\alpha,\psi}_{a^+}$ is the $\Psi$-Caputo fractional derivative, $\mathrm{g}_1 : \mho \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $g_2 : \mho \times \mathbb{R} \to \mathbb{R}$ are continuous with $g_1$ and $g_2$ are identically zero at the origin and $g_2(t,0) = 0$. Their reasoning is based on Dhage's fixed point theorem.

Using Banach's contraction mapping principle and Leray–Schauder nonlinear alternative fixed point theorem, Abbas [2] proved some existence and uniqueness of solutions of the following boundary value problem for coupled systems of $\psi$-Caputo fractional differential equations with four-point boundary conditions:

$$\begin{cases} {}^c\mathbb{D}^{\alpha,\psi}_{a^+}\varpi(t) = f(t,\varpi(t),y(t)), \quad t \in [0,1], 1 < \alpha < 2, \\ {}^c\mathbb{D}^{\beta,\psi}_{a^+} y(t) = g(t,\varpi(t),y(t)), \quad t \in [0,1], 1 < \beta < 2, \\ \varpi(0) = y(0) = 0, \\ \varpi(1) = \lambda\varpi(\eta), y(1) = \mu y(\xi), 0 < \eta,\xi < 1, \lambda,\mu > 0, \end{cases}$$

where ${}^c\mathbb{D}^{\alpha,\psi}_{a^+}, {}^c\mathbb{D}^{\beta,\psi}_{a^+}$ denote the $\psi$-Caputo fractional derivatives of order $\alpha,\beta$ and $f,g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Motivated by the above papers, we discuss the following nonlinear fractional differential hybrid system subject to periodic boundary conditions of the form:

$$\begin{cases} {}^c\mathbb{D}^{\alpha_i,\psi}_{a^+}(\varpi_i(t)g_i(t)) = f_i(t,\varpi_1(t),\varpi_2(t)), \quad t \in J := [a,b], \\ \varpi_i(a) = \varpi_i(b), \varpi_i'(a) = \varpi_i'(b), i = 1,2, \end{cases} \tag{1}$$

where ${}^c\mathbb{D}^{\alpha_i,\psi}_{a^+}$ are the $\psi$-Caputo fractional derivatives of order $\alpha_i \in (1,2)$, $0 < a < b$, $f_i \in C([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g_i \in C([a,b], \mathbb{R} \setminus \{0\})$. Our work in this paper is a direct continuation of the research mentioned in [40], where we build on the existing framework and by taking the results of [2] into considerations, to address the existence and Ulam stability results for the coupled system (1) by applying Banach's contraction principle and Scheafer's fixed point theorem.

The following is how this paper is organized. Section 2 contains definitions and lemmas that will be utilized throughout the work. Section 3 derives the existence and uniqueness results for the coupled system (1). The fourth section discusses the Ulam-Hyers stability results for our problem. In the final part, we present an example to demonstrate our main results.

## 2. Preliminaries

Let $\psi \in C^1(J,\mathbb{R})$ be an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in J$. Now, we start by defining $\psi$-fractional integral and derivative as follows:

**Definition 2.1:** ([20]) *The $\psi$–Riemann-Liouville fractional integral of order $\alpha > 0$ for an integrable function $\varpi : J \to \mathbb{R}$ is given by*

$$\mathbb{I}_{a^+}^{\alpha;\psi}\varpi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varpi(s)ds,$$

where $\Gamma$ is the Gamma function.

One can deduce that

$$D_t\left(\mathbb{I}_{a^+}^{\alpha;\psi}\varpi(t)\right) = \psi'(t)\mathbb{I}_{a^+}^{\alpha-1;\psi}\varpi(t), \alpha > 1,$$

where $D_t = \frac{d}{dt}$.

**Definition 2.2:** ([10]) *For* $n-1<\alpha<n$ $(n\in\mathbb{N})$ *and* $\varpi,\psi\in C^n(J,\mathbb{R})$, *the* $\psi$ *-Caputo fractional derivative of a function* $\varpi$ *of order* $\alpha$ *is given by*

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\varpi(t) = \mathbb{I}_{a^+}^{n-\alpha;\psi}\left(\frac{D_t}{\psi'(t)}\right)^n\varpi(t),$$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$, $n=\alpha$ for $\alpha\in\mathbb{N}$.

From the above definition, we can express $\psi$ -Caputo fractional derivative by formula

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\varpi(t) = \begin{cases} \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(\frac{D_t}{\psi'(s)}\right)^n\varpi(s)ds, & \text{if } \alpha\notin\mathbb{N}, \\ \left(\frac{D_t}{\psi'(\tau)}\right)^n\varpi(t), & \text{if } \alpha\in\mathbb{N}. \end{cases}$$

**Lemma 2.3:** ([10, 20]) *For* $\alpha,\beta>0$, *and* $\varpi\in C(J,\mathbb{R})$, *we have*

$$\mathbb{I}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{\beta;\psi}\varpi(t) = \mathbb{I}_{a^+}^{\alpha+\beta;\psi}\varpi(t), t\in J.$$

**Lemma 2.4:** ([10, 20]) *Let* $\alpha>0$. *If* $\varpi\in C(J,\mathbb{R})$, *then*

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi}\mathbb{I}_{a^+}^{\alpha;\psi}\varpi(t) = \varpi(t), t\in J,$$

and if $\varpi\in C^{n-1}(J,\mathbb{R})$, then

$$\mathbb{I}_{a^+}^{\alpha;\psi}\,{}^c\mathbb{D}_{a^+}^{\alpha;\psi}\varpi(t) = \varpi(t) - \sum_{k=0}^{n-1}\frac{\left(\frac{D_t}{\psi'(t)}\right)^k\varpi(a)}{k!}[\psi(t)-\psi(a)]^k, \quad t\in J.$$

**Lemma 2.5:** ([10, 20]) *For* $t>a, \alpha\geq 0,\ \beta>0$. *Then*

- $\mathbb{I}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}$;
- ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1}$;
- ${}^c\mathbb{D}_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^k = 0$, for all $k\in\{0,\dots,n-1\}, n\in\mathbb{N}$.

Let us recall the following existence result of solution of the linear hybrid fractional periodic system (see, [40]).

**Lemma 2.6:** *The solution of the periodic hybrid system*

$$\begin{cases} {}^{c}\mathbb{D}_{a^+}^{\alpha,\psi}(\varpi(t)g(t)) = f(t), \quad \alpha \in (1,2), t \in J, \\ \varpi(a) = \varpi(b), \varpi'(a) = \varpi'(b), \end{cases} \tag{2}$$

is given by

$$\varpi(t) = \frac{\mathbb{I}_{a^+}^{\alpha,\psi} f(t) - c_0 - c_1[\psi(t) - \psi(a)]}{g(t)}, \tag{3}$$

where

$$c_0 = \frac{g(a)(g(a))\psi'(b) - g(b))\psi'(a))\mathbb{I}_{a^+}^{\alpha,\psi} f(b)}{(g(b) - g(a))(g(b)\psi'(a) - g(a))\psi'(b)) + (g(a)g'(b) - g'((a)g(b))(\psi(b) - \psi(a))}$$
$$- \frac{(g(a))^2\psi'(b)[\psi(b) - \psi(a)]\mathbb{I}_{a^+}^{\alpha-1,\psi} f(b)}{(g(b) - g(a))(g(b))\psi'(a) - g(a))\psi'(b)) + (g(a)g'(b) - g'(a)g(b))(\psi(b) - \psi(a))},$$

and

$$c_1 = \frac{g(a)\psi'(b)(g(a) - g(b))\mathbb{I}_{a^+}^{\alpha-1,\psi} f(b)}{(g(b) - g(a))(g(b)\psi'(a) - g(a)\psi'(b)) + (g(a)g'(b) - g'(a)g(b))(\psi(b) - \psi(a))}$$
$$+ \frac{(g(a)g'(b) - g(b)g'(a))\mathbb{I}_{a^+}^{\alpha,\psi} f(b)}{(g(b) - g(a))(g(b)\psi'(a) - g(a)\psi'(b)) + (g(a)g'(b) - g'(a)g(b))(\psi(b) - \psi(a))}.$$

*Proof.* By Lemma 2.4, we can find that the general solution of (2) is given by

$$\varpi(t)g(t) = \mathbb{I}_{a^+}^{\alpha,\psi} f(t) - c_0 - c_1[\psi(t) - \psi(a)]. \tag{4}$$

Now, we need to determine the values of the constants $c_0$ and $c_1$. We begin by differentiating equation (4) and using Leibniz rule to obtain

$$\varpi'(t)g(t) + \varpi(t)g'(t) = \psi'(t)\mathbb{I}_{a^+}^{\alpha-1,\psi} f(t) - c_1\psi'(t).$$

Then

$$\varpi(a)g(a) = -c_0,$$
$$\varpi(b)g(b) = \mathbb{I}_{a^+}^{\alpha,\psi} f(b) - c_0 - c_1[\psi(b) - \psi(a)].$$

Furhter, we have

$$\varpi'(a)g(a) + \varpi(a)g'(a) = -c_1\psi'(a)$$
$$\varpi'(b)g(b) + \varpi(b)g'(b) = \psi'(b)\mathbb{I}_{a^+}^{\alpha-1,\psi} f(b) - c_1\psi'(b).$$

The boundary conditions imply that

$$c_0 = \frac{g(a)}{g(a) - g(b)}\mathbb{I}_{a^+}^{\alpha,\psi} f(b) - c_1\frac{g(a)}{g(a) - g(b)}[\psi(b) - \psi(a)],$$

and

$$c_1 = \frac{g(a)\psi'(b)}{g(a)\psi'(b) - g(b)\psi'(a)} \mathbb{I}_{a^+}^{\alpha-1,\psi} f(b) + \frac{c_0}{g(a)}\left(\frac{g(a)g'(b) - g(b)g'(a)}{g(a)\psi'(b) - g(b)\psi'(a)}\right).$$

Solving in $c_0$ and $c_1$, we obtain their values. This finishes the proof.

**Corollary 2.7:** *By the precedent theorem, we can deduce that the solution of the periodic hybrid nonlinear system (1) is given by*

$$\varpi_i(t) = \frac{\mathbb{I}_{a^+}^{\alpha_i,\psi} f_i(t,\varpi_1(t),\varpi_2(t)) + \xi_i(t) - \tilde{\xi}_i(t)}{g_i(t)}, \tag{5}$$

where

$$\xi_i(t) = [g_i(a)(g_i(a)\psi'(b) - g_i(b)\psi'(a)) - (g_i(a)g_i'(b) - g_i(b)g_i'(a))(\psi(t) - \psi(a))]$$
$$\times \frac{\mathbb{I}_{a^+}^{\alpha_i,\psi} f_i(b,\varpi_1(b),\varpi_2(b))}{(g_i(b) - g_i(a))(g_i(b)\psi'(a) - g_i(a)\psi'(b)) + (g_i(a)g_i'(b) - g_i'(a)g_i(b))(\psi(b) - \psi(a))},$$

and

$$\tilde{\xi}_i(t) = \frac{g_i(a)\psi'(b)[(\psi(b) - \psi(t))g_i(a) + (\psi(t) - \psi(a))g_i(b)]\mathbb{I}_{a^+}^{\alpha_i-1,\psi} f_i(b,\varpi_1(b),\varpi_2(b))}{(g_i(b) - g_i(a))(g_i(b)\psi'(a) - g_i(a)\psi'(b)) + (g_i(a)g_i'(b) - g_i'(a)g_i(b))(\psi(b) - \psi(a))}.$$

The system will be fully nonlinear system if we let $g_i(t) = g_i(t,\varpi_1(t),\varpi_2(t))$ by which we can obtain some results such as applying Banach fixed point theorem with some specific conditions. The existence results are obtained by using many fixed point theorems applied on the integral nonlinear equation (5).

## 3. Mains Results

By taking $\alpha_1 = \alpha_2 = \alpha$, we will rewrite the coupled system (1) in the following simple form:

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha,\psi}(\varpi_1(t)g_1(t)) = f_1(t,\varpi_1(t),\varpi_2(t)), \\ {}^c\mathbb{D}_{a^+}^{\alpha,\psi}(\varpi_2(t)g_2(t)) = f_2(t,\varpi_1(t),\varpi_2(t)), \alpha \in (1,2), \\ \varpi_i(a) = \varpi_i(b), \varpi_i'(a) = \varpi'(b), i = 1,2. \end{cases} \tag{6}$$

**Remark 3.1:** *A solution of the coupled system (6) is a function $\varpi^* = (\varpi_1,\varpi_2)$ where $\varpi_1,\varpi_2$ are continuous functions satisfying the problem (6).*

For $i = 1,2$, we set

$$\begin{cases} d_i = (g_i(b) - g_i(a))(g_i(b)\psi'(a) - g_i(a)\psi'(b)) + (g_i(a)g_i'(b) - g_i'(a)g_i(b))(\psi(b) - \psi(a)), \\ A_i = \dfrac{g_i(a)(g_i(a)\psi'(b) - g_i(b)\psi'(a))}{d_i}, \\ A_i^* = \dfrac{g_i(a)g_i'(b) - g_i(b)g_i'(a)}{d_i}, \\ B_i = \dfrac{g_i(a)\psi'(b)}{d_i}, \\ \Delta_i = \left(\dfrac{1}{\Gamma(\alpha+1)} + \dfrac{(|A_i| + |A_i^*|(\psi(b) - \psi(a)))}{\Gamma(\alpha+1)} + \dfrac{|B_i|(|g_i(a)| + |g_i(b)|)}{\Gamma(\alpha)}\right). \end{cases}$$

Let $\Omega = C \times C$ be the product Banach space which we equiped with the norm $\|(\varpi_1,\varpi_2)\|_\Omega = \|\varpi_1\|_C + \|\varpi_2\|_C$, where $C$ denotes the Banach space of all continuous functions from $J$ into $\mathbb{R}$.

Set $X = \{(\varpi_1,\varpi_2) \in \Omega$ with ${}^c\mathbb{D}^{\alpha,\psi}_{a^+}(\varpi_i g_i) \in C, i = 1,2\}$.

Let $T$ be the operator defined from $X$ into $X$ by:

$$T(\varpi_1,\varpi_2)(t) = (T_1(\varpi_1,\varpi_2)(t), T_2(\varpi_1,\varpi_2)(t)), t \in J, \tag{7}$$

where

$$T_i(\varpi_1,\varpi_2)(t) = \frac{1}{g_i(t)} H_i(\varpi_1,\varpi_2)(t),\ i = 1,2,$$

and

$$\begin{aligned} H_i(\varpi_1,\varpi_2)(t) \quad &= \mathbb{I}^{\alpha,\psi}_{a^+} f_i(t,\varpi_1(t),\varpi_2(t)) - (A_i - A_i^*(\psi(t)-\psi(a)))\mathbb{I}^{\alpha,\psi}_{a^+} f_i(b,\varpi_1(b),\varpi_2(b)) \\ &+ B_i((\psi(b)-\psi(t))g_i(a) - (\psi(t)-\psi(a))g_i(b))\mathbb{I}^{\alpha-1,\psi}_{a^+} f_i(b,\varpi_1(b),\varpi_2(b)), \end{aligned}$$

where $f_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous function such that $\| f_i \| < +\infty, i = 1,2$. For $i = 1,2$ we assume that $f_i$ and $g_i$ satisfy the following assumptions:

- The functions $g_i$ are continuous on $J$ and there exists a positive real numbers $M_1, M_2$ such that

$$|g_1(t)| \geq M_1,\ and\ |g_2(t)| \geq M_2,\ for\ all\ t \in J.$$

- There exist $k_i > 0$, such that

$$|f_i(t,x_1,y_1) - f_i(t,x_2,y_2)| \leq k_i\left[|x_1 - x_2| + |y_1 - y_2|\right],$$

for all $t \in J$, and $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

**Remark 3.2:** *In all what follow, we will write $\varpi$ to indicate the ordered couple $(\varpi_1,\varpi_2)$.*

**Lemma 3.3:** *The operator $T$ given in (7) is well defined.*

*Proof.* We will prove that $T_1\varpi$, $T_2\varpi$ are continuous functions on $J$ and

$${}^c\mathbb{D}^{\alpha,\psi}_{a^+}(T_1\varpi_1 g_1), {}^c\mathbb{D}^{\alpha,\psi}_{a^+}(T_2\varpi_2 g_2) \in C(J).$$

Let $(t_n)$ be a sequence in $J$ which converges to $t_0$ in $J$. Then,

$$\begin{aligned} H_1\varpi(t_n) - H_1\varpi(t_0) = &[\mathbb{I}^{\alpha,\psi}_{a^+} f_1(t_n,\varpi_1(t_n),\varpi_2(t_n)) \\ &- (A_1 - A_1^*(\psi(t_n)-\psi(a)))\mathbb{I}^{\alpha,\psi}_{a^+} f_1(b,\varpi_1(b),\varpi_2(b)) \\ &+ B_1((\psi(b)-\psi(t_n))g_1(a) - (\psi(t_n)-\psi(a))g_1(b))\mathbb{I}^{\alpha-1,\psi}_{a^+} f_1(b,\varpi_1(b),\varpi_2(b))] \\ &- [\mathbb{I}^{\alpha,\psi}_{a^+} f_1(t_0,\varpi_1(t_0),\varpi_2(t_0)) - (A_1 - A_1^*(\psi(t_0)-\psi(a)))\mathbb{I}^{\alpha,\psi}_{a^+} f_1(b,\varpi_1(b),\varpi_2(b)) \\ &+ B_1((\psi(b)-\psi(t_0))g_1(a) - (\psi(t_0)-\psi(a))g_1(b))\mathbb{I}^{\alpha-1,\psi}_{a^+} f_1(b,\varpi_1(b),\varpi_2(b))]. \end{aligned}$$

We have

$$\begin{aligned} &((\psi(b)-\psi(t_n))g_1(a) - (\psi(t_n)-\psi(a))g_1(b)) - ((\psi(b)-\psi(t_0))g_1(a) - (\psi(t_0)-\psi(a))g_1(b)) \\ &= g_1(a)(\psi(t_0)-\psi(t_n)) - g_1(b)(\psi(t_n)-\psi(t_0)) \underset{t_n \to t_0}{\to} 0, \end{aligned}$$

and

$$[(A_1 - A_1^*(\psi(t_n) - \psi(a))) - (A_1 - A_1^*(\psi(t_0) - \psi(a)))] = -A_1^*(\psi(t_n) - \psi(t_0)) \underset{t_n \to t_0}{\to} 0.$$

It is easily checked that

$$\begin{aligned}
&\left| \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t_n, \varpi_1(t_n), \varpi_2(t_n)) - \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t_0, \varpi_1(t_0), \varpi_2(t_0)) \right| \\
&\quad \le \frac{\| f_1 \|}{\Gamma(\alpha)} \int_{t_0}^{t_n} \psi'(s)(\psi(t_n) - \psi(s))^{\alpha-1} ds \\
&\quad + \frac{\| f_1 \|}{\Gamma(\alpha)} \int_a^{t_0} \psi'(s)(\psi(t_n) - \psi(s))^{\alpha-1} ds - \frac{\| f_1 \|}{\Gamma(\alpha)} \int_a^{t_0} \psi'(s)(\psi(t_0) - \psi(s))^{\alpha-1} ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left| \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t_n, \varpi_1(t_n), \varpi_2(t_n)) - \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t_0, \varpi_1(t_0), \varpi_2(t_0)) \right| \\
&\quad \le \frac{\| f_1 \|}{\Gamma(\alpha+1)} (\psi(t_n) - \psi(t_0))^{\alpha} + \frac{\| f_1 \|}{\Gamma(\alpha+1)} [(\psi(t_n) - \psi(a))^{\alpha} - (\psi(t_0) - \psi(a))^{\alpha}],
\end{aligned}$$

which implies that

$$\left| \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t_n, \varpi_1(t_n), \varpi_2(t_n)) - \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t_0, \varpi_1(t_0), \varpi_2(t_0)) \right| \underset{t_n \to t_0}{\to} 0.$$

Consequently,

$$\left| H_1\varpi(t_n) - H_1\varpi(t_0) \right| \underset{t_n \to t_0}{\to} 0,$$

that is $H_1\varpi \in C(J)$. From assumption $(S_1)$, we deduce that $T_1\varpi$ is continuous on $J$. On the other hand,

$${}^c\mathbb{D}_{a^+}^{\alpha,\psi}(T_1\varpi(t) g_1(t)) = {}^c\mathbb{D}_{a^+}^{\alpha,\psi}(H\varpi(t)) = f_1(t, \varpi_1(t), \varpi_2(t)),$$

that is ${}^c\mathbb{D}_{a^+}^{\alpha,\psi}(T_1\varpi g_1) \in C(J)$.

In the same way we prove that $T_2\varpi, {}^c\mathbb{D}_{a^+}^{\alpha,\psi}(T_1\varpi g_1) \in C(J)$. Therefore, $(T_1\varpi, T_2\varpi) \in \Omega$ and ${}^c\mathbb{D}_{a^+}^{\alpha,\psi}(T_i\varpi g_i) \in C(J), i = 1, 2$. The proof is completed.

The following theorem present our first main existence and uniqueness result.

**Theorem 3.4:** *If the hypotheses $(S_1)$ and $(S_2)$ are hold, and if*

$$0 < \left( \frac{k_1 \Delta_1}{M_1} + \frac{k_2 \Delta_2}{M_2} \right) < (\psi(b) - \psi(a))^{-\alpha}, \tag{8}$$

*then the coupled system (6) has a unique solution.*

*Proof.* Our main tool is Banach's contraction principle. We prove that $T$ is a contraction. Indeed, let $\varpi$ and $\omega \in C(J)$. For any $t \in J$, we have

$$\begin{aligned}
\left| H_1\varpi(t) - H_1\omega(t) \right| &\le \left| \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t, \varpi_1(t), \varpi_2(t)) - \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t, \omega_1(t), \omega_2(t)) \right| \\
&+ \left| (A_1 - A^*(\psi(t) - \psi(a))) \right| \left| \mathbb{I}_{a^+}^{\alpha,\psi} f_1(b, \varpi_1(b), \varpi_2(b)) - \mathbb{I}_{a^+}^{\alpha,\psi} f_1(b, \omega_1(b), \omega_2(b)) \right| \\
&+ \left| B_1((\psi(b) - \psi(t)) g_1(a) - (\psi(t) - \psi(a)) g_1(b)) \right| \\
&\times \left| \mathbb{I}_{a^+}^{\alpha-1,\psi} f_1(b, \varpi_1(b), \varpi_2(b)) - \mathbb{I}_{a^+}^{\alpha-1,\psi} f_1(b, \omega_1(b), \omega_2(b)) \right|.
\end{aligned}$$

We observe the following

$$\sup_{t\in J}\left|\mathbb{I}_{a^+}^{\alpha,\psi} f_1(t,\varpi_1(t),\varpi_2(t)) - \mathbb{I}_{a^+}^{\alpha,\psi} f_1(t,\omega_1(t),\omega_2(t))\right| \leq \frac{k_1}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^\alpha \parallel \varpi-\omega \parallel_X,$$

$$\sup_{t\in J}\left|(A_1 - A_1^*(\psi(t)-\psi(a)))\right|\left|\mathbb{I}_{a^+}^{\alpha,\psi} f_1(b,\varpi_1(b),\varpi_2(b)) - f_1(b,\omega_1(b),\omega_2(b))\right|$$

$$\leq \frac{k_1}{\Gamma(\alpha+1)}\left(|A| + |A^*||\psi(b)-\psi(a)|\right)\times(\psi(b)-\psi(a))^\alpha \parallel \varpi-\omega \parallel_X,$$

$$\sup_{t\in J}\left|B_1((\psi(b)-\psi(t))g_1(a) - (\psi(t)-\psi(a))g_1(b))\right|$$

$$\times\left|\mathbb{I}_{a^+}^{\alpha-1,\psi} f_1(b,\varpi_1(b),\varpi_2(b)) - f(b,\omega_1(b),\omega_2(b))\right|$$

$$\leq \frac{k_1}{\Gamma(\alpha)}(\psi(b)-\psi(a))^\alpha |B_1|\left(|g_1(a)| + |g_1(b)|\right) \parallel \varpi-\omega \parallel_X,$$

and

$$\parallel H_1\varpi - H_1\omega \parallel_C \leq k_1\left(\frac{1}{\Gamma(\alpha+1)} + \frac{(|A_1| + |A_1^*|(\psi(b)-\psi(a)))}{\Gamma(\alpha+1)} + \frac{|B_1|(|g_1(a)|+|g_1(b)|)}{\Gamma(\alpha)}\right)$$

$$\times(\psi(b)-\psi(a))^\alpha \parallel \varpi-\omega \parallel_X .$$

We conclude that

$$\parallel H_1\varpi - H_1\omega \parallel_C \leq k_1\Delta_1(\psi(b)-\psi(a))^\alpha \parallel \varpi-\omega \parallel_X . \tag{9}$$

From inequality (9) and the assumption $(S_1)$ we obtain

$$\left\|T_1\varpi - T_1\omega\right\|_C \leq \frac{k_1\Delta_1}{M_1}(\psi(b)-\psi(a))^\alpha \parallel \varpi-\omega \parallel_X . \tag{10}$$

Following the same steps, we get

$$\left\|T_2\varpi - T_2\omega\right\|_C \leq \frac{k_2\Delta_2}{M_2}(\psi(b)-\psi(a))^\alpha \parallel \varpi-\omega \parallel_X . \tag{11}$$

Setting $\eta := \left(\frac{k_1\Delta_1}{M_1} + \frac{k_2\Delta_2}{M_2}\right)(\psi(b)-\psi(a))^\alpha$, and taking into account (10), (11) and (8), we obtain

$$\left\|T\varpi - T\omega\right\|_X = \left\|T_1\varpi - T_1\omega\right\|_C + \left\|T_2\varpi - T_2\omega\right\|_C \leq \eta \parallel \varpi-\omega \parallel_X,$$

with $0<\eta<1$. Then, $T$ is a contraction and the proof is completed.

**Remark 3.5:** *If in addition we assume that there exists* $y_1, y_2 \in J$ *such that* $f_1(y_1,0,0)\neq 0$ *and* $f_2(y_2,0,0)\neq 0$, *then the solution of the problem (6) is non trivial.*

Our second main result is to prove that the problem (6) has at least a non trivial solution using a variant of Schaefer's fixed point theorem.

Assume the following hypothesis:

- There exist $\tilde{k}_i, \varkappa_i > 0$, such that

$$|f_i(t,x,y)| \leq \tilde{k}_i(|x|+|y|) + \varkappa_i,$$

  for all $t\in J$, and $x,y\in\mathbb{R}$.

Let $\gamma_i; i=1,2$ be two positive constants such that

$$\gamma_i = \frac{(\psi(b)-\psi(a))^{\alpha}}{M_i\Gamma(\alpha+1)}[1+|A_i|+|A_i^*|(\psi(b)-\psi(a))+\alpha|B_i|(|g_i(a)|+|g_i(b)|)]; i=1,2.$$

**Theorem 3.6:** *Assume that the hypotheses* $(S_1)$ *and* $(S_3)$ *hold. If*

$$\max\left\{\left(\frac{\tilde{k}_1\Delta_1}{M_1}+\frac{\tilde{k}_2\Delta_2}{M_2}\right)\left(\psi(b)-\psi(a)\right)^{\alpha}, \tilde{k}_1\gamma_1+\tilde{k}_2\gamma_2\right\}<1, \tag{12}$$

then problem (6) has at least a non trivial solution.

*Proof.* The proof is established in some steps.

**Step 1:** The continuity of the operator $T$ follows from the continuity of the function $f_i$ and $g_i$. So, we need demonstrate the compactness of $T$.

Let $D=\left\{\varpi \in X;\ \|\varpi\|_X \le r\right\}$ be a subset of $X$ where

$$r \ge \frac{\left(\frac{\varkappa_1\Delta_1}{M_1}+\frac{\varkappa_2\Delta_2}{M_2}\right)\left(\psi(b)-\psi(a)\right)^{\alpha}}{1-\left(\frac{\tilde{k}_1\Delta_1}{M_1}+\frac{\tilde{k}_2\Delta_2}{M_2}\right)\left(\psi(b)-\psi(a)\right)^{\alpha}}.$$

We have to prove that $T$ maps the bounded set $D$ into bounded set $D$. Let $\varpi \in D$, then we have

$$\begin{aligned} H_1\varpi(t) &= \mathbb{I}_{a^+}^{\alpha,\psi}(f_1(t,\varpi_1(t),\varpi_2(t))) - (A_1 - A_1^*(\psi(t)-\psi(a)))\mathbb{I}_{a^+}^{\alpha,\psi}(f_1(b,\varpi_1(b),\varpi_2(b))) \\ &\quad + B_1((\psi(b)-\psi(t))g_1(a) - (\psi(t)-\psi(a))g_1(b))\mathbb{I}_{a^+}^{\alpha-1,\psi}(f_1(b,\varpi_1(b),\varpi_2(b))). \end{aligned}$$

For all $t \in J$, by hypothesis $(S_3)$ we have

$$\|H_1\varpi\|_C \le \Delta_1(\tilde{k}_1\|\varpi\|_X + \varkappa_1)(\psi(b)-\psi(a))^{\alpha},$$

which implies

$$\|T_1\varpi\|_C \le \frac{\Delta_1}{M_1}(\tilde{k}_1 r + \varkappa_1)(\psi(b)-\psi(a))^{\alpha},$$

and

$$\|T_2\varpi\|_C \le \frac{\Delta_2}{M_2}(\tilde{k}_2 r + \varkappa_2)(\psi(b)-\psi(a))^{\alpha}.$$

Therefore,

$$\|T\varpi\|_X \le \left(\frac{\tilde{k}_1\Delta_1}{M_1}+\frac{\tilde{k}_2\Delta_2}{M_2}\right)(\psi(b)-\psi(a))^{\alpha} r + \left(\frac{\varkappa_1\Delta_1}{M_1}+\frac{\varkappa_2\Delta_2}{M_2}\right)(\psi(b)-\psi(a))^{\alpha}, \tag{13}$$

and by (12) we have

$$\|T\varpi\|_X \le r.$$

That is $T(D) \subset D$.

**Step 2:** We will prove that $T(D)$ is an equicontinuous set of $X$.

Let $t, t_0 \in J$ with $t_0 < t$, and $\varpi \in D$ we have,

$$\begin{aligned} T_1\varpi(t) - T_1\varpi(t_0) = &\frac{1}{g_1(t)}[\mathbb{I}^{\alpha,\psi}_{a^+} f_1(t, \varpi_1(t, \varpi_2(t)) - (A_1 - A_1^*(\psi(t) - \psi(a)))\mathbb{I}^{\alpha,\psi}_{a^+} f_1(b, \varpi_1(b), \varpi_2(b)) \\ &+ B_1((\psi(b) - \psi(t))g_1(a) - (\psi(t) - \psi(a))g_1(b))\mathbb{I}^{\alpha-1,\psi}_{a^+} f_1(b, \varpi_1(b), \varpi_2(b))] \\ &- \frac{1}{g_1(t_0)}[\mathbb{I}^{\alpha,\psi}_{a^+} f_1(t_0, \varpi_1(t_0), \varpi_2(t_0)) - (A_1 - A_1^*(\psi(t_0) - \psi(a)))\mathbb{I}^{\alpha,\psi}_{a^+} \\ &\times f_1(b, \varpi_1(b), \varpi_2(b)) \\ &+ B_1((\psi(b) - \psi(t_0))g_1(a) - (\psi(t_0) - \psi(a))g_1(b))\mathbb{I}^{\alpha-1,\psi}_{a^+} f_1(b, \varpi_1(b), \varpi_2(b))]. \end{aligned}$$

Thus

$$\begin{aligned} \left|T_1\varpi(t) - T_1\varpi(t_0)\right| &= \left|\frac{1}{g_1(t)}\mathbb{I}^{\alpha,\psi}_{a^+} f_1(t, \varpi_1(t), \varpi_2(t)) - \frac{1}{g_1(t_0)}\mathbb{I}^{\alpha,\psi}_{a^+} f_1(t_0, \varpi_1(t_0), \varpi_2(t_0))\right| \\ &\le \frac{(\tilde{k}_1 r + \varkappa_1)\,|\,g_1(t_0) - g_1(t)\,|}{M_1^2\Gamma(\alpha+1)}\left|(\psi(t) - \psi(a))^\alpha - (\psi(t_0) - \psi(a))^\alpha\right|. \end{aligned}$$

Therefore, we can obtain

$$\left|T_1\varpi(t) - T_1\varpi(t_0)\right| \underset{t \to t_0}{\to} 0. \tag{14}$$

The same process lead to

$$\left|T_2\varpi(t) - T_2\varpi(t_0)\right| \underset{t \to t_0}{\to} 0. \tag{15}$$

Then, $T(D)$ is equicontinuous in $X$. Since $T$ is a continuous operator which maps bounded sets of $X$ into uniformly bounded and equicontinuous sets in $X$ therefore, by Ascoli-Arzela theorem it follows that $T: X \to X$ is a compact operator.

**Step 3:** In order to apply Schaefer's fixed theorem, it remains to prove that the set

$$\Gamma = \{\varpi \in X : \ \lambda T\varpi = \varpi, \ 0 < \lambda < 1\} \text{ is bounded.}$$

Let $\varpi \in \Gamma$. Hence $\lambda T_1\varpi = \varpi_1$ and $\lambda T_2\varpi = \varpi_2$.

Since $|\lambda| < 1$ and using $(S_1), (S_3)$ and (12), we have

$$\|\varpi_1\|_C = \|\lambda T_1\varpi\|_C \le \frac{(\tilde{k}_1\|\varpi\|_X + \varkappa_1)(\psi(b) - \psi(a))^\alpha}{M_1\Gamma(\alpha+1)}\Big[1 + |A_1| + |A_1^*|(\psi(b) - \psi(a)) + \alpha\,|B_1|(|g_1(a)| + |g_1(b)|)\Big],$$

and

$$\|\varpi_2\|_C = \|\lambda T_2\varpi\|_C \le \frac{(\tilde{k}_2\|\varpi\|_X + \varkappa_2)(\psi(b) - \psi(a))^\alpha}{M_2\Gamma(\alpha+1)}\Big[1 + |A_2| + |A_2^*|(\psi(b) - \psi(a)) + \alpha\,|B_2|(|g_2(a)| + |g_2(b)|)\Big].$$

Thus

$$\|\varpi\|_X = \|\varpi_1\|_C + \|\varpi_2\|_C \leq \frac{(\tilde{k}_1 \|\varpi\|_X + \varkappa_1)(\psi(b)-\psi(a))^\alpha}{M_1\Gamma(\alpha+1)}\Big[1+|A_1|+|A_1^*|(\psi(b)-\psi(a))+\alpha|B_1|(|g_1(a)|+|g_1(b)|)\Big]$$
$$+\frac{(\tilde{k}_2 \|\varpi\|_X + \varkappa_2)(\psi(b)-\psi(a))^\alpha}{M_2\Gamma(\alpha+1)}\Big[1+|A_2|+|A_2^*|(\psi(b)-\psi(a))+\alpha|B_2|(|g_2(a)|+|g_2(b)|)\Big]$$
$$\leq (\tilde{k}_1\gamma_1 + \tilde{k}_2\gamma_2)\|\varpi\|_X + (\varkappa_1\gamma_1 + \varkappa_2\gamma_2).$$

Thus, by (12) we have

$$\|\varpi\|_X \leq \frac{\varkappa_1\gamma_1 + \varkappa_2\gamma_2}{1-(\tilde{k}_1\gamma_1 + \tilde{k}_2\gamma_2)},$$

that is

$$\|\varpi\| \leq \rho,$$

where $\rho$ is a positive number satisfying

$$\rho \geq \frac{\varkappa_1\gamma_1 + \varkappa_2\gamma_2}{1-(\tilde{k}_1\gamma_1 + \tilde{k}_2\gamma_2)}.$$

Consequently, Scheafer's fixed point theorem gives the solution of (6).

## 4. Ulam-Hyers Stability

Following [13, 30], we present the generalized Ulam-Hyers stability and the Ulam-Hyers stability of the problem (6).

**Definition 4.1:** ([13, 30]) *The fractional boundary value problem (2) is generalized Ulam-Hyers stable if there exists $\phi_f \in C(\mathbb{R}_+,\mathbb{R}_+), \phi_f(0)=0$, such that for each $\varepsilon>0$ and for each solution $v \in C$ of the inequality*

$$|^c\,\mathbb{D}_{a^+}^{\alpha,\psi}(\varpi(t)g(t)) - f(t)| \leq \varepsilon, t \in J,$$

there exists a solution $\varpi \in C$ of the fractional boundary value problem (2) with

$$|\varpi(t)-\varpi(t)| \leq \psi_f(\varepsilon), t \in J.$$

If $\phi_f(\varepsilon)=e\varepsilon$ with $e>0$, then the fractional boundary value problem (2) is Ulam-Hyers stable.

**Theorem 4.2:** *If all assumptions of Theorem (3.4) are met, then problem (6) is Ulam-Hyers stable.*

*Proof.* Let $\varepsilon$ be a real positive number and $\varpi$ the unique solution of the problem (6). Let $v=(v_1,v_2)\in X$ be a sollution of the coupled system of inequalities

$$\begin{cases} |^c\,\mathbb{D}_{a^+}^{\alpha,\psi}(v_1(t)g_1(t)) - f_1(t,v_1(t),v_2(t))| \leq \varepsilon, t\in J, \alpha\in(1,2) \\ \\ |^c\,\mathbb{D}_{a^+}^{\alpha,\psi}(v_2(t)g_2(t)) - f_1(t,v_1(t),v_2(t))| \leq \varepsilon, t\in J, \alpha\in(1,2) \\ \\ v_i(a)=v_i(b)=\varpi_i(a)=\varpi_i(b), \\ v_i'(a)=v_i'(b)=\varpi_i'(a)=\varpi_i'(b), i=1,2, \end{cases} \tag{16}$$

where $\varpi = (\varpi_1, \varpi_2) \in X$ is the unique solution of the coupled system (6). then, by using

$$v_i(a) = v_i(b) = \varpi_i(a) = \varpi_i(b), v_i'(a) = v_i'(b) = \varpi_i'(a) = \varpi_i'(b) \tag{17}$$

and by integrating the inequalities (18) and (19) bellow

$$|{}^c\mathbb{D}_{a^+}^{\alpha,\psi}(v_1(t)g_1(t)) - f_1(t, v_1(t), v_2(t))| \leq \varepsilon, t \in J, \alpha \in (1,2) \tag{18}$$

$$|{}^c\mathbb{D}_{a^+}^{\alpha,\psi}(v_2(t)g_2(t)) - f_1(t, v_1(t), v_2(t))| \leq \varepsilon, t \in J, \alpha \in (1,2) \tag{19}$$

we obtain

$$\left|v_1(t) - T_1 v(t)\right| \leq \frac{1}{|g_1(t)|}\varepsilon \mathbb{I}_{a^+}^{\alpha,\psi}(1) \leq \frac{1}{M_1\Gamma(\alpha+1)}(\psi(b) - \psi(a))^\alpha \varepsilon,$$
$$\left|v_2(t) - T_2 v(t)\right| \leq \frac{1}{|g_2(t)|}\varepsilon \mathbb{I}_{a^+}^{\alpha,\psi}(1) \leq \frac{1}{M_2\Gamma(\alpha+1)}(\psi(b) - \psi(a))^\alpha \varepsilon.$$

On the other hand,

$$\begin{aligned}\|v - \varpi\|_X &\leq \|v - Tv\|_X + \|Tv - \varpi\|_X \leq (\|v_1 - T_1 v\|_X + \|v_2 - T_2 v\|_X) + \|Tv - T\varpi\|_X \\ &\leq \left(\frac{1}{M_1} + \frac{1}{M_2}\right)\frac{1}{\Gamma(\alpha+1)}(\psi(b) - \psi(a))^\alpha \varepsilon + \eta\|v - \varpi\|_X,\end{aligned}$$

where we have used assumptions of Theorem (3.4) and the fact that the operator $T$ is a contraction. Therefore,

$$(1-\eta)\|v - \varpi\|_X \leq \left(\frac{1}{M_1} + \frac{1}{M_2}\right)\frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)}\varepsilon,$$

since $0 < \eta < 1$, then

$$\|v - \varpi\|_X \leq \left(\frac{1}{M_1} + \frac{1}{M_2}\right)\frac{(\psi(b) - \psi(a))^\alpha}{(1-\eta)\Gamma(\alpha+1)}\varepsilon.$$

Consequently, there exists a positive continuous function $\phi_{f_i}(\varepsilon)$ such that

$$\|v - \varpi\|_X \leq \phi_{f_i}(\varepsilon), \tag{20}$$

with $\phi_{f_i}(\varepsilon) = \left(\frac{1}{M_1} + \frac{1}{M_2}\right)\frac{(\psi(b) - \psi(a))^\alpha}{(1-\eta)\Gamma(\alpha+1)}\varepsilon$ and $\phi_{f_i}(0) = 0.$ Then the problem (6) is Ulam-Hyers stable.

## 5. Applications

We close this paper by the following example.

**Example 5.1:** *Consider the following problem:*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\frac{1}{2},\psi}(\varpi_i(t)g_i(t)) = f_i(t, \varpi_1(t), \varpi_2(t)), \quad t \in J := [0,1], \\ \varpi_i(0) = \varpi_i(1), \varpi_i'(0) = \varpi_i'(1), i = 1,2, \end{cases} \tag{21}$$

where $\alpha_1 = \alpha_2 = \frac{1}{2}$, $a = 0$, $b = 1$.

For $t \in J, \varpi_1, \varpi_2 \in \mathbb{R}$, set

$$f_1(t,\varpi_1,\varpi_2) = \frac{\sin(t)(1+\varpi_1+\varpi_2)}{105e^{-t+2}(2+|\varpi_1|)},$$
$$f_2(t,\varpi_1,\varpi_2) = \frac{\cos(t)(1+\varpi_1+\varpi_2)}{275e^{-t+4}(1+|\varpi_1|)},$$
$$g_1(t) = \frac{\sqrt{2}}{133}(t^2+5\sin(t)+3),$$

and

$$g_2(t) = \frac{1}{233}(3t^3+\cos(t)+2).$$

It is clear that the functions $f_i$ and $g_i$ are continuous and that $g_i(t) \neq 0$ for all $t \in [0,1]$. And, since

$$|g_1(t)| \geq \frac{9\sqrt{2}}{133}, \text{ and } |g_2(t)| \geq \frac{6}{233}, \text{ for all } t \in J.$$

then the hypothesis $(S_1)$ is verified with $M_1 = \frac{9\sqrt{2}}{133}$ and $M_2 = \frac{6}{233}$. For $t \in J$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}$, we have

$$|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \leq \frac{1}{105e}\left[|x_1-x_2| + |y_1-y_2|\right],$$

and

$$|f_2(t,x_1,y_1) - f_i(t,x_2,y_2)| \leq \frac{1}{278e^3}\left[|x_1-x_2| + |y_1-y_2|\right].$$

Thus, the hypothesis $(S_2)$ is satisfied with $k_1 = \frac{1}{105e}$ and $k_2 = \frac{1}{278e^3}$.

Further, we have

$$d_1 = \left(\frac{\sqrt{2}+\sin(1)\sqrt{2}}{133}\right)^2 + \left(\frac{30(\cos(1)+1)}{17689} - \frac{16(4+5\sin(1))}{17689}\right) \approx -0.00442798229593582,$$
$$d_2 = \left(\frac{2+\cos(1)}{233}\right)^2 + \left(\frac{3(11-\sin(1))}{54289} - \frac{10+2\cos(1)}{54289}\right) \approx 0.000476120728675036,$$
$$A_1 = \frac{\frac{3\sqrt{2}}{133}\left(\frac{3\sqrt{2}}{133} - \frac{4\sqrt{2}+\sin(1)\sqrt{2}}{133}\right)}{d_1} \approx 0.141061004750783,$$
$$A_2 = \frac{\frac{3}{233}\left(\frac{3}{233} - \frac{5+\cos(1)}{233}\right)}{d_2} \approx -0.294834109281285,$$
$$A_1^* = \frac{\left(\frac{30(\cos(1)+1)}{17689} - \frac{16(4+5\sin(1))}{17689}\right)}{d_1} \approx 1.08658658244547,$$
$$A_2^* = \frac{\left(\frac{3(11-\sin(1))}{54289} - \frac{10+2\cos(1)}{54289}\right)}{d_2} \approx 0.750344077448057,$$

$$B_1 = \frac{3\sqrt{2}}{133 d_1} \approx -7.20408346432856,$$

$$B_2 = \frac{3}{233 d_2} \approx 27.042587531354,$$

$$\Delta_1 = \left[ \frac{2}{\sqrt{\pi}} + \frac{2(|A_1| + |A_1^*|)}{\sqrt{\pi}} + \frac{|B_1|\left(\frac{3\sqrt{2}}{133} + \frac{4\sqrt{2}+\sin(1)\sqrt{2}}{133}\right)}{\sqrt{\pi}} \right] \approx 2.85244557564326,$$

$$\Delta_2 = \left[ \frac{2}{\sqrt{\pi}} + \frac{2(|A_2| + |A_2^*|)}{\sqrt{\pi}} + \frac{|B_2|\left(\frac{3}{233} + \frac{5+\cos(1)}{233}\right)}{\sqrt{\pi}} \right] \approx 2.86692357704738,$$

$$\left(\frac{k_1\Delta_1}{M_1} + \frac{k_2\Delta_2}{M_2}\right) - (\psi(b) - \psi(a))^{-\alpha} = \left(\frac{133 \times \Delta_1}{945 e\sqrt{2}} + \frac{233\Delta_2}{1668 e^3}\right) - 1 \approx -0.8756309100221 < 0.$$

As all the assumptions of Theorem 3.4 are verified, then we can deduce that problem (21) admits a unique solution defined on $[0,1]$.

**Remark 5.2:** *If we rather employ Theorem 3.6 to find and demonstrate the existence results for our problem (21), then we can simply take* $\tilde{k}_i = \varkappa_i = k_i$ *to satisfy the hypothesis* $(S_3)$. *Then, we can deduce our existence result based on Theorem 3.6.*

**Remark 5.3:** *Since the assumptions of Theorem 3.4 are met, then Theorem 4.2 implies that the problem (21) is Ulam-Hyers stable.*

## Declaration

*Acknowledgement*

J. Alzabut is thankful to Prince Sultan University and OSTİM Technical University for their endless support.

*Author contributions*

The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

*Funding*

Not available.

*Data availability statement*

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

*Conflict of interest*

It is declared that all authors have no competing interests.

*Ethical approval*

This article does not contain any studies with human participants or animals performed by any of the authors.

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