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Fractional differential equations with an approximate solution using the natural variation iteration method

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Abstract

The fractional natural variation iteration analysis is used in this study to analyze partial differential equations using the Caputo fractional operator (FNVIM). The FNVIM approach, which is a type of fractional Variation iteration with the natural transform, is used to generate the approximate analytical solutions. Illustrative scenarios show off the great accuracy and fast convergence of this innovative technique. The results show that the suggested approach may be used to solve nonlinear fractional differential equations. Furthermore, we show that FNVIM is more effective, transparent, and accurate in handling a large class of nonlinear equations utilizing the Caputo fractional operator, which makes it extremely valuable in physics and engineering.

Key words: Fractional differential equations; Variation iteration method, Natural transform; Caputo fractional operators.

Mathematics Subject Classification (2010): 35R11; 74H10

1. Introduction

Over the past century, significant advancements have taken place in both the theory and applications of fractional equations. In more and more domains of research, including the unification of absorption and wave propagation phenomena, mechanical systems, continuous-time random walks anomalous diffusive and subdiffusive systems, and others, these equations are being utilized to represent problems. The most significant benefit of employing fractional equations in these applications, as well

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as others, is their non-locality. It is well knowledge that the fractional order differential operator is a non-local operator, in contrast to the integer order differential operator, which is a local operator. Accordingly, a system's subsequent that has on all of its previous states in addition to its present state.

Many experts utilize these models frequently to develop natural issues that are well recognized for these occurrences, reduce the regulating design without compromising hereditary behaviors, and easily explain its complex structures [1–6].

The most approximate and empirical procedures, such as the fractional variation iteration technique, the fractional differential transformation methodology, and the technique for extending fractional series, using a fractional Laplace transform and the fractional Sumudu variation. Khan et al. [4] originally developed the natural transform technique (NTM), he gave specific applications and qualities of the natural transformation, which they designated to as the N-transform [7–29], which have been effectively applied.

The variation iteration approach has gained popularity and has been used to solve a variety of nonlinear issues. The capacity and adaptability of this approach to correctly and properly solve nonlinear equations is its key characteristic. Additionally, it has been recently highlighted that the variation iteration approach, together with other analytical techniques, is thought to be an efficient way for addressing a variety of non-linear problems without the use of general limiting assumptions.

We are attempting to present the FNVIM, a coupling method of the FVIM and NT, and utilize it to resolve nonlinear fractional PDEs. The following sections make up the remaining portion of this work: Some definitions for fractional calculus are provided in Section 2. Section 3 discusses the fundamental definition of natural transforms. The FNVIM with CFO analysis is carried out in section 4. Section 5 demonstrates FNVIM applications. The study's conclusion can be found in Section 6.

2. Preliminaries

This section goes over several fractional calculus principles and symbols that will come in handy during this inquiry [1, 2, 25].

Definition 2.1: Suppose $v(\zeta) \in R$,$\zeta > 0$, which is in the space $C_m, m \in R$ if there exists

$$\{\rho,(\rho > m), s.t. v(\zeta) = \zeta^{\rho} v_1(\zeta), where v_1(\zeta) \in C[0,8)\}$$

and $v(\zeta)$ is known as in the space C_m^n when $v^n \in C_m$, $m \in N$.

Definition 2.2: The fractional integral operator of order $\gamma \geq 0$ for Riemann Liouville of $v(\zeta) \in C_m, m \geq -1$ is given by the form

$$I^{\gamma} v(\zeta) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^{\zeta} (\zeta - \xi)^{\gamma-1} v(\xi) d\xi, & \gamma > 0, \zeta > 0 \\ I^0 v(\zeta) = v(\zeta) , & \gamma = 0 \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the recognizable Gamma function. The following are the characteristics of the operator I^{γ}: For $v \in Cm$, $m \geq -1$, $\gamma, \sigma \geq 0$, then

1. $I^{\gamma} I^{\sigma} v(\zeta) = I^{\gamma+\sigma} v(\zeta)$
2. $I^{\gamma} I^{\sigma} v(\zeta) = I^{\sigma} I^{\gamma} v(\zeta)$

Definition 2.3. In the understanding of Caputo, $v(\zeta)$'s fractional derivative is as follows:

$$D^{\gamma} v(\zeta) = I^{n-\gamma} D^n v(\zeta) = \frac{1}{\Gamma(n-\gamma)} \int_0^{\zeta} (\zeta - \xi)^{n-\gamma-1} v^{(n)}(\xi) d\xi \ , \tag{2.2}$$

such that $n-1 < \gamma \leq n, n \in N, \zeta > 0$ and $v \in C_{-1}^n$

Definition 2.4. The following formula gives the Mittag-Leffler function E_γ if it satisfies the following: For each $\gamma > 0$, then:

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{Z^\gamma}{\Gamma(n\gamma+1)} \tag{2.3}$$

3. Natural Transform definition

We present some context for the natural transform approach [25] in this section. Definition 3.1. The function $v(\zeta)$ for $\zeta \in R$ has a natural transform defined by

$$N[v(\zeta)] = R(\eta,\ell) = \int_{-\infty}^{\infty} e^{-\eta\zeta}\, v(\ell\zeta)d\zeta, \quad \eta,\ell \in (-\infty,\infty) \tag{3.1}$$

We denote that the Natural transform of the time function $v(\zeta)$ is $N[v(\zeta)]$, and the variables η and ℓ are the Natural transform elements. Furthermore, define $v(\zeta)H(\zeta)$ as on the axis of positive real, if $H(\zeta)$ is Heaviside function, and $\zeta \in (0,\infty)$. Consider

$$A = \{v(\zeta) : \exists M, t_1, t_2 > 0, with\ |v(\zeta)| \le Me^{\frac{|\zeta|}{tj}}, for\, \zeta \in (-1)^j \times [0,\infty), j \in Z^+\}$$

The natural transform, often known as the NT, is defined as follows:

$$N[v(\zeta)H(\zeta)] = N^+[v(\zeta)] = R^+\ (\eta,\ell) = \int_{0}^{\infty} e^{-\eta\zeta}\, v(\ell\zeta)d\zeta, \quad \eta,\ell \in (-\infty,\infty) \tag{3.2}$$

4. Fractional Natural Variation Iteration Method (FNVIM)

Suppose that the general fractional nonlinear PDEs with Caputo fractional operator

$$D_\zeta^\gamma v(\ell,\zeta) + Rv(\ell,\zeta) + Fv(\ell,\zeta) = ð(\ell,\zeta) \qquad m-1 < \gamma \le m \tag{4.1}$$

depending on the initial condition

$$v(\ell, 0) = ð(\ell), \tag{4.2}$$

s.t. the derivative of $v(\ell,\zeta)$ is $D_\zeta^\gamma v(\ell,\zeta)$ in Caputo sense, R is linear differential operator, F nonlinear differential operator, and the source phrase is $ð(\ell,\zeta)$. Now, by taking NT on both sides of (4.1)

$$N[D_\zeta^\gamma v(\ell,\zeta)] + N[Rv(\ell,\zeta) + Fv(\ell,\zeta)] = N[ð(\ell,\zeta)] \tag{4.3}$$

$$\frac{S^\gamma}{U^\gamma} v(\ell,\zeta) - \sum_{k=0}^{n} \frac{S^{\gamma-(k+1)}}{U^{\gamma-k}} v(\ell,0) + N[Rv(\ell,\zeta) + Fv(\ell,\zeta) - ð(\ell,\zeta)] = 0 \tag{4.4}$$

The iteration formula:

$$v_{n+1}(\ell,\zeta) = v_n + \lambda(\xi)\left[\frac{S^\gamma}{U^\gamma} v_n - \sum_{k=0}^{n} \frac{S^{\gamma-(k+1)}}{U^{\gamma-k}} v(\ell,0) + N[Rv_n(\ell,\zeta) + Fv_n(\ell,\zeta) - ð(\ell,\zeta)]\right] \tag{4.5}$$

Where $\lambda(\xi)$ Lagrange multiplier

Taking variation of (4.5)

$$\delta[v_{n+1}(\ell,\zeta)]=\delta[v_n]+\lambda(\xi)\delta\left[\frac{S^\gamma}{U^\gamma}v_n-\sum_{k=0}^{n}\frac{S^{\gamma-(k+1)}}{U^{\gamma-k}}v(\ell,0)+N[Rv_n(\ell,\zeta)+Fv_n(\ell,\zeta)-\eth(\ell,\zeta)]\right] \tag{4.6}$$

By using computation

$$\delta[v_{n+1}]=\delta[v_n]+\lambda(\xi)\frac{S^\gamma}{U^\gamma}\delta[v_n] \tag{4.7}$$

We impose the condition $\frac{\delta[v_{n+1}]}{\delta[v_n]}=0$

$$\delta[v_n]\left[1+\lambda(\xi)\frac{S^\gamma}{U^\gamma}\right]=0 \tag{4.8}$$

$$1+\lambda(\xi)\frac{S^\gamma}{U^\gamma}=0 \tag{4.9}$$

Hence, from (4.9) we get

$$\lambda(\xi)=-\frac{U^\gamma}{S^\gamma} \tag{4.10}$$

$$v_{n+1}(\ell,\zeta)=v_n-v_n+\frac{1}{S}v(\ell,0)-\frac{U^\gamma}{S^\gamma}N\ [Rv_n(\ell,\zeta)+Fv_n(\ell,\zeta)-\eth(\ell,\zeta)] \tag{4.11}$$

By applying Natural inverse to (4.11) after placing the value of $\lambda(\xi)$, its follow:

$$v_{n+1}(\ell,\zeta)=v(\ell,0)-N^{-1}\left[\frac{U^\gamma}{S^\gamma}N\ [Rv_n(\ell,\zeta)+Fv_n(\ell,\zeta)-\eth(\ell,\zeta)]\right] \tag{4.12}$$

The solution is provided by $v(\ell,\zeta)=\lim_{n\to\infty}v_n$

5. Applications

This section will demonstrate how to use the suggested approach for solving Cauchy reaction diffusion equations.

5.1 Example

Firstly, examine Cauchy reaction–diffusion equation which is indicated below

$$D_\zeta^\gamma v(\ell,\zeta)=v_{\ell\ell}(\ell,\zeta)-v(\ell,\zeta),\quad 0<\gamma\le 1 \tag{5.1}$$

with initial condition

$$v(\ell,0)=e^{-\ell}+\ell \tag{5.2}$$

Applying NT to each side of (5.1), and by using the differential property of FNVIM, we have

$$N[D_{\zeta}^{\gamma}v(\ell,\zeta)]=N[v_{\ell\ell}(\ell,\zeta)-v(\ell,\zeta)]$$
$$\frac{S^{\gamma}}{U^{\gamma}}v(\ell,\zeta)-\frac{S^{\gamma-1}}{U^{\gamma}}v(\ell,0)=N[v_{\ell\ell}(\ell,\zeta)-v(\ell,\zeta)] \tag{5.3}$$

$$v_{n+1}(\ell,\zeta)=v_n+\lambda(\xi)\left[\frac{S^{\gamma}}{U^{\gamma}}v(\ell,\zeta)-\frac{S^{\gamma-1}}{U^{\gamma}}v(\ell,0)-N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial\ell^2}-v_n(\ell,\zeta)\right]\right] \tag{5.4}$$

Applying (4.10) to (5.4), we get

$$v_{n+1}(\ell,\zeta)=\frac{1}{S}\,v(\ell,0)+\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial\ell^2}-v_n(\ell,\zeta)\right] \tag{5.5}$$

Taking the inverse Natural transform to (5.5), we obtain

$$v_{n+1}(\ell,\zeta)=v(\ell,0)+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial\ell^2}-v_n(\ell,\zeta)\right]\right] \tag{5.6}$$

Now, comparing (5.6), we get

$$\begin{aligned}
v_0 &= v(\ell,0)=e^{-\ell}+\ell\\
v_1 &= e^{-\ell}+\ell+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[e^{-\ell}-e^{-\ell}-\ell\right]\right]\\
&= e^{-\ell}+\ell-\ell\left[\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right]\\
v_2 &= e^{-\ell}+\ell+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[e^{-\ell}-e^{-\ell}-\ell+\ell\left(\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)\right]\right]\\
&= e^{-\ell}+\ell+N^{-1}\left[-\ell\frac{U^{\gamma}}{S^{\gamma+1}}+\ell\frac{U^{2\gamma}}{S^{2\gamma+1}}\right]\\
&= e^{-\ell}+\ell-\ell\left[\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right]+\ell\left[\frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\right]\\
&\ \vdots\\
v_n &= (-1)^n\frac{\ell\zeta^{n\gamma}}{\Gamma(n\gamma+1)}
\end{aligned} \tag{5.7}$$

Therefore, we have

$$\begin{aligned}
v(\ell,\zeta)&=\lim_{n\to\infty}\sum_{n=0}^{\infty}v_n\\
&=e^{-\ell}+\ell\left[1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}+\frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\right]\\
&=e^{-\ell}+\ell E_{\gamma}\,(-\zeta^{\gamma})
\end{aligned} \tag{5.8}$$

If $\gamma=1$, and by applying Taylor, the approximation yields

$$v(\ell,\zeta)=e^{-\ell}+\ell\left[1-\zeta+\frac{\zeta^2}{2!}-\cdots\right] \tag{5.9}$$

This actually represents the clear results to equation (5.1) in the scenario where $\gamma=1$. Because of this, the estimated solution eventually approaches the precise solution. Figures 1, 2 and 3 for various values of γ demonstrate the approximate solution of the estimated average and accurate values developed by FNVIM.

5.2 Example

We examine Cauchy reaction–diffusion equation which is indicated below

$$D_\zeta^\gamma v(\ell,\zeta)=v_{\ell\ell}(\ell,\zeta)-(1+4\ell^2)v(\ell,\zeta),\ 0<\gamma\le 1 \tag{5.10}$$

with initial condition

$$v(\ell,0)=e^{\ell^2} \tag{5.11}$$

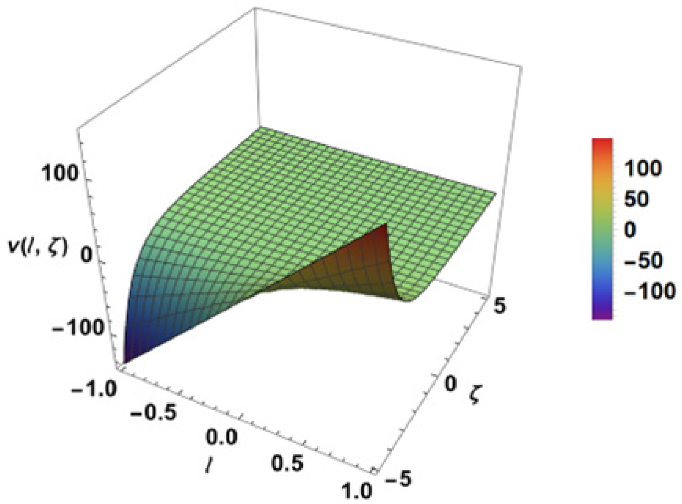

Figure 1: The surface graph of the approximate solution $v(l,\zeta)$ of (5.1) when $\gamma=1$.

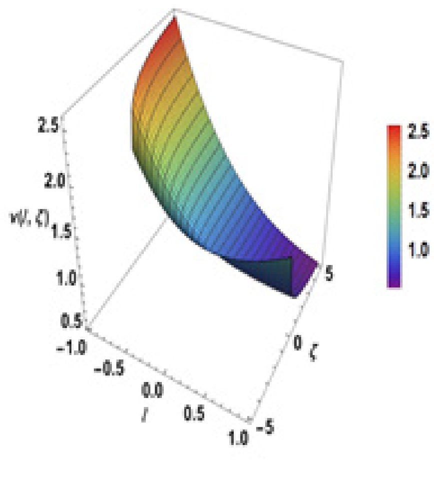

Figure 2: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.1) when $\gamma=0.4$

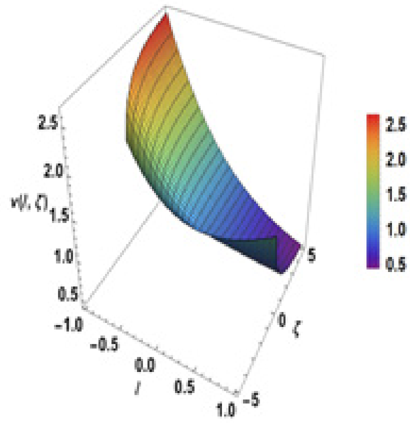

Figure 3: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.1) when $\gamma = 0.6$

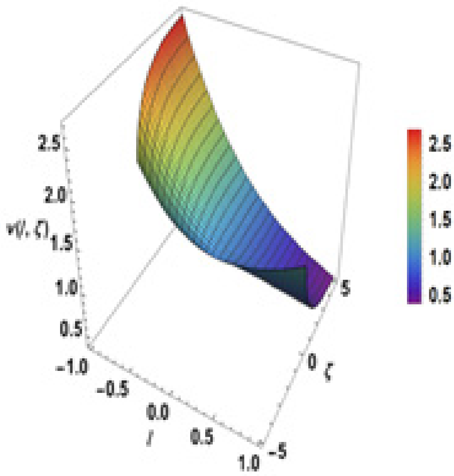

Figure 4: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.1) when $\gamma = 0.8$

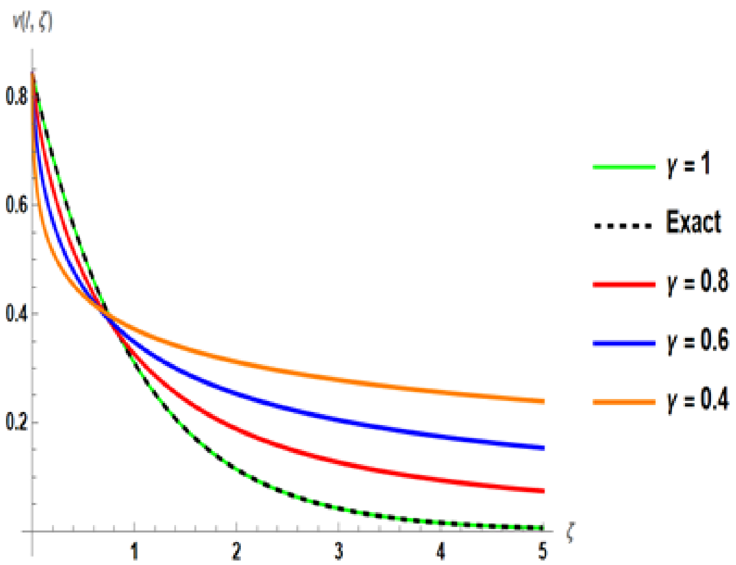

Figure 5: The approximate and exact solutions of $v(\ell,\zeta)$ of (5.1) for different values of γ.

Applying NT to each side of (5.10), and by using the differential property of FNVIM, we have

$$
\begin{aligned}
N[D_\zeta^\gamma v(\ell,\zeta)] &= N\left[\frac{\partial^2 v(\ell,\zeta)}{\partial \ell^2} - (1+4\ell^2)v(\ell,\zeta)\right] \\
\frac{S^\gamma}{U^\gamma}v(\ell,\zeta) - \frac{S^{\gamma-1}}{U^\gamma}v(\ell,0) &= N\left[\frac{\partial^2 v(\ell,\zeta)}{\partial \ell^2} - (1+4\ell^2)v(\ell,\zeta)\right]
\end{aligned}
\tag{5.12}
$$

$$
v_{n+1}(\ell,\zeta) = v_n + \lambda(\xi)\left[\frac{S^\gamma}{U^\gamma}v(\ell,\zeta) - \frac{S^{\gamma-1}}{U^\gamma}v(\ell,0) - N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial \ell^2} - (1+4\ell^2)v_n(\ell,\zeta)\right]\right] \tag{5.13}
$$

Applying (4.10) to (5.13), we get

$$
v_{n+1}(\ell,\zeta) = \frac{1}{S}\, v(\ell,0) + \frac{U^\gamma}{S^\gamma} N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial \ell^2} - (1+4\ell^2)\, v_n(\ell,\zeta)\right] \tag{5.14}
$$

Taking the inverse Natural transform to (5.14), then

$$
v_{n+1}(\ell,\zeta) = v(\ell,0) + N^{-1}\left[\frac{U^\gamma}{S^\gamma} N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial \ell^2} - (1+4\ell^2)v_n(\ell,\zeta)\right]\right] \tag{5.15}
$$

Now, comparing (5.15), we get

$$
\begin{aligned}
v_0 &= v(\ell,0) = e^{\ell^2} \\
v_1 &= e^{\ell^2} + N^{-1}\left[\frac{U^\gamma}{S^\gamma} N[4\ell^2 e^{\ell^2} + 2e^{\ell^2} - e^{\ell^2} - 4\ell^2 e^{\ell^2}]\right] \\
&= e^{\ell^2} + N^{-1}\left[\frac{U^\gamma}{S^{\gamma+1}} e^{\ell^2}\right] \\
&= e^{\ell^2} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)} e^{\ell^2} \\
v_2 &= e^{\ell^2} + N^{-1}\left[\frac{U^\gamma}{S^\gamma} N\left[4\ell^2 e^{\ell^2} + 2e^{\ell^2} + 2e^{\ell^2}\frac{\zeta^\gamma}{\Gamma(\gamma+1)} - e^{\ell^2} - e^{\ell^2}\frac{\zeta^\gamma}{\Gamma(\gamma+1)} - 4\ell^2 e^{\ell^2}\right]\right] \\
&= e^{\ell^2} + N^{-1}\left[\frac{U^\gamma}{S^{\gamma+1}} e^{\ell^2} + \frac{U^{2\gamma}}{S^{2\gamma+1}} e^{\ell^2}\right] \\
&= e^{\ell^2} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)} e^{\ell^2} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)} e^{\ell^2} \\
&\ \vdots \\
v_n(\ell,\zeta) &= \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)} e^{\ell^2}
\end{aligned}
\tag{5.16}
$$

Therefore, we have

$$
\begin{aligned}
v(\ell,\zeta) &= \lim_{n\to\infty}\sum_{n=0}^{\infty} v_n \\
v(\ell,\zeta) &= e^{\ell^2}\left[1 + \frac{\zeta^\gamma}{\Gamma(\gamma+1)} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{\zeta^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots + \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}\right] = e^{\ell^2} E_\gamma(\zeta^\gamma)
\end{aligned}
\tag{5.18}
$$

If $\gamma=1$, $v(\ell,\zeta)=e^{\ell^2+\zeta}$ is the accurate result of (5.10).

Figure 6 represents the clear results to equation (5.10) in the scenario where $\gamma=1$. Because of this, the estimated solution eventually approaches the precise solution. Figures 7, 8, 9 and 10 for various values of γ demonstrate the approximate solution of the estimated average and accurate values developed by FNVIM.

5.3 Example

We consider Cauchy reaction–diffusion equation which is indicated below

$$D_\zeta^\gamma v(\ell,\zeta)=v_{\ell\ell}(\ell,\zeta)-(\ell,\zeta)+v(\ell,\zeta)v_{\ell\ell}(\ell,\zeta)-v^2(\ell,\zeta)+v(\ell,\zeta), \quad 0<\gamma\le 1 \tag{5.19}$$

w.r.t initial condition

$$v\,(\ell,\,0)=e^{\ell} \tag{5.20}$$

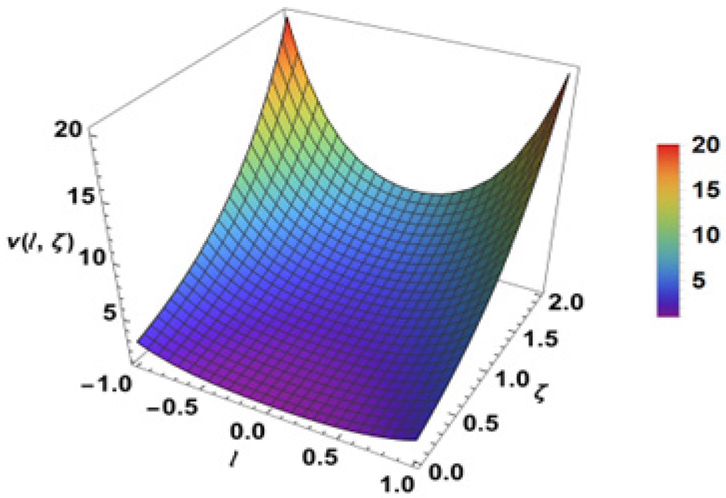

Figure 6: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.10) when $\gamma=1$.

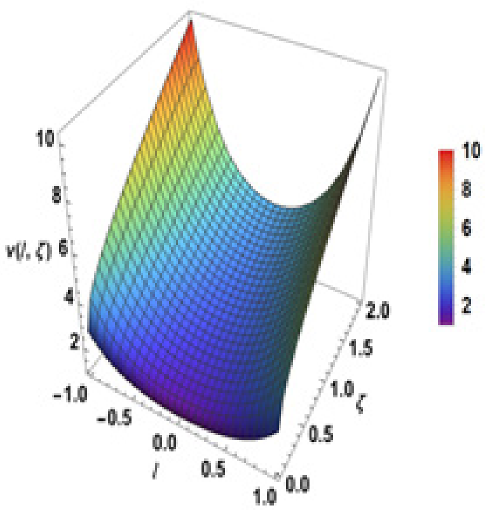

Figure 7: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.10) when $\gamma=0.4$

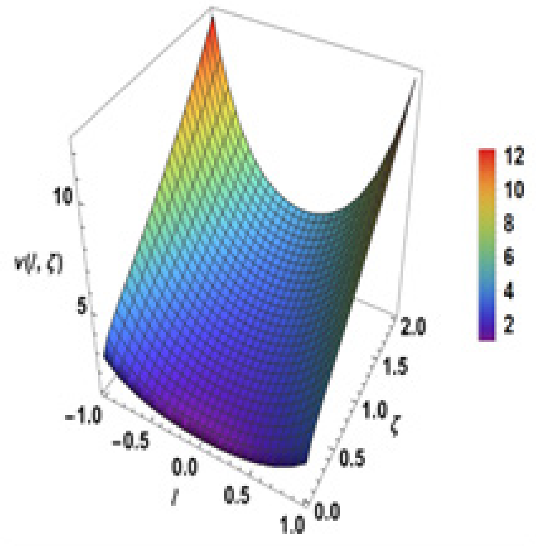

Figure 8: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.10) when $\gamma = 0.6$

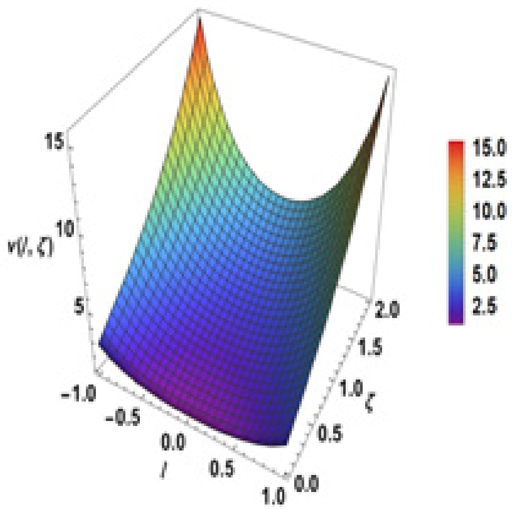

Figure 9: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.10) when $\gamma = 0.8$

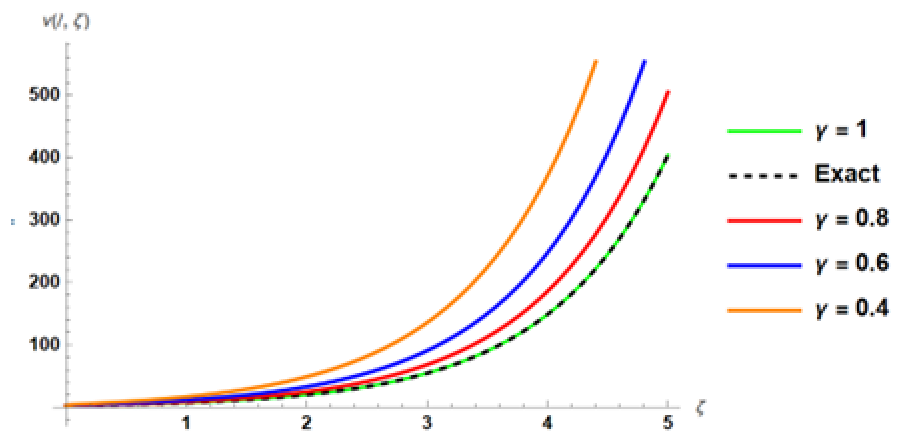

Figure 10: The approximate and exact solutions of $v(\ell,\zeta)$ of (5.10) for different values of γ.

Applying NT to each side of (5.19), and by using the differential property of FNVIM, we have

$$N[D_\zeta^\gamma v(\ell,\zeta)] = N[v_{\ell\ell} - v_\ell + v v_{\ell\ell} - v^2 + v]$$
$$\frac{S^\gamma}{U^\gamma} v(\ell,\zeta) - \frac{S^{\gamma-1}}{U^\gamma} v(\ell,0) = N[v_{\ell\ell} - v_\ell + v v_{\ell\ell} - v^2 + v] \tag{5.21}$$

$$v_{n+1}(\ell,\zeta) = v_n + \lambda(\xi)\left[\frac{S^\gamma}{U^\gamma} v_n - \frac{S^{\gamma-1}}{U^\gamma} v(\ell,0) - N\left[\frac{\partial^2 v_n}{\partial \ell^2} - \frac{\partial v_n}{\partial \ell} + v_n \frac{\partial^2 v_n}{\partial \ell^2} - (v_n)^2 + v_n\right]\right] \tag{5.22}$$

Applying

$$\lambda(\xi) = -\frac{U^\gamma}{S^\gamma}$$

to (5.22), we get

$$v_{n+1}(\ell,\zeta) = \frac{1}{S} v(\ell,0) + \frac{U^\gamma}{S^\gamma} N\left[\frac{\partial^2 v_n}{\partial \ell^2} - \frac{\partial v_n}{\partial \ell} + v_n \frac{\partial^2 v_n}{\partial \ell^2} - (v_n)^2 + v_n\right] \tag{5.23}$$

Taking the inverse Natural transform to (5.23), then

$$v_{n+1}(\ell,\zeta) = v(\ell,0) + N^{-1}\left[\frac{U^\gamma}{S^\gamma} N\left[\frac{\partial^2 v_n}{\partial \ell^2} - \frac{\partial v_n}{\partial \ell} + v_n \frac{\partial^2 v_n}{\partial \ell^2} - (v_n)^2 + v_n\right]\right] \tag{5.24}$$

Now, comparing (5.24), we get

$$\begin{aligned}
v_0 &= v(\ell,0) = e^\ell \\
v_1 &= e^\ell + N^{-1}\left[\frac{U^\gamma}{S^\gamma} N(e^\ell)\right] \\
&= e^\ell + N^{-1}\left[\frac{U^\gamma}{S^{\gamma+1}} e^\ell\right] \\
&= e^\ell + \frac{\zeta^\gamma}{\Gamma(\gamma+1)} e^\ell \\
v_2 &= e^\ell + N^{-1}\left[\frac{U^\gamma}{S^\gamma} N\left[e^\ell + \frac{\zeta^\gamma}{\Gamma(\gamma+1)} e^\ell\right]\right] \\
&= e^\ell + N^{-1}\left[\frac{U^\gamma}{S^{\gamma+1}} e^\ell + \frac{U^{2\gamma}}{S^{2\gamma+1}} e^\ell\right] \\
&= e^\ell + \frac{\zeta^\gamma}{\Gamma(\gamma+1)} e^\ell + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)} e^\ell \\
&\vdots \\
v_n(\ell,\zeta) &= \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)} e^\ell
\end{aligned} \tag{5.25}$$

Therefore, we have

$$\begin{aligned} v(\ell,\zeta) &= \lim_{n\to\infty}\sum_{n=0}^{\infty} v_n \\ &= e^{\ell}\left[1+\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}+\frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{\zeta^{3\gamma}}{\Gamma(3\gamma+1)}+\cdots+\frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}\right] \\ &= e^{\ell^2} E_{\gamma}(\zeta^{\gamma}) \end{aligned} \quad (5.26)$$

If $\gamma=1, v(\ell,\zeta)=e^{\ell+\zeta}$, which is the accurate result of (5.19).

Figure 11 represents the clear results to equation (5.19) in the scenario where $\gamma=1$. Because of this, the estimated solution eventually approaches the precise solution. Figures 12, 13, 14, and 15 for various values of γ demonstrate the approximate solution of the estimated average and accurate values developed by FNVIM.

6. Conclusion

This work has been using the FNVIM to successfully give an analytical estimation approach to the nonlinear of FPDEs. The FNVIM provides both exact solution considerations and results in the form of convergent series with easily estimated constituents.

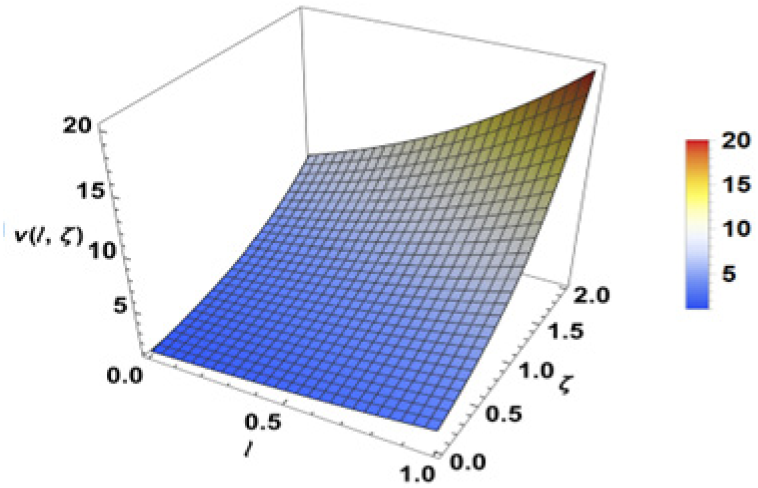

Figure 11: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.19) when $\gamma=1$.

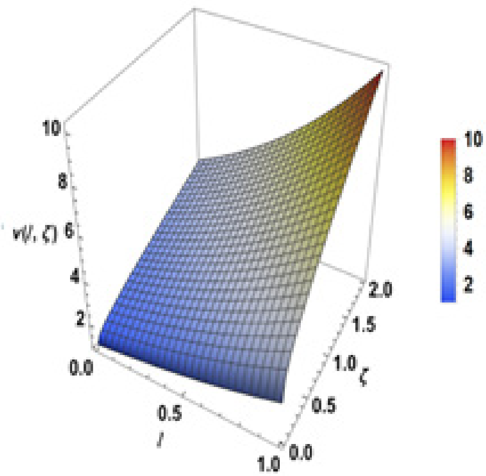

Figure 12: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.19) when $\gamma=0.4$.

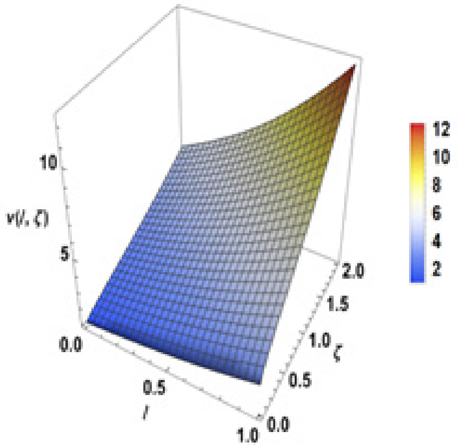

Figure 13: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.19) when $\gamma = 0.6$

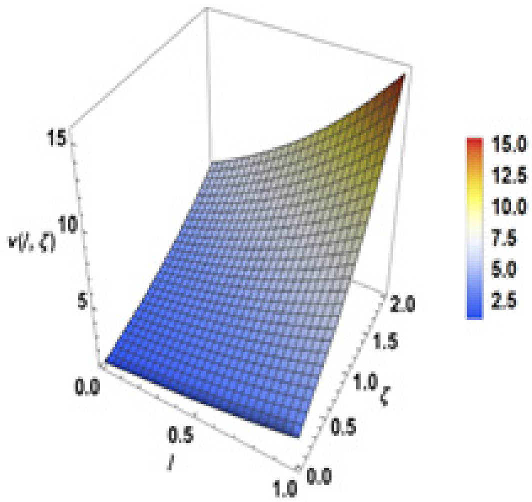

Figure 14: The surface graph of the approximate solution $v(\ell,\zeta)$ of (5.19) when $\gamma = 0.8$

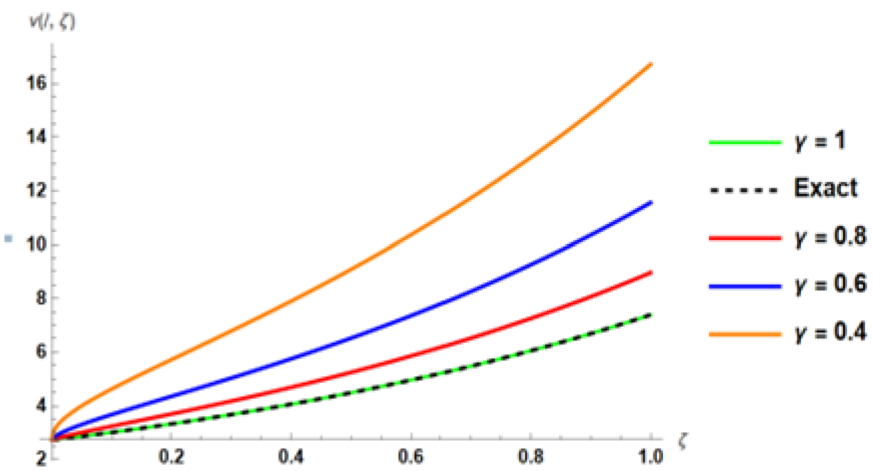

Figure 15: The approximate and exact solutions of $v(\ell,\zeta)$ of (5.19) for different values of γ.

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