Results in Nonlinear Analysis 6 (2023) No. 4, 7–12
https://doi.org/10.31838/rna/2023.06.04.002
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Parametric models of reliability functions

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Abstract

In this work, we study a family of parameters-dependent reliability functions. As a result, it can be represented as a parametric surface, where each point represents a different reliability function $R(t,\lambda)$ and can be parameterized by n-real valued variable $\lambda = (\lambda^1, \ldots, \lambda^n) \in I \subseteq R^n$, where I be an open subset. Then we prove that the reliability manifold N is a parametric model of one-dimensional.

1. Introduction

It was in 1975 that Efron first proposed the concept of "statistical curvature." This was the pioneering concept that established differential geometry's importance for statistical analysis. Later, using differential geometry techniques, Amari created an elegant representation of Fisher's theory of information loss. In this study, we introduce the concept of reliability models, which refers to a particular type of structure that can be attached to a certain kind of family of reliability functions. Parametric reliability models are the focus of this paper. Each member of a family is identified by its own set of parameters. When the parameters provide a smooth description of the family of functions, we can think of this as a multidimensional surface. In the study of the properties that do not depend on the choice of model coordinates. we need log-likelihood function $l(R(t,\lambda))$ which is a useful mapping defined by:

$$l(R(t,\lambda)) = lnl(R(t,\lambda))$$

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Received July 30, 2023; Accepted August 19, 2023; Online September 9, 2023

and its derivatives are

$$\partial^j l(R(t,\lambda)) = \frac{\partial \ln R(t,\lambda)}{\partial \lambda^j} = \gamma_j l(R(t,\lambda)), 1 \le j \le n$$

This paper contains some basic definitions and examples of parametric model and the parameterization of parametric model. When $R(t,\lambda)$ is sufficiently smooth in $\lambda = (\lambda^1,\ldots,\lambda^n)$ it is natural to introduce in parametric model N, the structure of an n-dimensional manifold, where λ plays the role of a coordinate system. We need smooth structure when we want to do calculus on manifolds. Hence we will define smooth manifold, which is ordinary manifold with an additional structure.

2. Mathematical Background

Definition 2.1: [1] The reliability of a component (or a system) is

$$\begin{aligned} R(t) &= P(T > t) = 1 - P(T \le t) \\ &= 1 - F(t) \end{aligned}$$

Therefore, we can write the reliability function in terms of pdf as follows:

$$R(t) = 1 - \int_0^t f(x)dx = \int_t^\infty f(x)dx \ldots \tag{1}$$

Conversely, we can write the pdf in terms of R(t) as follows:

$$f(t) = -\frac{d(R(t))}{dt}$$

Where F (t) $= P(T < t) = \int_0^t f(x)dx$ is the failure distribution function or Failure Function.

The reliability R(t) at time t, has the following properties:

$$R(t) \in [0,1]$$

1. Since $F(0) = 0,\ F(\infty) = 1,$ therefore

   $R(0) = 1$ and $R(0) = 0$ this implies that $0 \le R(t) \le 1$

2. R(t) is a decreasing function of time t.

The probability of failure of a given system in a particular time interval $[t_1, t_2]$ can be written in terms of the reliability function as:

$$\begin{aligned} \int_{t_1}^{t_2} f(x)dx &= \int_{t_1}^{\infty} f(x)dx - \int_{t_2}^{\infty} f(x)dx \\ &= R(t_1) - R(t_2) \end{aligned}$$

Using the exponential distribution, the pdf can be written in the form:

$$f(t) = \lambda e^{-\lambda t}$$

here λ is a parameter of the exponential distribution.

Therefore, the reliability function of the exponential distribution can be derived based on equation (1) as follows:

$$R(t) = 1 - \int_0^t \lambda e^{-\lambda x} dx$$

So, the reliability function becomes as follows:

$$R(t) = e^{-\lambda t}$$

here λ is a failure rate.

Definition 2.2: [2] *The functions* $\{\gamma_n(t,\lambda)\}$ *are* ***linearly independent*** *if* $W(t,\gamma_1(t,\lambda),\ldots,\gamma_n(t,\lambda)) \neq 0$.

In else they are called linearly dependent.

Definition 2.3: [3] A function $\psi : I \subseteq R^n \to \psi(I) \subseteq R^n$ is called **diffeomorphism function** if and only if it is one-to-one, onto, differentiable and has a differentiable inverse.

Definition 2.4: [5] Let $U \subset R^n$ and $M \subseteq R^m$ be open set. A function $f : U \to M$ is called a **smooth function** (or C^∞ – function) if and only if it is infinity differentiable, i.e. all it's partial derivatives.

$$\partial^\ell f = \frac{\partial^{\ell_1+\ell_2+\ldots+\ell_n}}{(\partial x^1)^{\ell_1}(\partial x^2)^{\ell_2}\ldots(\partial x^n)^{\ell_n}} f$$

exist and are continuous for all positive integer ℓ.

3. Parametric Models

Definition 3.1: [1] The family $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ is said to be **parametric model** if there exists a mapping

$g : I \to N$ which is satisfy the following conditions:

1. g is one-to-one (i.e. if $g(\lambda^1) = g(\lambda^2)$ then $\lambda^1 = \lambda^2$
2. The Wronskian determinate

$$\begin{vmatrix} \gamma_1(t,\lambda)\ldots. & \gamma_n(t,\lambda) \\ \gamma_1'(t,\lambda)\ldots. & \gamma_n'(t,\lambda) \\ \gamma_1^{(n-1)}(t,\lambda)\ldots. & \gamma_1^{(n-1)}(t,\lambda) \end{vmatrix} \neq 0 \ ,\forall\lambda$$

We can write it by $W(t,\gamma_1(t,\lambda),\ldots,\gamma_n(t,\lambda)) \neq 0$

where $\gamma_j(t,\lambda) = \dfrac{\partial R(t,\lambda)}{\partial \lambda^j}$ and $\gamma^{(k)}(t,\lambda) = \dfrac{\partial^{(k)}\gamma_j(t,\lambda)}{\partial t}$

Noted that the condition (2) states the regularity by the parametric model.

Now, some definitions and theorems are produced.

Definition 3.2: Let $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ be the image of one-to-one mapping $g : I \to N$ such that $g(\lambda) = R(t,\lambda)$, and its inverse function $\varphi : N \to I \subseteq R^n$ such that $\varphi(R(t,\lambda)) = \lambda$.

Then the mapping φ assigns a parameter λ with each reliability $R(t,\lambda)$ called a **coordinate system for parametric model** we can denoted it by (I,φ)

Definition 3.3: Let I and $\psi(I)$ are two open subsets in R^n if $g : I \to N$ such that $g(\lambda) = R(t,\lambda)$ and $\psi : I \to \psi(I)$ be diffeomorphism function. Then the function $\psi_0 g : N \to \psi(I)$ is **another coordinate** system and the parametric model can be written as:

$$N = \{R(t,\psi^{-1}(\rho)) : \rho \in \psi(I)\}$$

Theorem 3.4: The condition of regularity (2) of the parametric model $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ holds if and only if for any $\lambda \in I$, the set $\{\{\partial_1 l(R(t,\lambda), \partial_2 l(R(t,\lambda), \ldots, \partial_n lR(t,\lambda)\}$ is a system of n linearly independent functions of t.

Proof:
Since $N = \{R(t,\lambda), t \in (0,\infty), \lambda \in R^n > 0\}$ be a parametric model, then $\{\gamma_n(t,\lambda)\}$ are independent functions.

Now, $\dfrac{\partial l(R(t,\lambda))}{\partial \lambda^j} = \dfrac{1}{R(t,\lambda)} \dfrac{\partial R(t,\lambda)}{\partial \lambda^j} = \dfrac{1}{R(t,\lambda)} \gamma_j(t,\lambda)$

Since the output is equal to the input multiplied by a constant, then we get the system $\left\{\dfrac{\partial l(R(t,\lambda))}{\partial \lambda^j}\right\}$ and $\gamma_j(t,\lambda)\}$ are proportional.

Theorem 3.5: Let $N = \{e^{-\lambda t} : \lambda \in I\}$ be a family of the reliability functions with exponential lifetime distributions of one parameter.Then N is a parametric model of one-dimensional.

Proof:
Let I be one-dimensional parameter space $I = (0,\infty)$, which is an open interval in R. The reliability with exponential lifetime distribution with one parameter λ is given by the formula $R(t,\lambda) = e^{-\lambda t}$.

Now the mapping $g : I \to N$ is one-to-one for all $t \in (0,\infty)$, if $\lambda_1, \lambda_2 \in I$ such that $\lambda_1 = \lambda_2$, then $R(t,\lambda) = e^{-\lambda_1 t} = e^{-\lambda_2 t} = R(t,\lambda_2)$ and

$$\frac{\partial R(t,\lambda)}{\partial \lambda} = -t\,e^{-\lambda t} \neq 0, \forall t \in (o,\infty).$$

So, $N = \{e^{-\lambda t} : \lambda \in I\}$ is a parametric model dimension $n = 1$.

The diffeomorphism $\psi : (0,\infty) \to (0,\infty)$ such that $\psi(\lambda) = \dfrac{1}{\lambda}$, induces the new parametrization $R(t,\rho) = e^{\frac{-t}{\rho}}$

So, the family $N = \{e^{-\lambda t} : \lambda \in I\}$ is one-dimensional and hence can be considered as a curve in the infinite dimensional space of functions $\{R(t,\lambda)\}$.

Theorem 3.6: The family of Reliability functions with exponential lifetime distributions of two parameters is a two-dimension surface parameterized by $(0,\infty) \times (0,\infty)$.

Proof:
Let $I = (0,\infty) \times (0,\infty)$ be two-dimensional parameter space.

The reliability with exponential lifetime distribution with two parameters given by

$$R(t,\lambda,\theta) = e^{-\lambda(t-\theta)}, \forall t \in (0,\infty), (\lambda,\theta) \in I = (0,\infty) \times (0,\infty)$$

where λ refer to the scale parameter and θ refer to the location parameter.

Let $g : I \to N$ and log-liklihood function is

$$\ell R(t,\lambda,\theta) = \ln R(t,\lambda,\theta) = \ln(e^{-\lambda(t-\theta)}) = -\lambda(t-\theta)$$

Let $\ell R(t,\lambda,\theta) = \ell R(t,\lambda',\theta')$,

then $-\lambda(t-\theta) = -\lambda'(t-\theta')$, which is can be written as

$$\lambda t - \lambda\theta = \lambda' t - \lambda'\theta'$$

Hence $\lambda = \lambda'$ and $\theta = \theta'$. Therefore, g is one-to-one mapping.

Now, Let $\gamma_1(t) = \partial_\lambda \ell(R(t,\lambda,\theta)) = \theta - t$

$$\gamma_2(t) = \partial_\theta \ell(R(t,\lambda,\theta)) = \lambda$$

$$\begin{vmatrix} \gamma_1(t) & \gamma_2(t) \\ \gamma_1'(t,\lambda) & \gamma_2'(t) \end{vmatrix} = \begin{vmatrix} \theta - t & \lambda \\ 1 & 0 \end{vmatrix} = -\lambda \neq 0$$

So the family $N = \{e^{-\lambda(t-\theta)}\}$ be two-dimension surface parameterized by I. Since $\{\gamma_j(t,\lambda)\}$ are independent function of t. Then $\left\{\dfrac{\partial l(R(t,\lambda))}{\partial \lambda^j}\right\}$ are independent functions of t.

3. Reliability Manifold

Definition 4.1: Let $N = \{R(t,\lambda)\}$ be a parametric model of dimension n. Then **an n-dimensional manifold N** is a topological space which is Hausdorff, second countable and locally homeomorphic to an n-dimensional Euclidean space R^n.

Definition 3.2: Let $U\ N$ be an open set and $\varphi : U \to \varphi(U) \subset R^n$ is a homeomorphism of the open set U in N onto open subset $\varphi(U)$ of R^n. The pair (U,φ) is called a **chart** or (coordinate system). Chart (U,φ) gives us coordinate which helps us to calculate on the manifold, and in order to calculate on the whole manifold, we need a lot of charts such that all charts cover the whole manifold. Hence such a collection called an atlas.

Definition 3.3: A collection of charts $(U_i,\varphi_i)_{i\in I}$ is called an **atlas** if

$$\bigcup_{i\in I} U_i = N.$$

Definition 3.4: A **differentiable (smooth) manifold** N is an n-dimensional (topological) manifold endowed with maximal C^k −atlas (C^k −smoothly structure)

Theorem 3.5: Let $(\lambda,\theta) \in I = (0,\infty)\times(0,\infty)$ be two-dimensional parameter space of reliability function $e^{-\lambda(t-\theta)}$. Then The set

$$N = \{R(t,\lambda,\theta) : R(t,\lambda,\theta) = e^{-\lambda(t-\theta)}, t \in (0,\infty), (\lambda,\theta) \in I\} \text{ is smooth manifold.}$$

Proof:
Since reliability function $e^{-\lambda(t-\theta)}$ be continuously differentiable function. Then, $e^{-\lambda(t-\theta)} \in C^\infty$, where

$$C^\infty = f{:}f \text{ has continuous partial derivatives of all orders}\}.$$

So $e^{-\lambda(t-\theta)}$ be a smooth function. Let $N = \{e^{-\lambda(t-\theta)} : t \in (0,\infty), (\lambda,\theta) \in I\}$

Then N is a two- dimensional manifold that is diffeomorphic to the upper half-plane in R^2. The entire manifold is covered by only one atlas consists of only one chart φ from the open subset U in N onto open subset $\varphi(U)$ of R^n (i.e. $\varphi : U \subseteq N \to \varphi(U) \subseteq R^n$). There are other possible coordinate systems in particular, consider.

$$m_1 = E(T) = \frac{1}{\lambda} e^{\lambda\theta}$$

$$m_2 = E(T^2) = \frac{2}{\lambda}$$

where E denotes the expectation of a random variable. Then (m_1, m_2) is a coordinate systems. An atlas (N,φ) can be extended to maximal C^∞ −atlas which is smooth structure. So we get N be smooth manifold of the reliability function $R(t,\lambda,\theta)$.

5. Conclusion

Several definitions of smooth manifolds with related theorems are needed in order to extract a reliability manifold N and prove that it is a parametric model in 2-dimension.

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