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Some fixed point results in metric spaces equipped with a graph and their applications

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Abstract

In this paper, we define the notion of $(F-H)_G$-contraction and utilize the same to obtain fixed point results in the setting of metric spaces equipped with a graph. Our contraction generalizes many known contractions in literature. As an application, we have proved the existence and uniqueness of solution of Volterra type integral equation. Finally, some suitable examples are given to validate our claim.

K*eywords and phrases*. Connected graph; F-contraction; G-contraction; $(F-H)$ contraction; $(F-H)_G$-contraction.

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1. Introduction

A prestigious metric fixed point theory gets attention with the Banach contraction principle [1]. Over the years due to its wide applications in different areas such as chemistry, physics, computer science and various branches of mathematics, it has been generalized in many ways (see [2, 3, 4, 5]). For some recent generalizations and applications of Banach contraction principle see [6, 7, 8, 9, 10, 11]. Wardowski [12] introduced the notion of an F-contraction as a generalization of Banach contraction as follows:

Definition 1.1. ([12]). Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

(F_1) F is strictly increasing,

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(F_2) for each sequence $\{\alpha_n\}$ in $(0, \infty)$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

(F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Then a self map S on a metric space X is called an F-contraction if there exists $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(Sa, Sb)) \leq F(d(a, b)), \tag{1.1}$$

for all a, b in X with $Sa \neq Sb$.

Note: Since $\tau > 0$, by using equation (1.1) we get

$$F(d(Sa, Sb)) < F(d(a, b))$$

for all a, b in X with $Sa \neq Sb$.

Using (F_1),

$$d(Sa, Sb) < d(a, b)$$

for all a, b in X with $Sa \neq Sb$. Thus, every F-contraction is a continuous mapping. Further this concept had been studied by many researchers in different aspects (see [13, 14, 15, 16, 17, 18, 19, 20, 31]). One of the most interesting generalization of Banach's result was given by Jleli *et al.* [21] by introducing a family $\mathcal{H}$ of functions $H : [0, \infty)^3 \to [0, \infty)$ with the following properties:

(H_1) $\max\{u, v\} \leq H(u, v, w)$, for all $u, v, w \in [0, \infty)$,
(H_2) $H(0, 0, 0) = 0$,
(H_3) H is continuous.

Example 1.1. The following functions belongs to the family $\mathcal{H}$, for all $u, v, w \in [0, \infty)$:

- $H(u, v, w) = u + v + w$,
- $H(u, v, w) = \max\{u, v\} + w$,
- $H(u, v, w) = u + v + uv + w$.

Definition 1.2. [21] Let (X, d) be a metric space, $H \in \mathcal{H}$ and $\varphi : X \to [0, \infty)$ be given functions. Then a self map S on X is called an $(H - \varphi)$-contraction with respect to the metric d if and only if

$$H(d(Sa, Sb), \varphi(Sa), \varphi(Sb)) \leq kH(d(a, b), \varphi(a), \varphi(b)),$$

for all a, b in X and for some constant $k \in (0, 1)$.

Definition 1.3. [21] Set $Fix\ S = \{x \in X : Sx = x\}$ and for a given function $\varphi : X \to [0, \infty)$, $Z_\varphi = \{x \in X : \varphi(x) = 0\}$.

- An element $p \in X$ is said to be a φ-fixed point of S if and only if $p \in Fix\ S \cap Z_\varphi$.
- The operator S is said to be φ-Picard operator if and only if
  (i) $Fix\ S \cap Z_\varphi = \{p\}$,
  (ii) $S^n x \to p$ as $n \to \infty$, for each $x \in X$.
- The operator S is said to be weakly φ-Picard operator if and only if
  (i) $Fix\ S \cap Z_\varphi \neq \varnothing$,
  (ii) the sequence $\{S^n x\}$ converges to φ-fixed point of S, for each $x \in X$.

Very recently, Vetro [20] generalized the domain of F-contraction with the help of $(H - \varphi)$-contraction to find the fixed point of the respective contraction mapping and also proved that under particular assumptions these fixed points are zeros of some given function φ.

Definition 1.4. [20] Let (X, d) be a metric space, then a self map S on X is called an $(F-H)$-contraction if there exist a function $F \in \mathcal{F}$, a function $H \in \mathcal{H}$, a real number $\tau > 0$ and a function $\varphi : X \to [0, \infty)$ such that

$$\tau + F(H(d(Sa, Sb), \varphi(Sa), \varphi(Sb))) \leq F(H(d(a, b), \varphi(a), \varphi(b))) \tag{1.2}$$

for all a, b in X with $H(d(Sa, Sb), \varphi(Sa), \varphi(Sb)) > 0$.

Jachymski [22] united the notion of Banach contraction with graph theory and introduced the concept of G-contraction on a metric space equipped with a graph G. He explained that connectivity of a graph is necessary and sufficient condition for any class of generalization of contraction of Banach [1] and it became a source to study the graphical fixed point theory (see, [23, 24, 25, 26, 27, 28, 29, 30, 31]). Batra and Vashishtha [24] discussed the combination of F-contraction and G-contraction to introduce the concept of F_G-contraction.

Definition 1.5. [24] Let (X, d) be a metric space equipped with a graph $G = (V_G, E_G)$, then a self map S on X is said to be an F_G-contraction if it satisfies the following properties:

- (c_1) $(a, b) \in E_G \Rightarrow (Sa, Sb) \in E_G$, for all $x, y \in X$.
- (c_2) There exists a number $\tau > 0$ such that $\tau + F(d(Sa, Sb)) \leq F(d(a, b))$, for all $a, b \in X$ with $(a, b) \in E_G$ and $Sa \neq Sb$.

Inspired by the work mentioned in above literature, we adopt the idea of an $(F-H)$-contraction [20] to introduced the notion of $(\mathrm{F}-\mathrm{H})_G$-contraction which is a generalization of F-contraction [12], F-weak contraction [32], F_G-contraction [24], $(H-\varphi)$-contraction [21], $(F-H)$-contraction [20], rational type $(F-H)$-contraction and almost $(\mathrm{F}-H)$-contraction [33]. As applications, we have investigated the existence and uniqueness of a solution of a volterra type integral equation. At last, some numerical examples are provided to exhibit the authenticity of our results. The remaining part of this paper is organized as follows: Section 2 deals with some preliminary definitions and properties in a metric space equipped with a graph. In section 3 we have introduced the notion of $(F-H)_G$-contraction and proved fixed point results for such contraction mappings in a metric space equipped with a graph. Section 4 deals with applications of our main results in proving the existence and uniqueness of solution of a volterra type integral equation. Finally, in section 5 we have provided numerical examples in support of our main theorems proved in sections 3 and 4.

2. Preliminaries

We start this section by defining some notations, terminologies and basic concept about graph theory that will be needed in our main results.

Throughout the entire article,

- $\mathbb{N}$ denote the set of all natural numbers.
- (X_G, d_G) denote the metric space equipped with a directed graph $G = (V_G, E_G)$ with $V_G = X_G$ and $\Delta \subseteq E_G$, where $\Delta = \{(a, a) : a \in X_G\}$. Assume G contains no parallel edges. The graph G is said to be weighted graph if each edge has assigned a numerical value, that is distance between two vertices participating in construction of an edge. If direction of edges of G is reversed, then it is called conversion of G and is denoted by G^{-1}. An undirected graph can be obtained from G by ignoring the direction of the edges of G. It is convenient to treat $\tilde{G}$ as a directed graph with its symmetric edges, i.e., $E_{\tilde{G}} = E_G \cup E_{G^{-1}}$. A path of length $n(n \in \mathbb{N} \cup 0))$ between any two vertices a to b of G is a finite sequence $\{a_i\}_{i=0}^{n}$ of vertices with $a_0 = a$, $a_n = b$ and $(a_{i-1}, a_i) \in E_G$, for all $i = 1, 2, \cdots, n$. If there is a path between every two vertices, then G is called connected graph and if $\tilde{G}$ is connected, then G is called weakly connected graph. A path with a as a initial vertex in X_G, possesses a subgraph G_a which contains all edges and vertices and

is called component of G containing a. Also, G_a is connected for all $a \in G$ and $V_{G_a} = [a]_G$, where $[a]_G$ is the equivalence class of the relation R defined on the vertex set V_G as:

$$bRc \text{ if } G \text{ possesses } a \text{ path from } b \text{ to } c.$$

- **Property(*):** [22] Let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in X. If $x_n \to x$ and $(x_n, x_{n+1}) \in E_G$, then there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ satisfying $(x_{k_n}, x) \in E_G$, for all $n \in \mathbb{N}$.
- Fix S denotes the collection of all fixed points of a self map S on X_G.

Definition 2.1. [22] Let (X_G, d_G) be a metric space. Then a self map S on X_G is said to be

- orbitally continuous if for any sequence $\{k_n\}_{n\in\mathbb{N}}$ of positive integers and for all a, b in X_G,

$$\lim_{n\to\infty} d_G(S^{k_n}a, b) = 0 \text{ implies } \lim_{n\to\infty} d_G(S(S^{k_n}a), Sb) = 0.$$

- G-continuous if for any sequence $\{a_n\}_{n\in\mathbb{N}}$ and $a \in X_G$,

$$\lim_{n\to\infty} d_G(a_n, a) = 0, (a_n, a_{n+1}) \in E_G \text{ implies } \lim_{n\to\infty} d_G(Sa_n, Sa) = 0,$$

for $n \in \mathbb{N}$
- orbitally G-continuous if for any sequence $(k_n)_{n\in\mathbb{N}}$ and all a, b in X_G

$$\lim_{n\to\infty} d_G(S^{k_n}a, b) = 0, \ (S^{k_n}a, S^{k_n+1}a) \in E_G \text{ implies } \lim_{n\to\infty} d_G(S(S^{k_n}a), Sb) = 0.$$

Lemma 2.1. [20] Let (X, d) be a metric space and S be an $F - H$-contraction with respect to the functions $F \in \mathcal{F}$, $H \in \mathcal{H}$, $\varphi : X \to [0, \infty)$ and the real number $\tau > 0$. If $\{x_n\}$ is a Picard sequence starting at x_0, then

$$\lim_{n\to\infty} H(d(a_n, a_{n-1}), \varphi(a_{n-1}), \varphi(a_n)) = 0, \tag{2.1}$$

and hence

$$\lim_{n\to\infty} d(a_n, a_{n-1}) = 0 \text{ and } \lim_{n\to\infty} \varphi(a_n) = 0. \tag{2.2}$$

3. Main Results

First, we introduce the almost $(F - H)_G$-contraction as follows:

Definition 3.1. Let (X_G, d_G) be a metric space, then a self map S on X_G is said to be an almost $(F - H)_G$-contraction if, there exist functions $F \in \mathcal{F}$, $H \in \mathcal{H}$, a real number $\tau > 0$ and $\varphi : X_G \to [0, \infty)$ such that, for all $a, b \in X_G$

$$(a, b) \in E_G \text{ implies } (Sa, Sb) \in E_G \tag{3.1}$$

$$\tau + F(H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb))) \le F(H(M_G(a, b), \varphi(a), \varphi(b))) + Ld_G(b, Sa), \tag{3.2}$$

for all $a, b \in X_G$ with $(a, b) \in E_G$ and $H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb)) > 0$,

where $M_G(a,b) = \max\left\{d_G(a,b), \dfrac{d_G(a,Sa)[1 + d_G(b,Sb)]}{1 + d_G(Sa,Sb)}\right\}$.

Remark 3.1. If a self map S on X_G satisfies equations (3.1) and (3.2) for a directed graph G, then it also holds for $\tilde{G}$ and G^{-1}. Thus, every almost $(F - H)_G$-contraction is an almost $(F - H)_{\tilde{G}}$-contraction and an $(F - H)_{G^{-1}}$-contraction both.

Lemma 3.1. Let (X_G, d_G) be a metric space and S be an almost $(F - H)_G$-contraction on X_G. Then for $a \in X_G$ and $b \in [a]_{\tilde{G}}$,

$$\lim_{r\to\infty} H(d_G(S^n a, S^n b), \varphi(S^n a), \varphi(S^n b)) = 0, \tag{3.3}$$

and hence

$$\lim_{r\to\infty} d_G(S^n a, S^n b) = 0 \text{ and } \lim_{n\to\infty} \varphi(S^n b) = 0. \tag{3.4}$$

Proof. Suppose $a \in X_G$ and $b \in [a]_G$. This implies that there exists a path $\{a_i\}_{i=0}^{N}$ with $N + 1$ vertices in such a way that $a = a_0$, $a_1 = Sa_0$, …, $a_N = Sa_{N-1} = b$ and $(a_{i-1}, a_i) \in E_G$. By Remark 3.1, S is also a generalized $(F - H)_{\tilde{G}}$-contraction, it follows that $(a_{i-1}, a_i) \in E_{\tilde{G}}$, for all $i = 1, 2, \cdots, N$. Consider the case, $S^k a_{i-1} = S^k a_i$, for some $i = 1, 2, \cdots, N$ and $k \in \mathbb{N}$, then $S^n a_{i-1} = S^n a_i$, for all $n \geq k$. Now, we claim that $H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)) = 0$.

Suppose, if possible, $H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)) > 0$. Then using (3.2), we get

$$\begin{aligned}
&\tau + F(H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))) = \tau + F(H(0, \varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_{i-1})))\\
&\leq F(H(M_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \varphi(S^{n-1} a_{i-1}), \varphi(S^n a_i))) + L d_G(S^{n-1} a_i, S^n a_{i-1})\\
&\leq F\Big(H\Big(\max\left\{d_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \frac{d_G(S^{n-1} a_{i-1}, S^n a_{i-1})[1 + d_G(S^{n-1} a_i, S^n a_i)]}{1 + d_G(S^n a_{i-1}, S^n a_i)}\right\},\\
&\varphi(S^{n-1} a_{i-1}), \varphi(S^n a_i))) + L d_G(S^{n-1} a_i, S^{n-1} S a_{i-1})\\
&\leq F\Big(H(\max\left\{d_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \frac{d_G(S^{n-1} a_{i-1}, S^{n-1} S a_{i-1})[1 + d_G(S^{n-1} a_i, S^n a_i)]}{1 + d_G(S^n a_{i-1}, S^n a_i)}\right\},\\
&\varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_i))) + L d_G(S^{n-1} a_i, S^{n-1} a_i)\\
&\leq F\Big(H(\max\left\{d_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \frac{d_G(S^{n-1} a_{i-1}, S^{n-1} a_i)[1 + d_G(S^{n-1} a_i, S^n a_i)]}{1 + d_G(S^n a_{i-1}, S^n a_i)}\right\},\\
&\varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_{i-1})))\\
&\leq F\Big(H(\max\left\{d_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \frac{d(S^{n-1} a_{i-1}, S^{n-1} a_i)[1 + d_G(S^{n-1} a_i, S^n a_i)]}{1 + d_G(S^n a_{i-1}, S^n a_i)}\right\},\\
&\varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_{i-1}))).
\end{aligned}$$

This yields

$$\tau + F(H(0, \varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_{i-1}))) \leq F(H(0, \varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_{i-1}))$$

which is a contradiction, since $\tau > 0$. So, $\mathrm{H}(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)) = 0$, for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N$. It follows that (3.3) holds and by using the Property (H_1), (3.4) also holds. Now consider the case $S^n a_{i-1} \neq S^n a_i$, for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N$. Again by the property (H_1), we have

$$H\left(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)\right) \geq d_G(S^n a_{i-1}, S^n a_i) > 0.$$

Condition (3.2) implies that,

$$\begin{aligned}
&\tau + F\left(H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))\right) \\
&\leq F\left(H(M_G(S^{n-1}a_{i-1}, S^{n-1}a_i), \varphi(S^{n-1}a_{i-1}), \varphi(S^{n-1}a_i))\right) + Ld_G(S^{n-1}a_i, S^n a_{i-1}) \\
&\leq F\Big(H(\max\left\{d_G(S^{n-1}a_{i-1}, S^{n-1}a_i), \frac{d_G(S^{n-1}a_{i-1}, S^n a_{i-1})[1 + d_G(S^{n-1}a_i, S^n a_i)]}{1 + d_G(S^n a_{i-1}, S^n a_i)}\right\}, \\
&\quad \varphi(S^{n-1}a_{i-1}), \varphi(S^{n-1}a_i))) + Ld_G(S^{n-1}a_i, S^{n-1}Sa_{i-1}) \\
&\leq F\Big(H(\max\left\{d_G(S^{n-1}a_{i-1}, S^{n-1}a_i), \frac{d_G(S^{n-1}a_{i-1}, S^{n-1}a_i)[1 + d_G(S^{n-1}a_i, S^n a_i)]}{1 + d_G(S^{n-1}a_i, S^n a_i)}\right\}, \\
&\quad \varphi(S^{n-1}a_{i-1}), \varphi(S^{n-1}a_i))) + Ld_G(S^{n-1}a_i, S^{n-1}a_i) \\
&\leq F\left(H(d_G(S^{n-1}a_{i-1}, S^{n-1}a_i)), \varphi(S^{n-1}a_{i-1}), \varphi(S^{n-1}a_i)\right),
\end{aligned}$$

for all $n \in \mathbb{N}$. So

$$\begin{aligned}
F\left(H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))\right) &\leq F\left(H(d_G(S^{n-1}a_{i-1}, S^{n-1}a_i), \varphi(S^{n-1}a_{i-1}))\right) - \tau \\
&< F\left(H(d_G(S^{n-1}a_{i-1}, S^{n-1}a_i), \varphi(S^{n-1}a_{i-1}))\right).
\end{aligned}$$

$$\text{i.e., } H\left(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)\right) < H\left(d_G(S^{n-1}a_{i-1}, S^{n-1}a_i), \varphi(S^{n-1}a_{i-1})\right),$$

for all $n \in \mathbb{N}$. Thus, the sequence $\left\{H\left(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)\right)\right\}$ is a decreasing sequence of positive real numbers. So there exists some $\mu \geq 0$ such that

$$\lim_{n\to\infty} H\left(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)\right) = \mu.$$

If $\mu = 0$, then the property $(H_1)$of the function H gives us

$$\lim_{n\to\infty} d_G(S^n a_{i-1}, S^n a_i) = 0 \text{ and } \lim_{n\to\infty} \varphi(S^n a_i) = 0.$$

Now, suppose $\mu > 0$. From condition (3.2), we have

$$\begin{aligned}
&\tau + F\left(H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))\right) \\
&\leq F\left(H(M_G(S^{n-1}a_{i-1}, S^{n-1}a_i))\right) + Ld_G(S^{n-1}a_i, S^n a_{i-1}),
\end{aligned}$$

implies

$$\begin{aligned}
&F\left(H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))\right) \\
&\leq F\left(H(d_G(S^{n-1}a_{i-1}, S^{n-1}a_i), \varphi(S^{n-1}a_{i-1}), \varphi(S^{n-1}a_i))\right) - \tau \\
&\vdots \\
&\leq F\left(H(d_G(a_{i-1}, a_i), \varphi(a_{i-1}), \varphi(a_i))\right) - n\tau,
\end{aligned}$$

for all $n \in \mathbb{N}$. By taking limit as $n \to +\infty$

$$\lim_{n\to+\infty} F\left(H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))\right) = -\infty.$$

By using the property (F_2), we get

$$\lim_{n\to+\infty} H\left(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)\right) = 0,$$

and hence

$$\lim_{n\to+\infty} d_G(S^n a_{i-1}, S^n a_i) = 0 \text{ and } \lim_{n\to+\infty} \varphi(S^n a_i) = 0.$$

Remark 3.2. Note that, in the proof of Lemma 3.1, only the conditions (F_1) and (F_2) of F-contraction have been encountered.

Theorem 3.2. Let (X_G, d_G) be a metric space, then the following statements are equivalent:

1. G is weakly connected.
2. For any almost $(F - H)_G$-contraction S on X_G, both the sequences $\{S^n a\}$ and$\{S^n b\}$ $(n \in \mathbb{N})$ are Cauchy and equivalent too.
3. For any almost $(F - H)_G$-contraction S on X_G, $Card(Fix\ S)$ can not exceed by 1.

Proof. (1) ⇒ (2): First suppose that G be weakly connected and S be an almost $(F - H)_G$ contraction on X. Let $a, b \in X_G$, then $X_G = [a]_{\tilde{G}}$. Take $b = Sa \in [a]_{\tilde{G}}$, then as in Lemma 3.1, there exists a path $\{a_i\}_{i=0}^N$ with $N + 1$ vertices in such a way that $a_0 = a$, $a_1 = Sa_0$, … , $a_N = b = Sa_{N-1}$ and for all $i = 1, 2, \dots N$, $(a_{i-1}, a_i) \in E_{\tilde{G}}$. If $S^{k+1} a = S^k a$, for some $k \in \mathbb{N}$, then $\{S^n a\}$ $(n \in \mathbb{N})$ immediately will be constant, and becomes Cauchy. So consider the case, $S^{n+1} x \neq S^n a$, for all $n \in \mathbb{N}$. If we write, for all $n \in \mathbb{N}$,

$$d_n = d_G(S^n a, S^{n+1} a) > 0 \text{ and } h_n = H(d_G(S^n a, S^{n+1} a), \varphi(S^n a), \varphi(S^{n+1} a)),$$

Also, for all $i = 1, 2, \dots, N$,

$$d_{n,i} = d_G(S^n a_{i-1}, S^n a_i) \text{ and } h_{n,i} = H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i)).$$

Using triangular inequality, we get

$$d_n = d_G(S^n a, S^{n+1} a) \leq \sum_{i=1}^{N} d_G(S^n a_{i-1}, S^n a_i) = \sum_{i=1}^{N} d_{n,i}. \tag{3.5}$$

By Lemma 3.1, we see that

$$0 < h_{n-1} = H\left(d_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_i)\right) \to 0 \text{ as } n \to +\infty.$$

The property (F_3) of the function F ensures that there exists $k \in (0, 1)$ such that

$$\lim_{n\to+\infty} h_{n,i}^{k_i} F(h_{n,i}) = 0.$$

Using (3.2), we get

$$\begin{aligned}
&\tau + F\left(H(d(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))\right) \\
&\leq F\left(H(M_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_i))\right) + L d_G(S^{n-1} a_i, S^n a_{i-1}), \\
&\leq F\left(H(d_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \varphi(S^{n-1} a_{i-1})\right),
\end{aligned}$$

which implies

$$\begin{aligned} &F\Big(H(d_G(S^n a_{i-1}, S^n a_i), \varphi(S^n a_{i-1}), \varphi(S^n a_i))\Big) \\ &\quad \le F\Big(H(d_G(S^{n-1} a_{i-1}, S^{n-1} a_i), \varphi(S^{n-1} a_{i-1}), \varphi(S^{n-1} a_i))\Big) - \tau \\ &\quad \le F\Big(H(d_G(S^{n-2} a_{i-1}, S^{n-2} a_i), \varphi(S^{n-2} a_{i-1}), \varphi(S^{n-2} a_i))\Big) - 2\tau \\ &\quad \vdots \\ &\quad \le F\Big(H(d_G(S^0 a_{i-1}, S^0 a_i), \varphi(S^0 a_{i-1}), \varphi(S^0 a_i))\Big) - n\tau. \end{aligned}$$

That is,

$$F\left(h_{n,i}\right) \le F\left(h_{(n-1),i}\right) - \tau \le \ldots \le F\left(h_{0,i}\right) - n\tau.$$

Hence,

$$\begin{aligned} &F\left(h_{n,i}\right) - F\left(h_{0,i}\right) \le -n\tau \\ &\Rightarrow \lim_{n\to+\infty} h_{n,i}^{k_i}\left(F(h_{n,i}) - F(h_{0,i})\right) \le \lim_{n\to+\infty} h_{n,i}^{k_i}\left(-n\tau\right) \\ &\Rightarrow 0 = \lim_{n\to+\infty} h_{n,i}^{k_i}\left(F(h_{n,i})\right) \le \lim_{n\to+\infty} h_{n,i}^{k_i}\left(-n\tau\right) \le 0 \\ &\Rightarrow \lim_{n\to+\infty} h_{n,i}^{k_i} n = 0. \end{aligned}$$

This ensures that the series $\sum_{n=1}^{\infty} h_{n,i}$ is convergent. By the property $(H_1)$of the function H, the series $\sum_{n=1}^{\infty} d_{n,i}$ is also convergent and hence by (3.5) the series $\sum_{n=1}^{\infty} d_n$ is convergent. For $n > m$,

$$\begin{aligned} d_G(S^m a, S^n a) &\le d_G(S^m a, S^{m+1} a) + d_G(s^{m+1} a, S^{m+2} a) + \ldots + d_G(s^{n-1} a, S^n a) \\ &= d_m + d_{m+1} + \ldots + d_{n-1} \\ &< \sum_{i=m}^{\infty} d_i. \end{aligned}$$

Hence $\{S^n a\}$ is Cauchy sequence. By Lemma 3.1, $d_G(S^n a, S^n b) \to 0$ as $n \to +\infty$, thus $\{S^n b\}$ is also Cauchy.

(2) ⇒ (3):

Let S possesses two distinct fixed points a and b in X_G. Then condition (2) trivially gives $a = b$, i.e., $Card(Fix\ S) \le 1$.

(3) ⇒ (1):

Suppose if possible, G is not weakly connected, then $\tilde{G}$ is not connected. Let $a_0 \in X_G$ such that $[a_0]_{\tilde{G}}$, and $X_G \setminus [a_0]_{\tilde{G}}$ both are two non empty disconnected components. So we can choose $b_0 \in X_G \setminus [a_0]_{\tilde{G}}$ and define

$$Sa = \begin{cases} a_0, \text{ if } a \in [a_0]_{\tilde{G}} \\ b_0, \text{ if } a \in X_G \setminus [a_0]_{\tilde{G}}, \end{cases}$$

This implies, $Fix\ S = \{a_0, b_0\}$. Next suppose, $(a, b) \in E_G$ be an arbitrary edge. Then $a, b \in [a_0]_{\tilde{G}}$ or $a, b \in X_G \setminus [a_0]_{\tilde{G}}$ and $[a]_{\tilde{G}} = [b]_{\tilde{G}}$ and hence in both the cases $Sa = Sb$ and $(Sa, Sb) \in E_G$. So equation (3.1) satisfied. Also, condition (3.2) holds with $Sa = Sb$ and S having two fixed points, yields a contradiction. Hence G is weakly connected.

Theorem 3.3. Let (X_G, d_G) be a metric space and S be an almost $(F-H)_G$ contraction on X_G with $Sa_0 \in [a_0]_{\tilde{G}}$, for some a_0 in X_G and $\tilde{G}a_0$ be component of $\tilde{G}$ possessing a_0. Then $[a_0]_{\tilde{G}}$ is an S-invariant and $S|_{[a_0]_{\tilde{G}}}$ is an $(F-H)_{\tilde{G}_{a_0}}$ contraction. Moreover, if $a, b \in [a_0]_{\tilde{G}}$ then the both the sequences $\{S^n a\}$ and $\{S^n b\}$ $(n \in \mathbb{N})$ are Cauchy and equivalent.

Proof. Suppose $a \in [a_0]_{\tilde{G}}$ be arbitrary, then there is a path $\{a_i\}_{i=0}^N$ with $N+1$ vertices in $\tilde{G}$ in such a way that $a_0, a_1, \ldots, a_N = a$ and $(a_{i-1}, a_i) \in E_{\tilde{G}}$, for all $i = 1, 2, \cdots, N$. By Remark 3.1, we observe that S is also an almost $(F-H)_{\tilde{G}}$-contraction, hence $(Sa_{i-1}, Ta_i) \in E_{\tilde{G}}$, for all $i = 1, 2, \cdots, N$. Consequently we have a path $\{Sa_i\}_{i=0}^N$ with $N+1$ vertices in $\tilde{G}$ in such a way that $Sa_0, Sa_1, \cdots, Sa_N = Sa$. This implies that $Sa \in [Sa_0]_{\tilde{G}}$ and since $Sa_0 \in [a_0]_{\tilde{G}}$, we obtain $[Sa_0]_{\tilde{G}} = [a_0]_{\tilde{G}}$, Thus $[a_0]_{\tilde{G}}$ is an S-invariant.

Now assume $(a,b) \in E_{\tilde{G}_{a_0}}$ be arbitrary, then there exists a path $\{a_i\}_{i=0}^N$ with $N+1$ vertices in $\tilde{G}$ in such a way that $a_0, a_1, \cdots, a_{N-1} = a$, $a_N = b$. Inductively continuing the process occurred in the first part of the proof, we get another path $\{Sa_i\}_{i=0}^N$ with $N+1$ vertices in $\tilde{G}$ such that $Sa_0, Sa_1, \ldots, Sa = Sa_{N-1}$, $Sb = Sa_N$. Since $Sa_0 \in [a_0]_{\tilde{G}}$, it follows that there is a path $\{b_i\}_{i=0}^M$ with $M+1$ vertices in $\tilde{G}$ in such a way that $a_0 = b_0, b_1, b_2, \ldots, b_M = Sa_0$ and consequently we have a path in $\tilde{G}$ from a_0 to Sb such that $a_0 = b_0, b_1, b_2, \ldots, b_M, Sa_1, Sa_2, \cdots, Sa_N = Sb$. Now, $(Sa_{N-1}, Sa_N) \in E_{\tilde{G}_{a_0}}$ immediately implies that $(Sa, Sb) \in E_{\tilde{G}_{a_0}}$. Since $E_{\tilde{G}_{a_0}} \subseteq E_{\tilde{G}}$ and condition (3.2) also true for the graph $\tilde{G}_{a0}$, therefore $S|_{[a_0]_{\tilde{G}}}$ is an $(F-H)_{\tilde{G}_{a_0}}$ contraction.

Since $\tilde{G}_{a0}$ is connected, thus for all $a, b \in [a_0]_{\tilde{G}}$, both the sequences $\{S^n a\}$ and $\{S^n b\}$ are Cauchy and equivalent.

Theorem 3.4. Let (X_G, d_G) be a complete metric space satisfying Property (*), S be an almost $(F-H)_G$-contraction on X_G and $(X_G)_S = \{a \in X_G : (a, Sa) \in E_G\}$. Then

1. $CardFix\ S = Card\{[a]_{\tilde{G}} : a \in (X_G)_S\}$.
2. $Fix\ S \neq \varnothing$ if and only if $(X_G)_S \neq \varnothing$.
3. S possesses a unique fixed point if and only if there is a point $a_0 \in (X_G)_S$ such that $(X_G)_S \subseteq [a_0]_{\tilde{G}}$.
4. For any $a \in (X_G)_S, S|_{[a]_{\tilde{G}}}$ is a PO with $\varphi(p) = 0$, where p is a fixed point of S.
5. If G is weakly connected and $(X_G)_S \neq \varnothing$, then S is a PO with $\varphi(p) = 0$, where p is a fixed point of S.
6. If $X'_G = \cup\{[a]_{\tilde{G}} : a \in (X_G)_S\}$, then $S|_{X'_G}$ is a WPO.
7. If $S \subseteq E_G$, then S is a WPO.

Proof. Let us begin the proof with (4) and (5). Suppose $a \in (X_G)_S$ be arbitrary, then $(a, Sa) \in E_G$ and thus $Sa \in [a]_{\tilde{G}}$. By Theorem 3.3, we have for any $b \in X_G$, sequences $\{S^n a\}_{n \in \mathbb{N}}$ and $\{S^n y\}_{n \in \mathbb{N}}$ both are Cauchy and equivalent and by completeness of X_G, there is $p \in X_G$ such that $\lim_{n\to+\infty} S^n a = p = \lim_{n\to+\infty} S^n b$. For $(a, Sa) \in E_G$, condition (3.1) implies $(S^n a, S^{n+1} a) \in E_G$, for all $n \in \mathbb{N}$ and then by Property (*), there exists a subsequence $\{S^{k_n} a\}_{n\in\mathbb{N}}$ such that $(S^{k_n} a, p) \in E_G$, for all $n \in \mathbb{N}$. Therefore, it can be observe that there exist a path $(a, Sa, S^2 a, \ldots, S^{k_1} a, p)$ from a to p in G and in $\tilde{G}$ both. Hence $p \in [a]_{\tilde{G}}$. By similar computations as in Theorem 3.5, $S|_{[a]_{\tilde{G}}}$ is a PO such that $\varphi(p) = 0$. This proves condition (4).

Next, suppose G is weakly connected and $a \in (X_G)_S$. This implies that $X_G = [a]_{\tilde{G}}$ and hence S is PO such that $\varphi(p) = 0$, which yields condition (5) holds. Condition (6) immediate comes from (4).

If $S \subseteq E_G$ then $X_G = (X_G)_S$ and thus $X'_G = X_G$. By (4), S is WPO on X_G. Hence condition (7) holds.

To verify condition (1), define a mapping $\Gamma : Fix\ S \to Q$ by

$$\Gamma(a) = [a]_{\tilde{G}}, \text{ for all } a \in Fix\ S,$$

where $\mathcal{Q} = \{[a]_{\tilde{G}} : a \in (X_G)_S\}$. To prove our claim, we only need to show that Γ is a one-one and onto. First suppose $a \in (X_G)_S$ be arbitrary and by condition (4), $S|_{[a]_{\tilde{G}}}$ is a *PO*. Let $\lim_{n\to+\infty} S^n a = p$, so $p \in Fix\, S \cap [a]_{\tilde{G}}$ and $\Gamma p = [p]_{\tilde{G}} = [a]_{\tilde{G}}$. Thus Γ is onto. Next if p_1 and p_2 be any two fixed points of S such that $[p_1]_{\tilde{G}} = [p_2]_{\tilde{G}}$, then $p_2 \in [p_1]_{\tilde{G}}$. By (4), $\lim_{n\to+\infty} S^n a_2 \in [a_1]_{\tilde{G}} \cap Fix\, S = \{a_1\}$. But $S^n a_2 = a_2$, for all $n \in \mathbb{N}$. So, we obtain $p_1 = p_2$. Therefore, Γ is a one-one and hence we obtained our claim.

Conditions (2) and (3) are the consequences of condition (1).

Theorem 3.5. Let (X_G, d_G) be a complete metric space satisfying Property (*) and S is an almost $(F-H)_G$ contraction then the following statements are equivalent:

1. G is weakly connected.
2. If there is some $a_0 \in X_G$ such that $(a_0, Sa_0) \in E_G$, then S has a unique fixed point p in X_G with $\varphi(p) = 0$.

Proof. Suppose G is weakly connected, then by Theorem 3.2 we have, $\{S^n a\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and completeness of X_G implies that there is a p in X_G such that

$$\lim_{n\to+\infty} S^n a = p.$$

Since φ is a lower semicontinuous function, we get

$$0 \le \varphi(p) \le \liminf_{n\to+\infty} \varphi(S^n a) = 0,$$

Thus,

$$\varphi(p) = 0.$$

Now we assert that p is a fixed point of S. Property (*) indicates that there exists a subsequence $\{S^n a\}_{n\in\mathbb{N}}$ of $\{S^{n_k} a\}_{n\in\mathbb{N}}$ with $(S^{n_k}a, p) \in E_G$. If $\{S^{n_k}a\} = p$ or $S^{n_k+1}a = Sp$, for all $k \in \mathbb{N}$, then p is fixed point of S. Otherwise, assume that $S^n a \neq p$ and $S^{n_k+1}a \neq Sp$, for all $n \in \mathbb{N}$. Property (H_1) ensures $H(d_G(S^{n+1}a, Sp),\ \varphi(S^{n+1}a), \varphi(Sp)) > 0$ and by using (3.2), we get

$$\begin{aligned}
\tau + F\Big(H(d_G(S^{n+1}a, Sp), \varphi(S^{n+1}a), \varphi(Sp))\Big) &\le F\Big(H(M_G(S^n a, p), \varphi(S^n a), \varphi(p))\Big) \\
&\qquad + Ld_G(p, S^{n+1}a) \\
\Rightarrow F\Big(H(d_G(S^{n+1}a, Sp), \varphi(S^{n+1}a), \varphi(Sp))\Big) &\le F\Big(H(M_G(S^n a, p), \varphi(S^n a), \varphi(p))\Big) - \tau \\
\Rightarrow F\Big(H(d_G(S^{n+1}a, Sp), \varphi(S^{n+1}a), \varphi(Sp))\Big) &< F(H(M_G(S^n a, p), \varphi(S^n a), \varphi(p))),
\end{aligned}$$

since $\tau > 0$ and this yields

$$H(d_G(S^{n+1}a, Sp), \varphi(S^{n+1}a), \varphi(Sp)) < H(d_G(S^n a, p), \varphi(S^n a), \varphi(p)),$$

for all $n \in \mathbb{N}$. Now

$$\begin{aligned}
d_G(p, Sp) &\le d_G(p, S^{n+1}a) + d_G(S^{n+1}a, Sp) \\
&\le d_G(p, S^{n+1}a) + H\Big(d_G(S^{n+1}a, Sp), \varphi(S^{n+1}a), \varphi(Sp)\Big) \\
&< d_G(p, S^{n+1}a) + H\Big(d_G(S^n a, p), \varphi(S^n a), \varphi(p)\Big),
\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in above inequality and using the fact that H is continuous in (0, 0, 0), we deduce that

$$d_G(p, Sp) \leq H(0,0,0) = 0,$$

that is, $Sp = p$. Next we prove the uniqueness of fixed point. Arguing by contradiction, we suppose that S admits two distinct fixed points p and q in X_G, then $Sp = p$ and $Sq = q$. By the property (H_1), we have $H(d_G(Sp, Sq), \varphi(Sp), \varphi(Sq)) \geq d_G(Sp, Sq) = d_G(p, q) > 0$ and again by using (3.2), we get

$$\begin{aligned}&\tau + F\big(H(d_G(Sp,Sq),\varphi(Sp),\varphi(Sq))\big)\\ &\quad = \tau + F\big(H(d_G(p,q),\varphi(p),\varphi(q))\big) \leq F\big(H(d_G(p,q),\varphi(p),\varphi(q))\big).\end{aligned}$$

Clearly, this is a contradiction and hence $p = q$. Therefore we obtain the claim.

Imposing that F is a continuous function and relaxing the hypothesis ($F3$), we establish the following result.

Theorem 3.6. Let (X_G, d_G) be a complete metric space satisfying Property (*) and S is an almost $(F - H)_G$ contraction, where continuous function F satisfies only the conditions (F_1) and (F_2), then the following statements are equivalent:

1. G is weakly connected.
2. If there is some $a_0 \in X_G$ such that $(a_0, Sa_0) \in E_G$, then S has a unique fixed point p in X_G with $\varphi(p) = 0$.

Proof. First suppose that G be weakly connected and S be an almost $(F - H)_G$ contraction on X_G. Let $a, b \in X_G$, then $X_G = [a]_{\tilde{G}}$. Take $b = Sa \in [a]_{\tilde{G}}$, then as in Lemma 3.1, there exists a path $\{a_i\}_{i=0}^{N}$ with $N + 1$ vertices in such a way that $a_0 = a$, $a_1 = Sa_0$, …, $a_N = b = Sa_{N-1}$ and $(a_{i-1}, a_i) \in E_{\tilde{G}}$, for all $i = 1, 2, \dots N$. If $S^{k+1} a = S^k a$, for some $k \in \mathbb{N}$, then $\{S^n a\}_{n \in \mathbb{N}}$ immediately will be constant and becomes Cauchy. So consider the case, $S^{n+1} x \neq S^n a$, for all $n \in \mathbb{N}$. We claim that $\{S^n a\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Arguing by contradiction, $\{S^n a\}_{n \in \mathbb{N}}$ is not a Cauchy sequence. Then there is a positive real number ε and two sequences $\{m_k\}$ and $\{n_k\}$ such that

$$n_k > m_k \text{ and } d_G(S^{m_k}, S^{n_k}) \geq \varepsilon > d_G(S^{m_k}a, S^{n_k-1}a), \text{for all } k \in \mathbb{N}.$$

Using Lemma 3.1 and Remark 3.2, we have

$$d_G(S^{n-1}a, S^n a) \to 0 \text{ and } \varphi(S^n a) \to 0 \text{ as } n \to +\infty.$$

This implies

$$\lim_{k\to+\infty} d_G(S^{m_k}a, S^{n_k}a) = \lim_{k\to+\infty} d_G(S^{m_k-1}a, S^{n_k-1}a) = \varepsilon.$$

The hypothesis $d_G(S^{m_k}a, S^{n_k}a) > \varepsilon$ infer that

$$H\Big(d_G(S^{m_k}a, S^{n_k}a), \varphi(S^{m_k}a), \varphi(S^{n_k}a)\Big) > 0,$$

for all $k \in \mathbb{N}$. Since H is a continuous function,

$$\begin{aligned}&\lim_{k\to+\infty} H\Big(d_G(S^{m_k-1}a, S^{n_k-1}a), \varphi(S^{m_k-1}a), \varphi(S^{n_k-1}a)\Big)\\ &\quad = \lim_{k\to+\infty} H\Big(d_G(S^{m_k}a, S^{n_k}a), \varphi(S^{m_k}a), \varphi(S^{n_k}a)\Big)\\ &\quad = H(\varepsilon, 0, 0) > 0.\end{aligned}$$

Using again (3.2), we deduce that

$$\tau + F\left(H(d_G(S^{m_k}a, S^{n_k}a), \varphi(S^{m_k}a), \varphi(S^{n_k}a))\right) \le F\left(H(d_G(S^{m_k-1}a, S^{n_k-1}a), \varphi(S^{m_k-1}a), \varphi(S^{n_k-1}a))\right),$$

for all $k \in \mathbb{N}$. Letting $k \to +\infty$ in the previous inequality. Since the function F is continuous,

$$\tau + F(H(\varepsilon,0,0)) \le F(H(\varepsilon,0,0)),$$

which leads to contradiction. It follows that $\{S^n a\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Further, by following the similar arguments as in Theorem 3.5, we obtain the unique fixed point of S. This completes the proof.

Theorem 3.7. Let (X_G, d_G) be a complete metric space and S be an orbitally G-continuous almost $(F - H)_G$-contraction on X_G. Let $(X_G)_S = \{a \in X_G : (a, Sa) \in E_G\}$. Then

1. *Fix* $S \neq \emptyset$ if and only if $(X_G)_S \neq \emptyset$.
2. For any $a \in (X_G)_S$ and b in $[a]_{\tilde{G}}$, the sequence $\{S^n b\}_{n \in \mathbb{N}}$ converges to a fixed point of S and $\lim\limits_{n\to+\infty} S^n b$ is independent of b.
3. If G be weakly connected and $(X_G)_S \neq \emptyset$, then S is a *PO*.
4. If $S \subseteq E_G$, then S is a *WPO*.

Proof. Let us begin the proof with condition (2). Suppose $a \in (X_G)_S$, then $(a, Sa) \in E_G$ and thus $(S^n a, S^{n+1} a) \in E_G$, for all $n \in \mathbb{N}$. By Theorem 3.2, for any $b \in [a]_{\tilde{G}}$, the sequences $\{S^n a\}_{n \in \mathbb{N}}$ and $\{S^n b\}_{n \in \mathbb{N}}$ are Cauchy and equivalent, for all $n \in \mathbb{N}$ and consequently converges to the some point p. Since S is orbitally G-continuous, $\lim\limits_{n\to+\infty} S(S^n) = Sp$. Concomitantly $\lim\limits_{n\to+\infty} S(S^n a) = \lim\limits_{n\to+\infty} S^{n+1} a = p$. So we have $Sp = p$ and hence condition (2) holds and simultaneously half of condition (1) holds. Furthermore, the rest part of the condition (1) can be obtain by using the fact that $E_G \supseteq \Delta$ such that $(p, Sp) = (p, p) \in E_G$, then $p \in (X_G)_S$, i.e., $(X_G)_S \neq \emptyset$. Next $S \subseteq E_G$ implies $(X_G)_S = X_G$. Therefore, (4) occurred as a direct implication of (2). At last, to prove (3), let $a_0 \in (X_G)_S$ and G be weakly connected, then $[a_0]_{\tilde{G}} = X_G$ and hence S is a *PO* immediately comes from (2).

Next theorem enhance the strength of continuity condition on S.

Theorem 3.8. Let (X_G, d_G) be a complete metric space and S an orbitally continuous almost $(F - H)_G$-contraction on X_G. Let $(X_G)_S = \{a \in X_G : (a, Sa) \in E_G\}$. Then

1. *Fix* $S \neq \emptyset$ if and only if there is an $a_0 \in X_G$ such that $Sa_0 \in [a_0]_{\tilde{G}}$.
2. Let $a \in X_G$ with $Sa \in [a]_{\tilde{G}}$, then for any $b \in [a]_{\tilde{G}}$ the sequence $\{S^n b\}_{n \in \mathbb{N}}$ converges to a fixed point of S and $\lim\limits_{n\to+\infty} S^n b$ is independent of b, for all $n \in \mathbb{N}$.
3. If G be weakly connected, then S is a *PO*.
4. For any $a \in X_G$ if $Sa \in [a]_{\tilde{G}}$, then S is a *WPO*.

Proof. Let us begin the proof with (2). Suppose $a \in X_G$ with $Sa \in [a]_{\tilde{G}}$ and let $b \in [a]_{\tilde{G}}$. By Theorem 3.2, we can observe that the sequences $\{S^n a\}_{n \in \mathbb{N}}$ and $\{S^n y\}_{n \in \mathbb{N}}$ are Cauchy and equivalent, for all $n \in \mathbb{N}$ and thus converges to the some point p in X_G. Since S is orbitally continuous, $\lim\limits_{n\to\infty} S(S^n)a = Sp$, and concomitantly $\lim\limits_{n\to\infty} S(S^n a) = S$. So we have $Sp = p$ and simultaneously half of the condition (1) obtained. Furthermore, rest of the condition (1) can be obtain by using the fact that $a \in [a]_{\tilde{G}}$, for any $a \in X_G$. Now to prove (3), let G is weakly connected, then for any $a \in X_G$ we have $X_G = [a]_{\tilde{G}}$ and also $Sa \in [a]_{\tilde{G}}$ implies S is a *PO*, immediately comes from (2). Finally, condition (4) is obtained as a consequences of (2).

Corollary 3.9. Let (X_G, d_G) be a complete metric space, then the following statements are equivalent:

1. G be weakly connected.
2. If S be an orbitally continuous almost $(F - H)_G$-contraction, then it is *PO*.
3. If S be an orbitally continuous $(F - H)_G$-contraction, then *Card* (*Fix S*) can not exceed by 1.

Hence if $\tilde{G}$ is disconnected then there exists at least one orbitally continuous $(F - H)_G$-contraction $S : X_G \to X_G$ which has at least two fixed points.

Proof. Theorem 3.8 (3) indicates that (1) ⇒ (2). And it is obvious that (2) ⇒ (3) and (3) ⇒ (1) can be obtain from the proof of (3) ⇒ (1) from Theorem 3.2, where S being orbitally continuous.

4. Application to nonlinear integral equations

During the last three decades the theory of differential and integral equation has been widely used in the various fields of science and engineering. In this section, to discuss the application of our main results we establish an existence theorem in a metric space with graph for the solution of the following Volterra type integral equation:

$$a(t) = \int_0^t K(t,s,a(s))ds, \tag{4.1}$$

Where $a \in [0, 1]$, $x \in C([0, 1])$ (the set of all continuous functions from [0, 1] into $\mathbb{R}$) and $K : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is given function. Let us consider $X_G = C[0, 1]$ equipped with metric d_G: $X_G \times X_G \to [0, \infty)$ defined by $d_G(x, y) = \max_{t \in [0, 1]} |a(t) - b(t)|$. Further, define a graph G by using a partial order relation, i.e., a, $b \in X_G$, $a \leq b$ if and only if $a(t) \leq b(t)$, for any $t \in [0, 1]$.

So, we have

$$E_G = \{(a,b) \in X_G \times X_G : a \leq b\},$$

$$E_{G^{-1}} = \{(a,b) \in X_G \times X_G : b \leq a\}.$$

Also, $\Delta \subseteq E(G)$ and (X_G, d_G) has Property (*).
It is a routine verification to check that (X_G, d_G) is a complete metric space with a directed graph G.

Definition 4.1. A lower solution for (4.1) is a function $a \in X_G$ such that

$$a(t) \leq \int_0^t K(t,s,a(s))ds, \qquad t \in [0,1].$$

Definition 4.2. An upper solution for (4.1) is a function $a \in X_G$ such that

$$a(t) \geq \int_0^t K(t,s,a(s))ds, \qquad t \in [0,1].$$

Theorem 4.1. *Assume that K is nondecreasing in the third variable and suppose that the following condition holds:*

1. $K : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
2. $|K(t,s,a(s)) - K(t,s,b(s))| \leq e^{-\tau + L d_G(b(t), \int_0^t K(t,s,a(s))dx)} M_G(a(t),b(t))$, *for all a, b ∈ C[0, 1] and for all t, s ∈ [0, 1].*

If the nonlinear integral equation (4.1) has a lower solution, then there exists a unique solution of (4.1).

Proof. Let G is weakly connected and S: $X_G \times X_G$ be defined by

$$(Sa)(t) = \int_0^t K(t,s,a(s))ds,$$

for all $a \in X_G$. Next define functions $H : [0, \infty)^3 \to [0, \infty)$

$$H(u, v, w) = u + v + w,$$

for all $u, v, w \in [0, \infty)$ and $\varphi : X_G \to [0, \infty)$ by

$$\varphi(a) = 0,$$

for all $a \in X_G$.
For any $(a, b) \in E_G$,

$$\begin{aligned}(Sa)(t) &= \int_0^t K(t,s,a(s))ds \\ &\le \int_0^t K(t,s,b(s))ds \\ &= (Sb)(t),\end{aligned}$$

which shows that $(Sa, Sb) \in E_G$, i.e., S is edge-preserving. Next, we show that S is an $(F - H)_G$-contraction. For $a, b \in X_G$ and $t \in [0, 1]$ we get

$$\begin{aligned}\left|(Sa)(t)-(Sb)(t)\right| &= \left|\int_0^t K(s,t,a(s))ds - \int_0^t K(s,t,b(s))ds\right| \\ &= \left|\int_0^t K(s,t,a(s))ds - K(s,t,b(s))ds\right| \\ &\le \int_0^t \left|K(s,t,a(s))ds - K(s,t,b(s))ds\right| \\ &\le \int_0^t e^{(-\tau+L|b(s)-(Sa)(s)|)} M_G(a,b)ds \\ &\le \int_0^t e^{(-\tau+Ld_G(b,Sa))} M_G(a,b)ds \\ &\le e^{(-\tau+Ld_G(b,Sa))} M_G(a,b)t \\ &\le e^{(-\tau+Ld_G(b,Sa))} M_G(a,b)t.\end{aligned}$$

This implies that
$\max_{t\in[0,1]} \left|(Sa)(t)-(Sb)(t)\right| \le e^{-(\tau+Ld_G(b,Sa))} M_G(a,b)$ and hence

$$\begin{aligned}&d(Sa,Sb) \le e^{(-\tau+Ld_G(b,Sa))} M_G(a,b) \\ &\Rightarrow \ln d(Sa,Sb) - \ln M_G(a,b) \le -\tau + Ld_G(b,Sa) \\ &\Rightarrow \quad \tau + \ln d(Sa,Sb) \le \ln M_G(a,b) + Ld_G(b,Sa).\end{aligned}$$

Thus, inequality (3.2) is satisfied with $F(\alpha) = \ln \alpha$ for all $\alpha > 0$. Also by using (4.1) we have $(x_0, Tx_0) \in E_G$. Thus, all the conditions of Theorem 3.6 are satisfied. Hence, the result is established.

Theorem 4.2. *Assume that K is nonincreasing in the third variable and suppose that the following condition holds:*

1. $K : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathrm{R}$ *is continuous.*
2. $|K(t,s,a(s)) - K(t,s,b(s))| \le e^{-\tau+Ld_G(b(t),\int_0^t K(t,s,a(s))ds)} M_G(a,b)$, *for all x, y ∈ C[0, 1] and for all t, s ∈ [0, 1].*

If the nonlinear integral equation (4.1) has a upper solution, then there exists a unique solution of (4.1).

Proof. Let G is weakly connected and following the proof of Theorem 4.1 with an analogous procedure, we can check that all the hypothesis of Theorem 3.6 is satisfied and hence there exists a unique solution of the integral equation (4.1).

5. Numerical Examples

In this section, an example is presented in which we define a contraction and prove that, it is an $(F-H)_G$-contraction. Additionally, we also demonstrate that, it is not an F-contraction, not an F-weak contraction, not an F_G-contraction, not an $(H-\varphi)$-contraction, not an $(F-H)$-contraction, not rational type $(F-H)$-contraction and not an almost $(F-H)$-contraction. Next, an example is given to support Theorem 3.6. Furthermore, this section is finished with an example in which existence of unique solution of integral equation (4.1) is obtained successfully.

Example 5.1. For $n \in \mathbb{N}$, consider $X_G = \{0, n\}$ is equipped with usual metric $d_G(a, b) = |a - b|$, and $V_G = X_G$, $E_G = \{\Delta \cup (0, n)\}$. Define $S : X_G \to X_G$ by

$$Sa = 2a, \qquad \text{for all } a \in X_G,$$

$F \in \mathcal{F}$ by

$$F(\alpha) = \ln \alpha, \qquad \text{for all } \alpha \in \mathbb{R}^+,$$

$H \in \mathcal{H}$ by

$$H(u,v,w) = u + v + w, \qquad \text{for all } u,v,w \in [0,\infty)$$

and $\varphi : X \to [0, +\infty)$ by

$$\varphi(t) = t, \qquad \text{for all } t \in [0,\infty).$$

It is easy to observe that S preserves edges. But $(a, b) \in E_G$ with $H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb)) > 0$ if and only if $(a, b) \in \{(n, n), (0, n): n \in \mathbb{N}\}$, i.e., there are following two cases arises:

Case 1: Let $a = 0$ and $b = n$, i.e., $(a, b) = (0, n) \in E_G$ then $(Sa, Sb) = (S0, Sn) = (0, 2n) \in E_G$. Also, $d_G(a, b) = d(0, n) = n$, $d_G(a, Sa) = d_G(0, S0) = 0$, $d_G(b, Sb) = d_G(n, Sn) = n$, $d_G(a, Sb) = d_G(0, Sn) = 2n$, $d_G(b, Sa) = d_G(n, S0) = n$. Now it is noted that

$$H\left(d_G(S0,Sn),\varphi(S0),\varphi(Sn)\right) = H\left(d_G(0,2n),\varphi(0),\varphi(2n)\right) = 4n > 0$$

and $F(H(d_G(S0, Sn), \varphi(S0), \varphi(Sn))) = \ln 4n$. Similarly, $F(H(M_G(0, n), \varphi(0), \varphi(n))) = \ln 2n$. This implies for $L = \frac{7}{10}$ and $\tau \in (0, 0.006]$, we have

$$\tau + F\left(H(d_G(Sa,Sb),\varphi(Sa),\varphi(Sb))\right) < F\left(H(M_G(0,n),\varphi(0),\varphi(n))\right) + Ld_G(b,Sa).$$

Case 2: Let $a = n$ and $b = n$, i.e., $(n, n) \in E_G$ then $(Sn, Sn) = (2n, 2n) \in E_G$, and

$$H\left(d_G(Sa,Sb),\varphi(Sa),\varphi Sb\right) = H\left(d_G(Sn,Sn),\varphi(Sn),\varphi(Sn)\right) = 4n > 0,$$

implies $F(H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb)) = \ln 4n$. Similarly, $F(H(d_G(a, b), \varphi(a), \varphi(b))) + L\, d_G(b, Sa) = \ln 2n + \frac{7}{10}n$. Thus for $L = \frac{7}{10}$ and $\tau \in (0, 0.006]$, we have

$$\tau + F\left(H(d_G(Sa,Sb),\varphi(Sa),\varphi(Sb))\right) < F\left(H(d_G(a,b),\varphi(a),\varphi(b))\right) + Ld_G(b,Sa).$$

Hence S is an $(F - H)_G$-contraction for $L = \frac{7}{10}$ and $\tau \leq 0.006$. But, if we define a metric d on $X = X_G$ by $d(a, b) = |a - b|$, for $a = 0$ and $b = n$ then

- S is not an F-contraction [30], since $F(d(Sa, Sb)) = \ln 2n$ and $F(d(a, b)) = \ln n$, which implies $\tau + F(d(Sa, Sb)) \nleq F(d(a, b))$.
- S is not an F-weak contraction [32], since

$$\begin{aligned}
&\max\left\{d(a,b), d(a,Ta), d(b,Tb), \frac{d(a,Sb)+d(b,Sa)}{2}\right\} \\
&= \max\left\{d(0,n), d(0,S0), d(n,Sn), \frac{d(0,Sn)+d(n,S0)}{2}\right\} \\
&= \max\left\{n, 0, n, \frac{3n}{2}\right\} \\
&= \frac{3n}{2},
\end{aligned}$$

which implies, $\tau + F(d(Sa,Sb)) \nleq \max\left\{d(a,b), d(a,Ta), d(b,Tb), \frac{d(a,Sb)+d(b,Sa)}{2}\right\}$.

- S is not an F_G-contraction [24], since $\tau + F(d(Sa, Sb)) \nleq F(d(a, b))$.
- S is not an $(H - \varphi)$-contraction [21], since $H(d(Sa, Sb), \varphi(Sa), \varphi(Sb)) = 4n$, and $H(d(a, b), \varphi(a), \varphi(b)) = 2n$, which implies

$$H\left(d(Sa,Sb), \varphi(Sa), \varphi(Sb)\right) \nleq kH\left(d(a,b), \varphi(a), \varphi(b)\right), \text{for } k \in (0,1).$$

- S is not an $(F - H)$-contraction [20], since $F(d(Sa, Sb), \varphi(Sa), \varphi(Sb)) = \ln 4n$, and $F(d(a, b), \varphi(a), \varphi(b)) = \ln 2n$, which implies

$$\tau + F\left(d(Sa,Sb), \varphi(Sa), \varphi(Sb)\right) \nleq F\left(d(a,b), \varphi(a), \varphi(b)\right).$$

- S is not rational type $(F - H)$-contraction, since

$$\begin{aligned}
M(a,b) &= \max\left\{d(a,b), \frac{d(a,Sa)[1+d(b,Sb)]}{1+d(Sa,Sb)}\right\} \\
&= \max\left\{d(0,n), \frac{d(0,S0)[1+d(n,Sn)]}{1+d(S0,Sn)}\right\} \\
&= n,
\end{aligned}$$

which implies $F(M(a, b), \varphi(a), \varphi(b)) = \ln 2n$ and thus $\tau + F(d(Sa, Sb), \varphi(Sa), \varphi(Sb)) \nleq F(M(a, b), \varphi(a), \varphi(b))$.

- Also, S is not an almost $(F - H)$-contraction [33], since

$$\begin{aligned}
F\left(H(d(a,b) + Ld(b,Sb), \varphi(a), \varphi(b))\right) &= F\left(H(d(0,n) + \frac{7}{10}d(n,Sn), \varphi(0), \varphi(n))\right) \\
&= F\left(H(n + \frac{7}{10}n, 0, n)\right) \\
&= \ln 2.7n,
\end{aligned}$$

which implies $\tau + F(d(Sa, Sb), \varphi(Sa), \varphi(Sb)) \nleq F(H(d(a, b) + L\, d(b, Sb), \varphi(a), \varphi(b)))$.

Next following example is given to support Theorem 3.6.

Example 5.2. Let (X_G, d_G) be a metric space, where $X_G = [0, 1]$ and $V_G = X_G$, $E_G = \{\Delta \cup (r, 0): r \in [0, 1]\}$ and metric $d_G : X_G \times X_G \to [0, \infty)$ is defined by $d_G(a, b) = |a - b|$. Define $S : X_G \to X_G$ by

$$Sa = \begin{cases} \dfrac{a^2}{2}, \text{ if } a \in [0,1) \\ \dfrac{2}{3}, \text{ if } a = 1. \end{cases}$$

Define $F \in \mathcal{F}$ by

$$F(\alpha) = \alpha^2$$

$H \in \mathcal{H}$ by

$$H(u, v, w) = u + v + w,$$

and $\varphi : X \to [0, \infty)$ by

$$\varphi(t) = t.$$

Clearly, G is weakly connected and S is edge preserving for all $(a, b) \in E_G$, but $H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb)) > 0$ holds only for the following cases:

Case-1: If $(a, b) = (r, r) \in E_G$, where $r \in (0, 1)$, then $Sa = \frac{r^2}{2} = Sb, d_G(a,b) = 0 = d_G(Sa,Sb), d_G(a,Sa) =$ $r - \frac{r^2}{2} = d_G(b,Sb) = d_G(b,Sa), \varphi(a) = r = \varphi(b)$ and $\varphi(Sa) = \frac{r^2}{2} = \varphi(Sb)$. So

$$\begin{aligned} H\left(d_G(Sa,Sb), \varphi(Sa), \varphi(Sb)\right) &= H\left(0, \frac{r^2}{2}, \frac{r^2}{2}\right) \\ &= r^2 > 0. \end{aligned}$$

This implies $F(H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb))) = r^4$.

Similarly, $F\left(H(M_G(a,b), \varphi(a), \varphi(b))\right) = 9r^2 + \frac{r^4}{4} - 3r^3$.

Case-2: If $(a, b) = (1, 1) \in E_G$, then $Sa = Sb = \frac{2}{3}, d_G(a,b) = 0 = d_G(Sa,Sb), d_G(a,Sa) = \frac{1}{3} = d_G(b,Sb) =$ $d_G(b,Sa), \varphi(a) = 1 = \varphi(b)$ and $\varphi(Sa) = \frac{2}{3} = \varphi(Sb)$. So

$$\begin{aligned} H\left(d_G(Sa,Sb), \varphi(Sa), \varphi(Sb)\right) &= H\left(0, \frac{2}{3}, \frac{2}{3}\right) \\ &= \frac{4}{3} > 0. \end{aligned}$$

This implies $F(H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb))) = \frac{16}{9}$.

Similarly, $F(H(M_G(a, b), \varphi(a), \varphi(b))) = 4$.

Case-3: If $(a, b) = (r, 0) \in E_G$, where $r \in (0, 1)$, then $Sa = \frac{r^2}{2}, Sb = 0$. In this case, $d_G(a,b) = r, d_G(Sa,Sb) =$ $\frac{r^2}{2} = d_G(b,Sa), \varphi(a) = r, \varphi(b) = 0, \varphi(Sa) = \frac{r^2}{2}$ and $\varphi(Sb) = 0$. So

$$\begin{aligned} H\left(d_G(Sa,Sb), \varphi(Sa), \varphi(Sb)\right) &= H\left(\frac{r^2}{2}, \frac{r^2}{2}, 0\right) \\ &= r^2 > 0. \end{aligned}$$

Table 1: Inequality (3.2)

Cases	r	τ	L	*Inequality (3.2)*
Case-1: $(a, b) = (r, r) \in E_G$	$r \in (0, 1)$	$\tau \in (0, 0.1]$	$L = \frac{9}{10}$	$\tau + r^4 \leq 9r^2 + \frac{r^4}{4} - 3r^3 + L(r - \frac{r^2}{2})$
Case-2: $(a, b) = (1, 1) \in E_G$	$r = 1$	$\tau \in (0, 0.1]$	$L = \frac{9}{10}$	$\tau + \frac{16}{9} \leq 4 + L\frac{1}{3}$
Case-3: $(a, b) = (r, 0) \in E_G$	$r \in (0, 1)$	$\tau \in (0, 0.1]$	$L = \frac{9}{10}$	$\tau + r^4 \leq 4r^2 + L\frac{r^2}{2}$

This implies $F(H(d_G(Sa, Sb), \varphi(Sa), \varphi(Sb))) = r^4$. By similar computation, we get $F(H(M_G(a, b), \varphi(a), \varphi(b))) = 4r^2$.

Table 1 gives a summary of inequality (3.2) for different values of $(a,b) \in E(G)$ and thus inequality (3.2) holds for all $(a,b) \in E(G)$. Consequently S is an $(F - H)_G$-contraction.

Also it has Property (*) and for $0 \in X_G$, we have $(0, S0) \in E_G$. Thus, all the conditions of Theorem 3.6 are fulfilled and hence S possesses a unique fixed point 0 in X_G.

Next we furnish a numerical example to exhibit the utility of Theorem 4.1.

Example 5.3. Consider the function $a \in X_G = C[0, 1]$ defined by $a(t) = t$, $t \in [0, 1]$. Then we show that, it is a lower solution in X_G for the following integral equation:

$$a(t) = \int_0^t e^{-\tau}\left(\ln(1 + a(s)) + 1\right) ds \tag{5.1}$$

and $e^{-\tau}\left((t+1)\ln(1 + a(t))\right)$ is the unique solution of (5.1).

Proof. First we define a mapping $S : X_G \to X_G$ by

$$(Sa)(t) = \int_0^t e^{-\tau}\left(\ln(1 + a(s)) + 1\right) ds$$

Now taking $K(t, s, a(s)) = e^{-\tau}\ln(s+1) + 1$ and $\tau \leq 0.001$. Then notice the following facts:

1. The function $K(t, s, a(s))$ is nondecreasing in the third variable.
2. By the computation, we get

$$\int_0^t e^{-\tau}\ln(1 + a(s)) ds = e^{-\tau}\left((t+1)\ln(1 + a(t))\right), \qquad t \in [0,1].$$

3. From Figure 1 it is clear that the following inequality holds:

$$t \leq \int_0^t e^{-\tau}\left(\ln(s + 1) + 1\right) ds \tag{5.2}$$

   which implies that $a(t) = t$ is a lower solution for (5.1).

4. From Figure 2, it is clear that the following inequality is true for all $a, b \in [0, 1]$:

$$\begin{aligned} \left|e^{-0.001}\ln(1+a) - \ln(1+b)\right| &\leq e^{(-0.001 + 15/16|b - \ln(1+a)|)}\,|a - b| \\ &\leq e^{(-0.001 + 15/16\max|b - \ln(1+a)|)}\max_{t\in[0,1]}|a - b| \end{aligned} \tag{5.3}$$

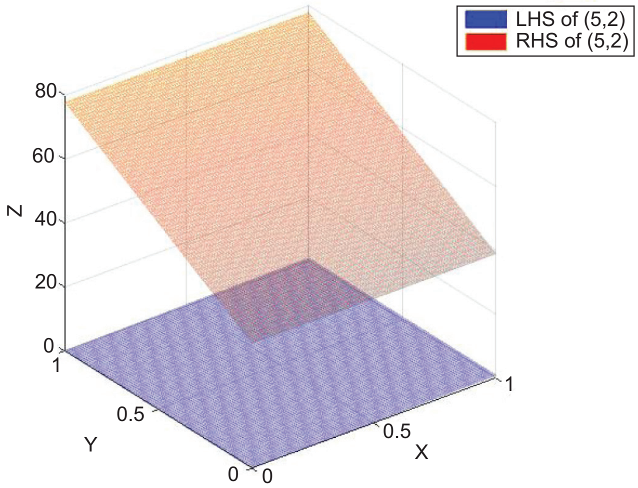

Figure 1: Inequality (5.2).

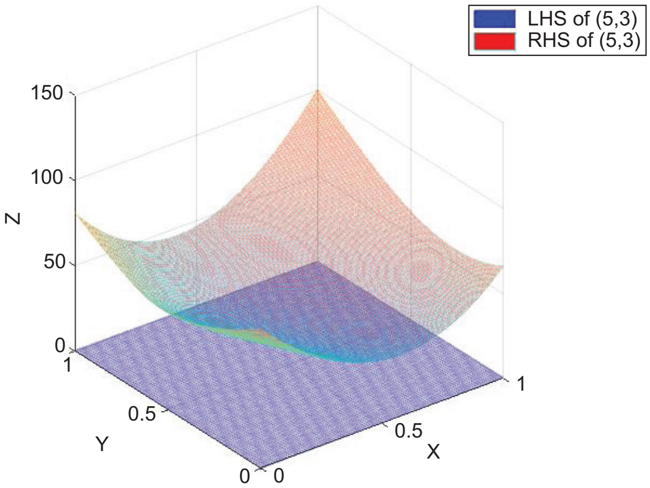

Figure 2: Inequality (5.3).

Hence all the condition of Theorem 3.6 are satisfied and hence there exists a unique fixed point, namely, $(t + 1)\ln(1 + \alpha(t))$ in $C[0, 1]$ of integral equation (5.1).

6. Conclusion

$(F - H)_G$- contraction introduced in this paper is a proper generalisation of F-contraction, F-weak contraction, F_G-contraction, $(H - \varphi)$-contraction, $(F - H)$-contraction, rational type $(F - H)$-contraction and almost $(F - H)$-contraction. Consequently, the existence results proved for $(F - H)_G$-contraction mappings in a metric space equipped with a graph includes many existing fixed point theorems in literature. Fixed point theorem for $(F - H)_G$-contraction mapping in a metric space equipped with a graph is utilised to prove the existence of unique solution of Volterra type integral equations. There is scope for further extensions and generalisations of the fixed point results proved in this work and utilising them in various mathematical problems arising in engineering and science.

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