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On new common fixed point results by using (*JCLR*) property in *b*-metric spaces

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Abstract

In this presented paper, we introduce the concept of $(JCLR)$ property in the sense of b-metric spaces. Further, we obtain common fixed point theorems for weakly compatible along with generalized (CLR) property. Our results extend and improve a very recent theorem in the related literature. An example is also given to support our main result.

Mathematics Subject Classification (2010): 47H09; 47H10

Key words: $(JCLR)$ property; weakly compatible; b-metric spaces

1. Introduction and Preliminaries

Throughout this paper, we denote by $\mathbb{N}, \mathbb{R}_+$ and $\mathbb{R}^n$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

The Banach contraction principle is an important tool of analysis and it is considered as the main source of metric fixed point theory. It guarantees the existence and uniqueness of fixed points of certain self mappings on metric spaces and provides a constructive method to find those fixed points.

Theorem 1.1: ([1]) *Let (X,d) be a complete metric spaces and $T: X \to X$ be a Banach-contraction mapping, i.e.,*

$$d(Tx,Ty) \le kd(x,y)$$

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for all $x, y \in X$, where $k \in (0,1)$. Then T has a unique fixed point.

This principle was extended and improved in many ways and various fixed point theorems were obtained. One way of extending and improving the Banach contraction principle is to replace the complete metric space (X,d) by b-metric spaces which is introduced by Bakhtin [1] (see also Czerwik [2]). Next, we recall some definitions from b-metric spaces as follows:

Definition 1.2: ([3]) *Let X be a nonempty set and $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions, for all $x, y, z \in X$*

1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then (X,d) is called a b-metric space with coefficient s.

As when $s = 1$, a b-metric space is a metric space, we infer that the family of b-metric spaces is larger than the one of metric spaces. Indeed, every metric space is a b-metric space.

Later, several researchers have studied many results in b-metric spaces (see in [4, 5, 2, 6] and references therein).

Next, we give the concepts of convergence in a b-metric space.

Definition 1.3: ([4]) *Let (X,d) be a b-metric space and $\{x_n\}$ be a sequence in X. If there exists $x \in X$ such that*

$$d(x_n, x) \to \textit{as } n \to \infty,$$

then a sequence $\{x_n\}$ is called b-convergent. In this case, we write $\lim_{n\to\infty} x_n = x$.

Proposition 1.4: ([4]) *In a b-metric space (X,d), the following assertions hold.*

($p1$) *A b-convergent sequence has a unique limit.*

($p2$) *In general, a b-metric is not continuous.*

Next, we need the following lemma about b-convergent sequences in the proof of our results.

Lemma 1.5: ([7]) *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$, $\{y_n\}$ be b-convergent to the points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$,

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

Fixed point theorems are statements containing sufficient conditions that ensure the existence of a fixed point. Therefore, one of the central concerns in fixed point theory is to find a minimal set of sufficient conditions which guarantee a fixed point or a common fixed point as the case may be. Common fixed point theorems for contractive type mappings necessarily require a commutativity condition, a condition on the ranges of the mappings, continuity of one or more mappings besides a contractive condition. Other way of extending and improving the Banach contraction principle is to replace the single-value mapping by multi-valued mapping.

In 2002, Aamri and El-Moutawakil [8] introduced the concept of (E.A) property for two single-valued mappings as follows:

Definition 1.6: ([8]) *Let (X,d) be a metric space and $f, g : X \to X$ be two mappings. Then f and g satisfy the property (E.A) if there exists a sequence $\{x_n\} \subseteq X$ such that*

$$\lim_{n\to\infty} fx_n = t = \lim_{n\to\infty} gx_n$$

for some $t \in X$.

Afterwards, Kamran [9] extended the property (E.A) to a hybrid pair of single-valued and multi-valued mappings as follows:

Definition 1.7: ([9]) *Let (X,d) be a metric space, $f: X \to X$ be a single-valued mapping, let $CB(X)$ be the class of all nonempty bounded closed subsets of X and $T: X \to CB(X)$ be a multi-valued mapping. Then a hybrid pair (f,T) satisfy the property (E.A) if there exists a sequence $\{x_n\} \subseteq X$ such that*

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$$

for some $t \in X$ and for some $A \in CB(X)$.

Recently, Sintunavarat and Kumam [10] introduced the new property which is so called "common limit in the range" for two single-valued mappings.

Definition 1.8: ([10]) *Let (X,d) be a metric space and $f,g: X \to X$ be two mappings. Then f and g satisfy the common limit in the range of f (in short, CLR_f property) if there exists a sequence $\{x_n\} \subseteq X$ such that*

$$\lim_{n\to\infty} fx_n = ft = \lim_{n\to\infty} gx_n$$

for some $t \in X$.

Later, Imdad et al. [11] extended the common limit in the range property to a hybrid pair of single-valued and multi-valued mappings.

Definition 1.9: ([11]) *Let (X,d) be a metric space, $f: X \to X$ be a single-valued mapping and $T: X \to CB(X)$ be a multi-valued mapping. Then a hybrid pair (f,T) satisfy the common limit in the range of f (in short, CLR_f property) if there exists a sequence $\{x_n\} \subseteq X$ such that*

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n$$

for some $u \in X$ and for some $A \in CB(X)$.

Recently, Jungck and Rhoades [12] defined the concept of weakly compatible mappings and showed that compatible mappings are weakly compatible but the converse in not true.

Definition 1.10: ([12]) *Let $f,g: X \to X$ be two given mappings. The pair (f,g) is said to be weakly compatible if*

$$fgx = gfx,$$

whenever $fx = gx$.

Definition 1.11: ([12]) *Let (X,d) be a metric space and $f,g: X \to X$ be two mappings. The pair (f,g) is said to be compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

In this paper, we introduce the notion of $(JCLR)$ property in b-metric space without the completeness. We establish some common fixed point results, and we also give some example for supporting the main results. Our theorems modify and generalize the several well-known results given by some authors in metric space. Finally, we apply our main results to prove the existence of periodic solution to delay differential equation.

2. Main Results

In this section, we obtain the unique common fixed point devoid the completeness of b-metric space by using weakly compatibility along with ($JCLR$) property.

We start with the following result.

Definition 2.1: *Let* (X,d) *be a b-metric space with the coefficient* $s \geq 1$ *and* $f,g,H,T : X \to X$ *be four mappings. The pairs* (f,H) *and* (g,T) *are said to satisfy the joint common limit in the range of H and T property (shortly,* $(JCLR_{HT})$ *property) if there exists the sequences* $\{x_n\}$ *and* $\{y_n\}$ *in X such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = Hu = Tu, \tag{1}$$

for some $u \in X$.

Theorem 2.2: *Let* (X,d) *be a b-metric space with* $s \geq 1$ *and* f,g,H *and T be four mappings from X into itself. Further, let the pairs* (f,H) *and* (g,T) *satisfy the* $(JCLR_{HT})$ *property and*

$$d(fx,gy) \leq a_1(x,y)d(Hx,Ty) + a_2(x,y)d(fx,Hx) + a_3(x,y)d(gy,Ty) + a_4(x,y)[d(fx,Ty) + d(gy,Hx)] \tag{2}$$

for all $x,y \in X$, where $a_1,a_2,a_3,a_4 : X \times X \to \left[0,\frac{1}{s}\right)$ and

$$\sup_{x,y\in X}\{a_1(x,y) + a_2(x,y) + a_3(x,y) + 2sa_4(x,y)\} \leq \lambda < \frac{1}{s},$$

whenever λ is a given number. Therefore, f,g,H and T have a coincidence point in X. If the pairs (f,H) and (g,T) are weakly compatible, then f,g,H and T have a unique common fixed point.

Proof. Since the pairs (f,H) and (g,T) satisfy the $(JCLR_{HT})$ property, there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = Hu = Tu \tag{3}$$

for some $u \in X$. Now, we will show that $gu = Tu$. Assume this to contrary that $gu \neq Tu$. By using (2.2), with $x = x_n$ and $y = u$, we obtain that

$$\begin{aligned} d(fx_n,gu) &\leq a_1(x_n,u)d(Hx_n,Tu) + a_2(x_n,u)d(fx_n,Hx_n) \\ &\quad + a_3(x_n,u)d(gu,Tu) + a_4(x_n,u)[d(fx_n,Tu) + d(gu,Hx_n)] \end{aligned} \tag{4}$$

Taking limit supremum as $n \to \infty$ and using Lemma 1.5 and (2.3), we have

$$\begin{aligned} \frac{1}{s}d(Tu,gu) &\leq \limsup_{n\to\infty}\left[a_3(x_n,u)d(gu,Tu) + a_4(x_n,u)d(gu,Hx_n)\right] \\ &\leq \limsup_{n\to\infty}\left[a_3(x_n,u)d(gu,Tu)\right] + \limsup_{n\to\infty}\left[a_4(x_n,u)d(gu,Hx_n)\right] \\ &\leq d(Tu,gu)\limsup_{n\to\infty} a_3(x_n,u) + sd(gu,Tu)\limsup_{n\to\infty} a_4(x_n,u) \\ &\leq d(Tu,gu)\left[\limsup_{n\to\infty} a_3(x_n,u) + s\limsup_{n\to\infty} a_4(x_n,u)\right] \\ &\leq \lambda d(Tu,gu). \end{aligned} \tag{5}$$

This implies that

$$\left(\frac{1}{s}-\lambda\right)d(Tu,gu)\leq 0. \tag{6}$$

Since $\lambda < \frac{1}{s}$, we get

$$d(Tu,gu)=0$$

and hence

$$Tu = gu. \tag{7}$$

Next, we prove that $fu = Tu$. Assume this to contrary that $fu \neq Tu$. By using (2.2), with $x = u$ and $y = y_n$, we obtain that

$$\begin{aligned} d(fu,gy_n) &\leq a_1(u,y_n)d(Hu,Ty_n)+a_2(u,y_n)d(fu,Hu) \\ &\quad +a_3(u,y_n)d(gy_n,Ty_n)+a_4(u,y_n)[d(fu,Ty_n)+d(gy_n,Hu)] \end{aligned} \tag{8}$$

By taking limit supremum as $n \to \infty$ and using Lemma 1.5 and (2.3), we have

$$\begin{aligned} \frac{1}{s}d(fu,Tu) &\leq \limsup_{n\to\infty}\left[a_2(u,y_n)d(fu,Tu)+a_4(u,y_n)d(fu,Ty_n)\right] \\ &\leq \limsup_{n\to\infty}\left[a_2(u,y_n)d(fu,Tu)\right]+\limsup_{n\to\infty}\left[a_4(u,y_n)d(fu,Ty_n)\right] \\ &\leq d(fu,Tu)\limsup_{n\to\infty}a_2(u,y_n)+sd(fu,Tu)\limsup_{n\to\infty}a_4(u,y_n) \\ &\leq d(fu,Tu)\left[\limsup_{n\to\infty}a_2(u,y_n)+s\limsup_{n\to\infty}a_4(u,y_n)\right] \\ &\leq \lambda d(fu,Tu). \end{aligned} \tag{9}$$

It yields that

$$\left(\frac{1}{s}-\lambda\right)d(fu,Tu)\leq 0. \tag{10}$$

Since $\lambda < \frac{1}{s}$, we obtain that

$$d(fu,Tu)=0$$

and so

$$fu = Tu. \tag{11}$$

From (2.3), (2.7) and (2.11), we have u is a coincident point of f,g,H and T.

Now, we assume that $z = fu = Tu = gu = Hu$. It follows from (f,H) is weakly compatible, we have

$$fHu = Hfu$$

and hence

$$fz = fHu = Hfu = Hz. \tag{12}$$

From the pair (g,T) is weakly compatible, we obtain that

$$gTu = Tgu$$

and then

$$gz = gTu = Tgu = Tz. \tag{13}$$

Next, we will show that $z = fz$. By using (2.2) with $x = z$ and $y = u$, we get

$$d(fz,gu) \le a_1(z,u)d(Hz,Tu) + a_2(z,u)d(fz,Hz) + a_3(z,u)d(gu,Tu) + a_4(z,u)[d(fz,Tu) + d(gu,Hz)] \tag{14}$$

Since $z = fu = Tu = gu = Hu$, we obtain that

$$\begin{aligned} d(fz,z) &\le a_1(z,u)d(fz,z) + a_4(z,u)[d(fz,z) + d(z,fz)] \\ &= [a_1(z,u) + 2a_4(z,u)]d(fz,z) \\ &\le \lambda d(fz,z) \end{aligned} \tag{15}$$

and hence

$$(1-\lambda)d(fz,z) \le 0. \tag{16}$$

From $\lambda < \frac{1}{s} < 1$, we have

$$d(fz,z) = 0$$

and so

$$z = fz.$$

Therefore,

$$z = fz = Hz. \tag{17}$$

We show that $z = gz$. To prove this, using (2.2) with $x = u$ and $y = z$, we get

$$d(fu,gz) \le a_1(u,z)d(Hu,Tz) + a_2(u,z)d(fu,Hu) + a_3(u,z)d(gz,Tz) + a_4(u,z)[d(fu,Tz) + d(gz,Hu)] \tag{18}$$

Since $z = fu = Tu = gu = Hu$, we have

$$\begin{aligned} d(z,gz) &\le a_1(u,z)d(z,gz) + a_4(u,z)[d(z,gz) + d(gz,z)] \\ &= [a_1(u,z) + 2a_4(u,z)]d(z,gz) \\ &\le \lambda d(z,gz) \end{aligned} \tag{19}$$

and then

$$(1-\lambda)d(z,gz) \le 0. \tag{20}$$

From $\lambda < \frac{1}{s} < 1$, we obtain that

$$d(z,gz) = 0.$$

It yields that

$$z = gz$$

and so

$$z = gz = Tz. \tag{21}$$

Therefore, we conclude that

$$z = fz = gz = Hz = Tz \tag{22}$$

and hence f,g,H and T have a common fixed point $z \in X$. Let v be an another common fixed point of f,g,H and T. Using (2.2) with $x = z$ and $y = v$, we have

$$d(fz,gv) \le a_1(z,v)d(Hz,Tv) + a_2(z,v)d(fz,Hz) + a_3(z,v)d(gv,Tv) + a_4(z,v)[d(fz,Tv) + d(gv,Hz)]. \tag{23}$$

This implies that

$$\begin{aligned} d(z,v) &\le a_1(z,v)d(z,v) + a_4(z,v)[d(z,v) + d(v,z)] \\ &= [a_1(z,v) + 2a_4(z,v)]d(z,v) \\ &\le \lambda d(z,v) \end{aligned} \tag{24}$$

and so

$$(1-\lambda)d(z,v) \le 0. \tag{25}$$

Therefore,

$$d(z,v) = 0$$

and then

$$z = v. \tag{26}$$

Hence, f,g,H and T have a unique common fixed point.

Corollary 2.3: *Let (X,d) be a b-metric space with $s \ge 1$ and f,g,H and T be four mappings from X into itself. Further, let the pairs (f,H) and (g,T) satisfy the $(JCLR_{HT})$ property and*

$$d(fx,gy) \le c_1 d(Hx,Ty) + c_2 d(fx,Hx) + c_3 d(gy,Ty) + c_4[d(fx,Ty) + d(gy,Hx)] \tag{27}$$

for all $x,y \in X$, where c_1,c_2,c_3,c_4 are real numbers such that

$$c_1 + c_2 + c_3 + 2sc_4 \le \lambda < \frac{1}{s}, \tag{28}$$

whenever λ is a given number. Therefore, f,g,H and T have a coincidence point in X. If the pairs (f,H) and (g,T) are weakly compatible, then f,g,H and T have a unique common fixed point.

Next, we give the example for support our main result.

Example 2.4: *Let $X = (2,12)$ and the mapping $d : X \times X \to [0,\infty)$ defined by*

$$d(x,y) = |x-y|^2$$

for all $x,y \in X$. Therefore, (X,d) is a b-metric space with the coefficient $s = 2$.

Define $f,g,H,T : X \to X$ by

$$fx = \begin{cases} 2 & if\ x \in \{2\} \cup (10,12); \\ 4 & if\ x \in (2,10], \end{cases} \quad gx = \begin{cases} 2 & if\ x \in [2,10]; \\ 3 & if\ x \in (10,12), \end{cases} \quad Hx = \begin{cases} 2 & if\ x = 2; \\ 12 & if\ x \in (2,10]; \\ \dfrac{x+2}{6} & if\ x \in (10,12), \end{cases}$$

and

$$Tx = \frac{x+4}{3}$$

for all $x \in X$.

To prove that the pairs (f,H) and (g,T) satisfy the $(JCLR_{HT})$ property. Let $\{x_n\} = \left\{10 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\{y_n\} = \left\{2 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ be two sequences in X. Then we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \frac{10 + \frac{1}{n} + 2}{6} = \lim_{n\to\infty} Hx_n = H2 = T2 = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} \frac{2 + \frac{1}{n} + 4}{3} = \lim_{n\to\infty} gy_n.$$

Therefore, the pairs (f,H) and (g,T) satisfy the $(JCLR_{HT})$ property.

Next, we will show that the pairs (f,H) and (g,T) are weakly compatible. Suppose that $fz = Hz$ and $gz = Tz$. Then $z = 2$ and so

$$fH2 = f2 = H2 = Hf2 \text{ and } gT2 = g2 = T2 = Tg2.$$

Define $a_1, a_2, a_3, a_4 : X \times X \to \left[0, \frac{1}{2}\right]$ by

$$a_1(x,y) = \frac{1}{15}, a_2(x,y) = \frac{1}{15}, a_3(x,y) = \frac{1}{15}, \text{ and } a_4(x,y) = \frac{1}{15}$$

for all $x, y \in X$. Then

$$\sup_{x,y\in X} \{a_1(x,y) + a_2(x,y) + a_3(x,y) + 2sa_4(x,y)\} = \sup_{x,y\in X} \left\{\frac{1}{15} + \frac{1}{15} + \frac{1}{15} + \frac{4}{15}\right\} = \frac{7}{15} < \frac{1}{2}.$$

Next, we will divide the proof that T satisfies inequality (2.2) into 7 cases.

Case I: For $x = 2$ and $y \in [2,10]$, we have

$$\begin{aligned} d(fx,gy) &= |2-2|^2 \\ &= 0 \\ &\le a_1(x,y)d(Hx,Ty) + a_2(x,y)d(fx,Hx) + a_3(x,y)d(gy,Ty) + a_4(x,y)[d(fx,Ty) + d(gy,Hx)]. \end{aligned}$$

Case II: For $x = 2$ and $y \in (10,12)$, we obtain that

$$\begin{aligned} d(fx,gy) &= |2-3|^2 \\ &= 1 \\ &\le \frac{162}{135} \\ &\le \frac{1}{15}\left|\frac{y-2}{3}\right|^2 + \frac{1}{15}\left|\frac{y-5}{3}\right|^2 + \frac{1}{15}\left[\left|\frac{y-2}{3}\right|^2 + 1\right] \\ &= a_1(x,y)d(Hx,Ty) + a_2(x,y)d(fx,Hx) + a_3(x,y)d(gy,Ty) + a_4(x,y)[d(fx,Ty) + d(gy,Hx)]. \end{aligned}$$

Case III: For $x \in (2,10]$ and $y = 2$, we get

$$
\begin{aligned}
d(fx,gy) &= |4-2|^2 \\
&= 4 \\
&\leq \frac{268}{15} \\
&= \frac{100}{15}+\frac{64}{15}+\frac{104}{15} \\
&= a_1(x,y)d(Hx,Ty)+a_2(x,y)d(fx,Hx)+a_3(x,y)d(gy,Ty)+a_4(x,y)[d(fx,Ty)+d(gy,Hx)].
\end{aligned}
$$

Case IV: For $x,y\in(2,10]$, we have

$$
\begin{aligned}
d(fx,gy) &= |4-2|^2 \\
&= 4 \\
&\leq \frac{1960}{135} \\
&\leq \frac{1}{15}\left|\frac{y-32}{3}\right|^2+\frac{64}{15}+\frac{1}{15}\left|\frac{y-2}{3}\right|^2+\frac{1}{15}\left[\left|\frac{y-8}{3}\right|^2+100\right] \\
&= a_1(x,y)d(Hx,Ty)+a_2(x,y)d(fx,Hx)+a_3(x,y)d(gy,Ty)+a_4(x,y)[d(fx,Ty)+d(gy,Hx)].
\end{aligned}
$$

Case V: For $x\in(2,10]$ and $y\in(10,12)$, we get

$$
\begin{aligned}
d(fx,gy) &= |4-3|^2 \\
&= 1 \\
&\leq \frac{1623}{135} \\
&\leq \frac{1}{15}\left|\frac{y-32}{3}\right|^2+\frac{64}{15}+\frac{1}{15}\left|\frac{y-5}{3}\right|^2+\frac{1}{15}\left[\left|\frac{y-8}{3}\right|^2+81\right] \\
&= a_1(x,y)d(Hx,Ty)+a_2(x,y)d(fx,Hx)+a_3(x,y)d(gy,Ty)+a_4(x,y)[d(fx,Ty)+d(gy,Hx)].
\end{aligned}
$$

Case VI: For $x\in(10,12)$ and $y=[2,10]$, we obtain that

$$
\begin{aligned}
d(fx,gy) &= |2-2|^2 \\
&= 0 \\
&= a_1(x,y)d(Hx,Ty)+a_2(x,y)d(fx,Hx)+a_3(x,y)d(gy,Ty)+a_4(x,y)[d(fx,Ty)+d(gy,Hx)].
\end{aligned}
$$

Case VII: For $x,y\in(10,12)$, we have

$$
\begin{aligned}
d(fx,gy) &= |2-3|^2 \\
&= 1 \\
&\leq \frac{568}{540} \\
&\leq \frac{1}{15}\left|\frac{x-2y-6}{6}\right|^2+\frac{1}{15}\left|\frac{x-10}{6}\right|^2+\frac{1}{15}\left|\frac{y-5}{3}\right|^2+\frac{1}{15}\left[\left|\frac{y-2}{3}\right|^2+\left|\frac{x-16}{6}\right|^2\right] \\
&= a_1(x,y)d(Hx,Ty)+a_2(x,y)d(fx,Hx)+a_3(x,y)d(gy,Ty)+a_4(x,y)[d(fx,Ty)+d(gy,Hx)].
\end{aligned}
$$

From Case I - Case VII, we have all conditions of Corollary 2.3 hold. Hence, f,g,H and T have a unique common fixed point. In this case, 2 is a common fixed point of f,g,H and T.

Corollary 2.5: *Let (X,d) be a b-metric space with $s\geq 1$ and $f:X\to X$ be a given mapping. Suppose that*

$$d(fx,fy)\leq a_1(x,y)d(x,y)+a_2(x,y)d(fx,x)+a_3(x,y)d(fy,y)+a_4(x,y)[d(fx,y)+d(fy,x)]$$

for all $x,y\in X$, where $a_1,a_2,a_3,a_4:X\times X\to\left[0,\frac{1}{s}\right)$ and

$$\sup_{x,y\in X}\{a_1(x,y)+a_2(x,y)+a_3(x,y)+2sa_4(x,y)\}\leq\lambda<\frac{1}{s},$$

whenever λ is a given number. Then, f has a unique fixed point.

3. Application to Delay Differential Equations

In this section, we apply our main result to guarantee the existence of periodic solution to delay differential equation by Corollary 2.5.

Consider the following problem

$$\begin{aligned}\frac{dh(t)}{dt}&=h(t)=g(t,h_t),t\in I,\\ h(0)&=\alpha(0)=h(\alpha)(b),\\ h(\theta)&=\alpha(\theta),\theta\in[-\tau,0],\end{aligned}\tag{29}$$

where $I=[0,b]$, $b>0$ and $g:I\times\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function and $h(t)\in\mathbb{R}^n$ for each $t\in I$. Also, $h_t\in C([-\tau,0],\mathbb{R}^n)$ is defined by

$$h_t(\theta)=h(\theta+t)\tag{30}$$

for all $-\tau\leq\theta\leq 0$, $\tau>0$ and $\theta:[-\tau,0]\to\mathbb{R}^n$ is a continuous function, that is, $\alpha\in C([-\tau,0],\mathbb{R}^n)$ (space of all continuous functions from $[-\tau,0]$ into $\mathbb{R}^n$). It is known that there exists a unique function $h\in C([-\tau,b],\mathbb{R}^n)$ which is absolutely continuous and differentiable on every compact interval $[0,b]$ for $b>0$ and it satisfies

$$\frac{dh(t)}{dt}=h(t)=g(t,h_t),t\in I\tag{31}$$

and

$$h(0)=\alpha(0),h(\theta)=\alpha(\theta),-\tau\leq\theta\leq 0,\tag{32}$$

for a Lipschitz continuous function g. The solution $h(\cdot,\alpha)$ continuously depends on α and g on compact intervals.

For an arbitrary $h\in C(0,b],\mathbb{R}^n)$, define

$$||h||=\sup_{t\in I}|h(t)|,\tag{33}$$

where

$$|h(t)|=|(h_1(t),h_2(t),\ldots,h_n(t))|=\sqrt{h_1^2(t)+h_2^2(t)+\cdots+h_n^2(t)}.\tag{34}$$

Then, $(C(0,b],\mathbb{R}^n),||\cdot||)$ is a Banach space endowed with the metric d defined by

$$d(h,k)=\sup_{t\in I}|h(t)-k(t)|^p, \tag{35}$$

where $h,k\in C(0,b],\mathbb{R}^n)$.

Lemma 3.1: *Problem (3.1) is equivalent to the following integral equation:*

$$h_t(\alpha)(\theta)=\begin{cases} e^{-\lambda\theta}\int_0^b G(t,t^*)(g(t^*,h_{t^*})+\lambda h(t^*))dt^* & if\ t+\theta\geq 0,\\ \alpha(t+\theta) & if\ t+\theta\leq 0,\end{cases}$$

where

$$G(t,t^*)=\begin{cases} \dfrac{e^{-\lambda(t-t^*)}}{1-e^{-p\lambda b}} & if\ 0\leq t^*\leq t+p\theta,\\ \dfrac{e^{-\lambda(t+b-t^*)}}{1-e^{-p\lambda b}} & if\ t+p\theta\leq t^*\leq b.\end{cases} \tag{36}$$

Define

$$K=\left\{h\in C([0,b],\mathbb{R}^n)\,|\,h(0)=\alpha(0)=h(b),\alpha\in C([-\tau,0],\mathbb{R}^n)\right\}$$

and

$$h(t)=e^{-\lambda\theta}\int_0^b G(t-\theta,t^*)(g(t^*,h_{t^*})+\lambda h(t^*))dt^*,$$

where $G(t-\theta,t^*)$ is defined in (3.8). For each $h\in K$, set $g(h)(t)=g(t,h_t)$, where

$$h_t(\theta)=\begin{cases} h(t+\theta) & if\ 0\leq t+\theta\leq b,\\ \alpha(t+\theta) & if\ -\tau\leq t+\theta\leq 0.\end{cases}$$

Define the operator $F:K\to K$ as follows:

$$(F(h))(t)=e^{-\lambda\theta}\int_0^b G(t-\theta,t^*)(g(h)(t^*)+\lambda h(t^*))dt^*.$$

Following assumptions that are necessary to prove the existence and uniqueness of solution of Problem (3.1).

Theorem 3.2: *Let* $a_1,a_2,a_3,a_4:K\times K\to\left[0,\frac{1}{s}\right)$ *be four mappings satisfy*

$$\sup_{h,k\in K}\left\{a_1(h,k)+a_2(h,k)+a_3(h,k)+2sa_4(h,k)\right\}<\frac{1}{s} \tag{37}$$

and there exists $\lambda>0$ such that

$$\begin{aligned} ||(g(t,h)+\lambda h(0))-(g(t,k)+\lambda k(0))||^p &\leq \left(\frac{(1-e^{-\lambda bq})^{\frac{p}{q}}}{b(1-e^{\lambda b})^p}\right)(a_1(h,k)|h(t)-k(t)|^p\\ &+a_2(h,k)|F(h)(t)-h(t)|^p+a_3(h,k)|F(k)(t)-k(t)|^p\\ &+a_4(h,k)[|F(h)(t)-k(t)|^p+|F(k)(t)-h(t)|^p]) \end{aligned} \tag{38}$$

for all $h,k\in K$. Then there exists a unique solution of Problem (3.1).

Proof. Let $h,k \in K$. By (3.10) and thought of Holder's inequality, we have

$$||F(h)(t)-F(k)(t)||^p = ||e^{-\lambda\theta}\int_0^b G(t-\theta,t^*)(g(h)(t^*)+\lambda h(t^*)-g(k)(t^*)-\lambda k(t^*))dt^*||^p$$

$$\le e^{-p\lambda\theta}\left(\int_0^b ||G(t-\theta,t^*)|| \, ||(g(h)(t^*)+\lambda h(t^*))-(g(k)(t^*)+\lambda k(t^*))||dt^*\right)^p$$

$$\le e^{-p\lambda\theta}\left[\left(\int_0^b ||G(t-\theta,t^*)||^q \, dt^*\right)^{\frac{1}{q}}\left(\int_0^b ||(g(h)(t^*)+\lambda h(t^*))-(g(k)(t^*)+\lambda k(t^*))||^p \, dt^*\right)^{\frac{1}{p}}\right]^p$$

$$\le \left(\frac{(1-e^{-\lambda bq})^{\frac{p}{q}}}{(1-e^{\lambda b})^p}\right)(a_1(h(t),k(t))|h(t)-k(t)|^p + a_2(h(t),k(t))|F(h)(t)-h(t)|^p$$

$$+a_3(h(t),k(t))|F(k)(t)-k(t)|^p + a_4(h(t),k(t))[|F(h)(t)-k(t)|^p + |F(k)(t)-h(t)|^p])$$

$$e^{-p\lambda\theta}(\int_0^b G(t-\theta,t^*)^q dt^*)^{\frac{p}{q}}$$

$$\le \left(\frac{(1-e^{-\lambda bq})^{\frac{p}{q}}}{(1-e^{\lambda b})^p}\right)(a_1(h(t),k(t))|h(t)-k(t)|^p + a_2(h(t),k(t))|F(h)(t)-h(t)|^p$$

$$+ a_3(h(t),k(t))|F(k)(t)-k(t)|^p + a_4(h(t),k(t))[|F(h)(t)-k(t)|^p + |F(k)(t)-h(t)|^p])$$

$$e^{-p\lambda\theta}\left[\int_0^{t+p\theta} G(t-\theta,t^*)^q dt^* + \int_{t+p\theta}^b G(t-\theta,t^*)^q dt^*\right]^{\frac{p}{q}}$$

$$\le \left(\frac{(1-e^{-\lambda bq})^{\frac{p}{q}}}{(1-e^{\lambda b})^p}\right)(a_1(h(t),k(t))||h(t)-k(t)||^p + a_2(h(t),k(t))|F(h)(t)-h(t)|^p$$

$$+ a_3(h(t),k(t))|F(k)(t)-k(t)|^p + a_4(h(t),k(t))[|F(h)(t)-k(t)|^p + |F(k)(t)-h(t)|^p])$$

$$e^{-p\lambda\theta}\left[\int_0^{t+p\theta}\left(\frac{e^{-\lambda(t-t^*)}}{1-e^{-p\lambda b}}\right)^q dt^* + \int_{t+p\theta}^b\left(\frac{e^{-\lambda(t+b-t^*)}}{1-e^{-p\lambda b}}\right)^q dt^*\right]^{\frac{p}{q}}$$

$$= a_1(h(t),k(t))|h(t)-k(t)|^p + a_2(h(t),k(t))|F(h)(t)-h(t)|^p$$

$$+ a_3(h(t),k(t))|F(k)(t)-k(t)|^p + a_4(h(t),k(t))[|F(h)(t)-k(t)|^p + |F(k)(t)-h(t)|^p]$$

$$\le a_1(h(t),k(t))d(h,k) + a_2(h(t),k(t))d(Fh,h) + a_3(h(t),k(t))d(Fk,k)$$

$$+ a_4(h(t),k(t))[d(Fh,k)+d(Fk,h)].$$

Therefore, all conditions of Corollary 2.5 are satisfied. Hence, Problem (3.1) has a unique solution.

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