Results in Nonlinear Analysis 7 (2024) No. 2, 127–139
https://doi.org/10.31838/rna/2024.07.02.012
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Stability of an unemployment model with a non-linear job creation

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Abstract

This study introduces a mathematical model for unemployment that incorporates nonlinear functions in the matching process and job creation. Derived from observed data and constraints on job creation, the nonlinearity enhances the model's realism. Through a geometric approach, sufficient conditions are identified to ensure a successful application of global stability analysis. The dynamics of the general system, including thresholds and global stability of the nontrivial equilibrium, are fully determined. Existence and stability of both trivial and non-trivial equilibria are rigorously proven using Lyapunov functions, Jacobian matrices, and the Lozinski measure. Numerical analysis illustrates the theoretical findings, emphasizing the global asymptotic stability of the nontrivial equilibrium under specific conditions.This work offers valuable insights into unemployment dynamics, bridging complex mathematics with practical economic models.

Keywords: Unemployment model; Global asymptotic stability; Geometric approach; Nonlinear dynamics.

Mathematics Subject Classification: 37B25, 34D23, 34D05

1. Introduction

The modeling of unemployment dynamics is central to understanding the intricate relationships between unemployed individuals, employed individuals, and available vacancies. Bilinear and

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Received February 6 2024; Accepted March 15, 2024; Online April 4, 2024

nonlinear unemployment models, represented as ordinary differential systems, have been widely explored in the literature. In the linear unemployment model, described by an ordinary differential system, the matching process between vacancies, V, and unemployed individuals, U, is represented by a bilinear function kUV. The following system as an example [1]:

$$\begin{cases} \dot{U} = A - kUV + \beta E - \mu U, \\ \dot{E} = kUV - \beta E - \alpha E, \\ \dot{V} = \sigma E - \delta V. \end{cases} \tag{1}$$

with A,β,μ,α,σ, and δ as positive constants. While interpretable, this model may fall short in capturing intricate nonlinear behavior.

Contrastingly, the nonlinear unemployment model employs a differential system with a nonlinear matching function, $m(U,V)$ [2, 3], and non-linear job creation, $\sigma(U)$ [4]. This approach assumes a job creation rate proportional to U until a limit is reached. The increased flexibility of this model allows for a more nuanced fit to diverse and complex data but can present greater analytical challenges. El Fadily et al. recently analyzed an unemployment model, incorporating nonlinear matching functions and a linear function of new job creation ($\sigma(E) = \sigma(E)$) as shown in the following system [5]:

$$\begin{cases} \dot{U} = A - m(U)V + \beta E - \mu U, \\ \dot{E} = m(U)V - \beta E - \alpha E, \\ \dot{V} = \sigma E - \delta V, \end{cases} \tag{2}$$

Where :

- $m(0) = 0$;
- m is continuously differentiable in the interior of $\mathbb{R}_+$ and monotone increasing function on $\mathbb{R}_+$.

Assumptions H0 and H1 imply that as the number of unemployed individuals rises, the matching of vacancies correspondingly increases, and conversely, the matching ceases when there are no unemployed individuals. Those assumptions are based on the works of Pissarides and Mortensen [6, 7].

Let $C(E)$ denote the job creation function, where E represents the number of employed individuals. Traditional labor economic models often presuppose a linear relationship between variables, yet this assumption may oversimplify the underlying complexities of the labor market. A non-linear formulation for $C(E)$ provides a more nuanced mathematical characterization that aligns with observed economic phenomena.Considering, for instance, the non-linear effects that emerge in search and matching models as expounded by Mortensen and Pissarides [7]. In their seminal work, job creation and destruction are found to exhibit non-linear behaviors due to frictions and heterogeneous matching between employers and job seekers. This non-linearity captures the intrinsic complexities of labor markets that are not reducible to linear dynamics.

Further, the concept of hysteresis, as discussed by Blanchard and Summers [8], introduces a dependence on past employment levels, leading to a non-linear persistence in unemployment rates. This historical dependence reflects the irreversibility and memory effects inherent in labor market adjustments and thus calls for a non-linear description in the form of $C(E)$. The empirical evidence across different labor market structures also underscores the non-linear patterns in employment growth [9]. Bassanini and Duval [10] elucidate how various policies and institutions may create non-linear effects in employment dynamics, adding another dimension to the intricacy of the function $C(E)$. Thus, non-linear modeling of $C(E)$ provides a more accurate and comprehensive view of job creation dynamics and encompasses a richer set of economic behaviors and offers a more holistic representation of underlying mechanisms, providing a substantial contrast with the limitations inherent in a linear approximation.

Following this rationale, a more generalized system was proposed, introducing a non-linear function of the matching process, as well as specific conditions on the job creation function:

$$\begin{cases} \dot{U} = A - m(U)V + \beta E - \mu U, \\ \dot{E} = m(U)V - \beta E - \alpha E, \\ \dot{V} = C(E) - \delta V, \end{cases} \tag{3}$$

where C is the function of new job creation satisfying specific conditions such as :

- $C(0) = 0$, $C'(0) > 0$;
- C is continuously differentiable in the interior of $\mathbb{R}_+$, increasing and concave on $\mathbb{R}_+$.

For the hypothesis (H_2) the initial condition $C(0) = 0$ reflects that no jobs can be created in the absence of employed individuals. The condition $C'(0) > 0$ is based on the economic assumption that initially, as employment starts to increase, job creation increases as well. This aligns with job creation dynamics, particularly at the beginning of employment changes [11]. It suggests that a minimal level of employment is necessary to stimulate job creation, and empirical studies often find that job creation is initially responsive to increases in employment but may not start immediately due to various market frictions or delays in the hiring process [6].

As for the hypothesis (H_2), the assumption that $C(E)$ is continuously differentiable ensures that the model is mathematically tractable and realistic, reflecting a smooth job creation process. Its increasing nature is consistent with the economic principle that higher employment generally leads to more job creation due to increased economic activity and demand for labor. The concavity of C(E) reflects diminishing returns to job creation, a widely observed phenomenon in labor economics, where each additional employed individual contributes less to job creation than the previous one, possibly due to market saturation, increased competition, or limited resources. This is supported by the works of [6] and the empirical observations noted in various studies, including those of [12], which illustrate diminishing returns in job creation across different sectors and high employment levels.

These conditions generalize the linear case, and the analysis herein explores the dynamics, thresholds, and global stability of this system. The results offer valuable insights into the labor market dynamics and contribute to a robust mathematical understanding of unemployment.

2. Equilibria

We consider the set,

$$T = \left\{ (U,E,V), \quad 0 \le U + E \le \frac{A}{\kappa_1}, 0 \le V \le \kappa_2 \right\} \tag{4}$$

T is bounded and positively invariant, where $\kappa_1 = \min\{\alpha, \mu\}$ and $\kappa_2 = \frac{C\left(\frac{A}{\kappa_1}\right)}{\delta}$. We define the basic reproduction number,

$$R_0 = \frac{m\left(\frac{A}{\mu}\right)C'(0)}{\delta(\alpha + \beta)} \tag{5}$$

The basic reproduction number R_0 serves as a critical threshold determinant. It encapsulates key parameters of the labor market dynamics, offering a clear criterion to distinguish between different systemic behaviors. It offers a single value that summarizes the complex interactions of various parameters in the model (job creation rate, matching function). [13]

We prove the following result of the existence and uniqueness of trivial and non-trivial equilibriums. Assume that hypotheses $(H_0) - (H_3)$ hold. Then System 3 admits two equilibria:

1. a trivial equilibrium $P_0 = \left(\frac{A}{\mu}, 0, 0\right)$. and
2. a positive equilibrium $P = \left(U^*, E^*, V^*\right)$, when $R_0 > 1$.

Proof. Consider the situation where (U, E, V) constitutes an equilibrium for system (1.2). Such an equilibrium can be described by the following equations:

$$\begin{cases} A - \mu U - m(U)V + \beta E = 0 \\ m(U)V - (\alpha + \beta)E = 0 \\ C(E) - \delta V = 0 \end{cases} \tag{6}$$

If E is zero, then it follows that V is also zero, leading to $U = \frac{A}{\mu}$ and the equilibrium point $P_0 = \left(\frac{A}{\mu}, 0, 0\right)$. In the case where E is not zero, the system 3 can be reformulated as:

$$\begin{cases} U = \frac{A}{\mu} - \frac{\alpha E}{\mu}, \\ V = \frac{C(E)}{\delta}, \\ \frac{C(E)}{\delta} m\left(\frac{A}{\mu} - \frac{\alpha E}{\mu}\right) - (\alpha + \beta)E = 0. \end{cases} \tag{7}$$

By defining the function f over the interval $\left[0, \frac{A}{\alpha}\right]$ as

$$f(E) = \frac{C(E)}{\delta} m\left(\frac{A}{\mu} - \frac{\alpha E}{\mu}\right) - (\alpha + \beta)E, \tag{8}$$

the task at hand is to demonstrate that the equation $f(E) = 0$ possesses a singular solution, thereby confirming the existence and uniqueness of the solution for System (2.2). With the continuous differentiability of f over $\left[0, \frac{A}{\alpha}\right]$, ensured by hypotheses H_1 and H_3, the derivative is given by

$$f'(E) = \frac{-\alpha}{\mu} m'\left(\frac{A}{\mu} - \frac{\alpha E}{\mu}\right)\frac{C(E)}{\delta} + m\left(\frac{A}{\mu} - \frac{\alpha E}{\mu}\right)\frac{C'(E)}{\delta} - (\alpha + \beta)$$

Under the condition $R_0 > 1$, it follows that $f'(0) > 0$, and thus, a value E_1 exists in the open interval $\left(0, \frac{A}{\alpha}\right)$ for which $f(E_1) > 0$. Additionally, the inequality $f\left(\frac{A}{\alpha}\right) = -(\alpha + \beta)\frac{A}{\alpha} < 0$ holds. Therefore, by applying the intermediate-value theorem, a unique value E^* is obtained in the interval $\left(E_1, \frac{A}{\alpha}\right)$ satisfying $f(E^*) = 0$. For the proof of uniqueness, it must be shown that $f'(E^*) < 0$.

Through elementary calculations and the consideration of the relationships $\alpha + \beta = \frac{f\left(U^*\right)V^*}{E^*}$ and $V^* = \frac{C\left(E^*\right)}{\delta}$, it is deduced that

$$m\left(\frac{A}{\mu} - \frac{\alpha E*}{\mu}\right)\frac{C'(E*)}{\delta} - (\alpha + \beta) = \frac{-m\left(U^*\right)}{\delta E^*}\left[C\left(E^*\right) - E^* C'\left(E^*\right)\right].$$

With the concavity of C and the condition $C\left(0\right) = 0$, as per hypotheses H_2 and H_3, it follows that

$$C\left(E^*\right) - E^* C'\left(E^*\right) > 0$$

and, given the increasing nature of the function m (see hypothesis (H_1)), the conclusion is reached that $f'(E^*) < 0$. This finalizes the proof.

Altering the assumption from C being concave with $C\left(0\right) = 0$ to the requirement that the ratio $\frac{C(E)}{E}$ is monotonically decreasing leads to the identical conclusion as presented in Theorem 1.

3. Global Stability of the Trivial Equilibrium

We will explore the behavior of the trivial equilibrium, P_0, in system (1.2) through linearization and the application of a specific Lyapunov function. The subsequent theorem encapsulates the key stability properties of P_0.

1. If $R_0 < 1$, then P_0 exhibits global asymptotic stability.
2. If $R_0 > 1$, then P_0 is characterized as unstable.

Proof. We commence by defining a Lyapunov functional candidate $L_1(t)$ for a given positive constant p:

$$L_1(t) = U - \frac{A}{\mu} - \frac{A}{\mu}\ln\left(\frac{\mu U}{A}\right) + \frac{\beta}{2(\mu+\alpha)}\left(U - \frac{A}{\mu} + E\right)^2 + E + pV.$$

The time derivative of this function can be expressed as:

$$\frac{dL_1(t)}{dt} = \left(1 - \frac{A}{\mu U}\right)\dot{U} + \frac{\beta}{\mu+\alpha}\left(U - \frac{A}{\mu} + E\right)(\dot{U} + \dot{E}) + \dot{E} + p\dot{V}.$$

Utilizing system 3, we derive the following:

$$\begin{aligned}\frac{dL_1(t)}{dt} &= \left(1 - \frac{A}{\mu U}\right)(A - m(U)V + \beta E - \mu U) + \frac{\beta}{\mu+\alpha}\left(U - \frac{A}{\mu} + E\right)(A - \mu U - \alpha E)\\ &\quad + m(U)V - (\alpha+\beta)E + p(C(E) - \delta V)\end{aligned}$$

Applying hypotheses H_2 and H_3, and recognizing that $C(E) \le C'(0)E$, we arrive at:

$$\begin{aligned}\frac{dL_1(t)}{dt} &\le -\left[\frac{\mu}{U} + \frac{\beta\mu E}{AU} + \frac{\mu^2\beta}{(\mu+\alpha)A}\right]\left(\frac{A}{\mu} - U\right)^2\\ &\quad - \frac{\alpha\beta\mu}{(\mu+\alpha)A}E^2\\ &\quad + C'(0)\left(p - \frac{\alpha+\beta}{C'(0)}\right)E\\ &\quad + \delta\left(\frac{m\left(\frac{A}{\mu}\right)}{\delta} - p\right)V.\end{aligned}$$

Considering $R_0 < 1$, we find that $\frac{dL_1(t)}{dt} < 0$, confirming the global asymptotic stability of P_0. To analyze the instability of P_0, we examine the Jacobian matrix $J(P_0)$ of system (1.2) at P_0:

$$J(P_0) = \begin{pmatrix} -\mu & \beta & -m\left(\frac{A}{\mu}\right) \\ 0 & -\alpha-\beta & m\left(\frac{A}{\mu}\right) \\ 0 & C'(0) & -\delta \end{pmatrix} \tag{9}$$

The eigenvalues of this matrix are given by $a_1 = -\mu < 0$, a_2, and a_3, satisfying:

$$a_2 + a_3 = -(\alpha+\beta)\delta\left(1 - R_0\right).$$

The existence of a positive eigenvalue when $R_0 > 1$ affirms the instability of P_0.

4. Local Stability of the Positive Equilibrium

The ensuing analysis focuses on the local behavior of the positive equilibrium P in system 3, a critical component for understanding its global behavior.

Assuming conditions (H_0) and (H_1), and given that $R_0 > 1$, the positive equilibrium P exhibits local asymptotic stability.

Proof. Consider the Jacobian matrix $J(P)$ for system (1.2) at equilibrium P, and define its characteristic polynomial as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

with coefficients:

$$\begin{aligned} a_1 &= \mu + \delta + m'(U^*)V^* + \alpha + \beta, \\ a_2 &= \delta\mu + \delta(\alpha+\beta) + \delta\alpha + m'(U^*)V^*\alpha - C'(E^*)m(U^*), \\ a_3 &= \alpha\delta m'(U^*)V^* - \mu\left(\delta\alpha + \delta\beta - C'(E^*)m(U^*)\right). \end{aligned}$$

Applying the established relationships $\alpha + \beta = \frac{m(U^*)V^*}{E^*}$ and $C(E^*) = \delta V^*$, we deduce:

$$\begin{aligned} \delta(\alpha+\beta) - m(U^*)C'(E^*) &= \frac{\delta m(U^*)V^*}{E^*} - m(U^*)C'(E^*), \\ &= \frac{C(E^*)m(U^*)}{E^*} - m(U^*)C'(E^*), \\ &= m(U^*)\frac{C(E^*) - E^*C'(E^*)}{E^*}. \end{aligned}$$

Given that m is monotonically increasing and C is concave with $C(0) = 0$, it follows that $a_1, a_2, a_3 > 0$ and $a_1a_2 - a_3 > 0$. Hence, by Hurwitz's criterion, P is locally asymptotically stable.

Altering the assumption that C is concave with $C(0) = 0$ to the stipulation that $\frac{C(E)}{E}$ is decreasing does not affect the conclusion of Theorem 3.

5. Global Stability of the Positive Equilibrium

The global stability of the positive equilibrium is a significant aspect, and to understand it, we'll first explore Proposition 5.1 that associates this stability with the condition $R_0 > 1$.

The system (1.2) will be uniformly persistent if and only if $R_0 > 1$.

Proof. According to an earlier result, if $R_0 > 1$, the trivial equilibrium $P_0 = \left(\frac{A}{\mu}, 0, 0\right)$ is not stable. Consequently, for $R_0 > 1$, system (1.2) will remain uniformly persistent (refer to Theorem 4.3 in [14].

Next, we define a matrix norm associated with a given vector norm $|.|$ in $\mathbb{R}^n$ (where $n \in \mathbb{N}$) and denote it also by $|.|$. This matrix norm is linked with the Lozinski measure as:

$$\mu(B) = \lim_{h\to 0^+} \frac{|I + hB| - 1}{h}, \tag{10}$$

where I represents the identity matrix.

The following theorem encapsulates our main result:

Assuming that hypotheses $(H_0) - (H_3)$ are valid and that $R_0 > 1$, the equilibrium P is globally asymptotically stable in T.

Proof. To demonstrate this theorem, it's adequate to select an appropriate vector norm in $\mathbb{R}^3$ and a matrix $A(x)$ satisfying:

$$\bar{q}_2 := \limsup_{t\to\infty} \sup_{x_0\in K} \frac{1}{t}\int_0^t \mu_1\left(B\left(x\left(s,x_0\right)\right)\right)ds < 0,$$

where μ is the Lozinski $\breve{i}$ measure, $x=(U,E,V)$, $B = A_g A^{-1} + AJ^{[2]}A^{-1}$, and the vector field of (1.2) is mapped by $x \mapsto g(x)$.

To prove this, we need to compute the Jacobian matrix J of system (1.2) at x and its second additive compound matrix $J^{[2]}$:

$$J = \begin{pmatrix} -\mu - m'(U)V & \beta & -m(U) \\ m'(U)V & -(\alpha+\beta) & m(U) \\ 0 & C'(E) & -\delta \end{pmatrix},$$

$$J^{[2]} = \begin{pmatrix} -\mu - m'(U)V - (\alpha+\beta) & m(U) & m(U) \\ C'(E) & -\mu - m'(U)V - \delta & 0 \\ 0 & m'(U)V & -(\alpha+\beta) \end{pmatrix}.$$

For $A(x) = diag\left\{1, \frac{E}{V}, \frac{E}{V}\right\}$, we find:

$$A_g A^{-1} = diag\left\{0, \frac{E'}{E} - \frac{V'}{V}, \frac{E'}{E} - \frac{V'}{V}\right\},$$

and further break down the matrix B:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

$$B_{11} = -\mu - m'(U)V - (\alpha+\beta), \quad B_{12} = \frac{V}{E}(m(U)m(U)),$$

$$B_{21} = \begin{pmatrix} \frac{E}{V}C'(E) \\ 0 \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} -\mu - \delta - m'(U)V + \frac{E'}{E} - \frac{V'}{V} & \beta \\ m'(U)V & -\alpha - \delta - \delta + \frac{E'}{E} - \frac{V'}{V} \end{pmatrix}.$$

Now, let $\bar{\mu}_1$ symbolize the Lozinski $\breve{i}$ measure corresponding to the norm $(u,v,w) \in \mathbb{R}^3 \mapsto \left|(u,v,w)\right| = \max\left\{|u|, |v|+|w|\right\}$. The following inequality is derived by conducting a component-wise decomposition of the Lozinski measure $\bar{\mu}_1(B)$ with respect to a pre-defined vector norm. The upper bounds g_1 and g_2 are formulated by incorporating the eigenvalues of the diagonal submatrices B_{11} and B_{22}, as well as the absolute magnitudes of the off-diagonal elements B_{12} and B_{21}:

$$\bar{\mu}_1(B) \le \sup\{g_1, g_2\},$$

where

$$g_1 = \mu_1(B_{11}) + |B_{12}|,$$
$$g_2 = \mu_1(B_{22}) + |B_{21}|.$$

From this, we derive the following:

$$\begin{aligned} \mu_1(B_{11}) &= -\mu - m'(U)V - (\alpha+\beta), \\ |B_{12}| &= m(U)\frac{V}{E}, \\ |B_{21}| &= \frac{E}{V}C'(E). \end{aligned}$$

Furthermore, by employing a method detailed in [15], we derive:

$$\mu_1(B_{22}) = \frac{E'}{E} - \frac{V'}{V} - \delta - \min\{\mu,\alpha\},$$

and consequently:

$$g_1 = -\mu - m'(U)V + m(U)\frac{V}{E} - (\alpha+\beta),$$
$$g_2 = \frac{E}{V}C'(E) + \frac{E'}{E} - \frac{V'}{V} - \delta - \min\{\mu,\alpha\}.$$

This leads us to:

$$\frac{E'}{E} = \frac{m(U)V}{E} - (\alpha+\beta),$$
$$\frac{V'}{V} = \frac{C(E)}{V} - \delta,$$

and after substituting, we have:

$$g_1 = \frac{E'}{E} - \mu - m'(U)V,$$
$$g_2 = \frac{E'}{E} - \min\{\mu,\alpha\} - \frac{1}{V}\big(C(E) - EC'(E)\big).$$

Given that m is increasing and C is concave with $C(0)=0$, we have $m' \geq 0$ and $C(E) - EC'(E) > 0$, thus:

$$g_1 \leq \frac{E'}{E} - \mu,$$
$$g_2 \leq \frac{E'}{E} - \min\{\mu,\alpha\}.$$

If we define $\eta = \min\{\mu,\alpha\}$, and since system (1.2) is uniformly persistent when $R_0 > 1$, we can find $c > 0$ and $t_0 > 0$ such that for $t > t_0$:

$$c \leq E(t) \leq \frac{A}{\eta}, \quad \text{and} \quad c \leq V(t) \leq \frac{A}{\eta},$$

for all initial conditions in K, and for $t > t_0$:

$$\frac{1}{t}\int_0^t \bar{\mu}_1(B)\,ds \leq \frac{1}{t}\int_0^{t_0} \bar{\mu}_1(B)\,ds + \frac{1}{t}\log\frac{E(t)}{E(t_0)} - \eta\frac{t-t_0}{t} \leq \frac{-\eta}{2},$$

for all initial conditions in K, implying:

$$\bar{q}_2 < 0.$$

This concludes the proof.

By modifying the assumption that C is concave with $C(0)=0$ to the assumption that $\frac{C(E)}{E}$ is monotonically decreasing, we reach the same conclusion as in Theorem 4.

6. Numerical Results

In this section, we provide numerical simulations to illustrate the effectiveness of our theoretical results. For this, we consider the following practical unemployment situations.

6.1. Non-linear modeling of labor market dynamics in Morocco

The study's model was used to simulate the employment dynamics in Morocco. The model parameters and initial conditions were chosen based on data gathered from the High Commission for Planning (HCP) of Morocco [16]. Below, we discuss each parameter and initial condition in detail.

The specific form $C(E) = s(1 - \exp(-b_1 E))$ was chosen for several compelling reasons.

First, this form ensures that $C(E)$ satisfies the natural boundary conditions $C(0) = 0$ and $C'(\infty) = 0$, which are realistic in the context of labor markets. When $E = 0$, no new jobs can be created, satisfying $C(0) = 0$. As E becomes very large, the marginal impact of adding more employed individuals on job creation diminishes, making $C'(\infty) = 0$ a realistic condition.

Second, this form contains an exponential term $\exp(-b_1 E)$, which introduces a natural slowing down effect as E increases. This captures the real-world phenomenon where initially, an increase in employment significantly boosts job creation, but as the number of employed individuals reaches a certain level, additional increases have a diminishing impact. This is in line with the law of diminishing returns, a well-established concept in economics [17]. The parameter $b1$ controls this diminishing effect, making the model more realistic for scenarios where an increase in employment has a progressively lesser impact on job creation.

Third, the form is analytically tractable, allowing for easier mathematical analysis and derivation of key performance indicators like R_0. This is essential for policy simulations and forecasting, which are common applications of such models.

Lastly, the parameters s and b_1 offer flexibility in adjusting the model to fit diverse labor market conditions, making it adaptable for empirical validation. The parameter s scales the overall job creation, while b_1 adjusts the sensitivity of job creation to changes in employment.

Therefore, the chosen form $C(E) = s(1 - \exp(-b_1 E))$ offers a balanced combination of realism, analytical convenience, and adaptability, making it a robust choice for modeling job creation dynamics in Morocco and potentially other labor markets.

- $m = 0.39$: The matching rate was set to 39%, aligning with the employment rate in Morocco.
- $\beta = 0.02$: This parameter represents the rate at which employed individuals become unemployed. It was set to 0.02 to reflect the transitional dynamics of the labor market.
- $\mu = 0.01$: This rate represents how quickly unemployed individuals exit the labor force, either by finding a job or for other reasons like retirement.
- $\alpha = 0.01$: This is the job destruction rate, indicating how quickly jobs become obsolete or are eliminated.
- $\delta = 0.01$: This is the rate at which job vacancies are filled or expire. The average time to fill a vacancy in Morocco is approximately 6 months, which justifies this value.
- $\sigma = 0.0001$: This value is based on $C'(0)$, the derivative of the job creation function at $E = 0$.
- $s = 0.05$: This is the job filling rate,
- $b1 = 0.1$: This parameter represents the non-linearity in the job creation function $C(E)$ and is set to 0.1 to capture the complexities of the labor market.
- $U_0 = 1.1$: The initial unemployed population was set to 1.1 million.
- $E_0 = 12$: The initial employed population was set to 12 million.
- $V_0 = 0.2$: The initial number of job vacancies was set to 200,000 to reflect the labor market conditions.
- $A = 0.3 \times 10^6$: The inflow rate of new job seekers into unemployment is 300,000.

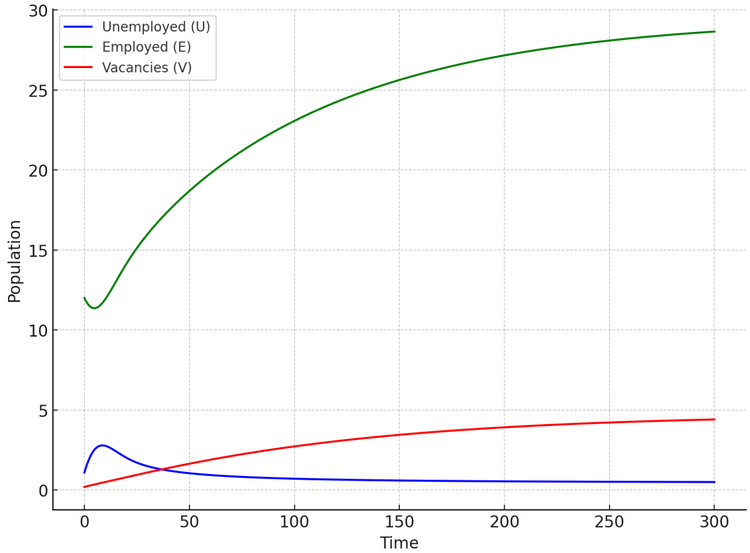

Figure 1: Labor Dynamics in Morocco.

The calculated value of R_0 is approximately 1.95, which above the threshold of 1, proving global asymptotic stability.

6.2. Linear vs Non-linear models: An empirical assessment

Let's compare the linear case of job creation function [5] with the nonlinear case. Both the linear and non-linear models were simulated to understand their predictive power in representing the labor market dynamics in Morocco. In the linear model, σ was determinedas $C'(0)$, the derivative of the job creation function $C(E)$ at $E = 0$. This ensures that both the linear and non-linear models start with the same "initial slope," making the comparison more insightful. It allows the linear model to approximate the non-linear model at least at the initial stages, thus providing a baseline for further divergence as time progresses. The choice of σ as $C'(0)$ in the linear model is a mathematical convenience that ensures both models start from the same initial conditions, facilitating a comparison.

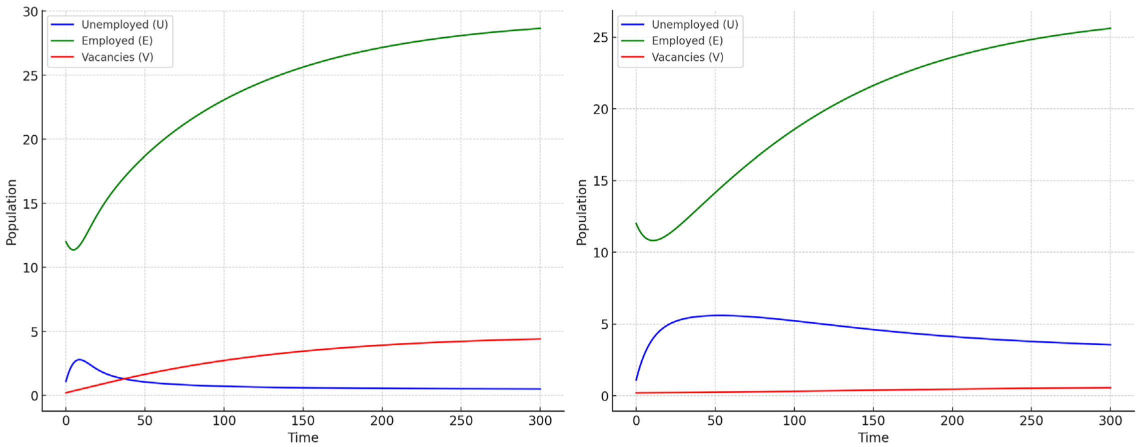

Figure 2: Comparative Labor Dynamics in Morocco, between linear model (right) and non-linear model (left)

Both the linear and non-linear models demonstrate a decrease in unemployment and an increase in employment over time. However, the non-linear model exhibits a more gradual rate of change, which could be a more realistic depiction of labor market dynamics. Real-world labor markets often face frictions, matching inefficiencies, and other complex factors that make instantaneous transitions unlikely.

On the other hand, the linear model shows more abrupt changes, which, while useful for capturing sudden shocks in the labor market, may not be as effective for modeling ongoing, steady-state behavior.

The comparative analysis reveals that while both models capture essential features of the labor market, the non-linear model exhibits a more realistic temporal evolution of employment and unemployment rates. The steep initial increases in both employment and unemployment rates in the linear model seem less aligned with empirical observations, making the non-linear model a more suitable choice for policy analysis.

6.3. Impact of Scaling Factor s and Non-linear Term b_1 on Labor Market Dynamics

To probe the sensitivity of the model to variations in the job creation function $C(E)$, we specifically examine the effects of the scaling factor s and the non-linear term b_1. These parameters were selected due to their direct influence on the rate of job creation, a critical aspect of any labor market.

The following sets of parameters are considered:

Table 1: Parameter sets used in the numerical simulations.

Set	A	m	β	μ	α	δ	s	b_1
1	0.3	0.39	0.02	0.01	0.01	0.01	0.05	0.1
2	0.3	0.39	0.02	0.01	0.01	0.01	0.05	0.2
3	0.3	0.39	0.02	0.01	0.01	0.01	0.1	0.1
4	0.3	0.39	0.02	0.01	0.01	0.01	0.1	0.2

based on this sets of parameters, numerical simulations were made,as seen in figure 3

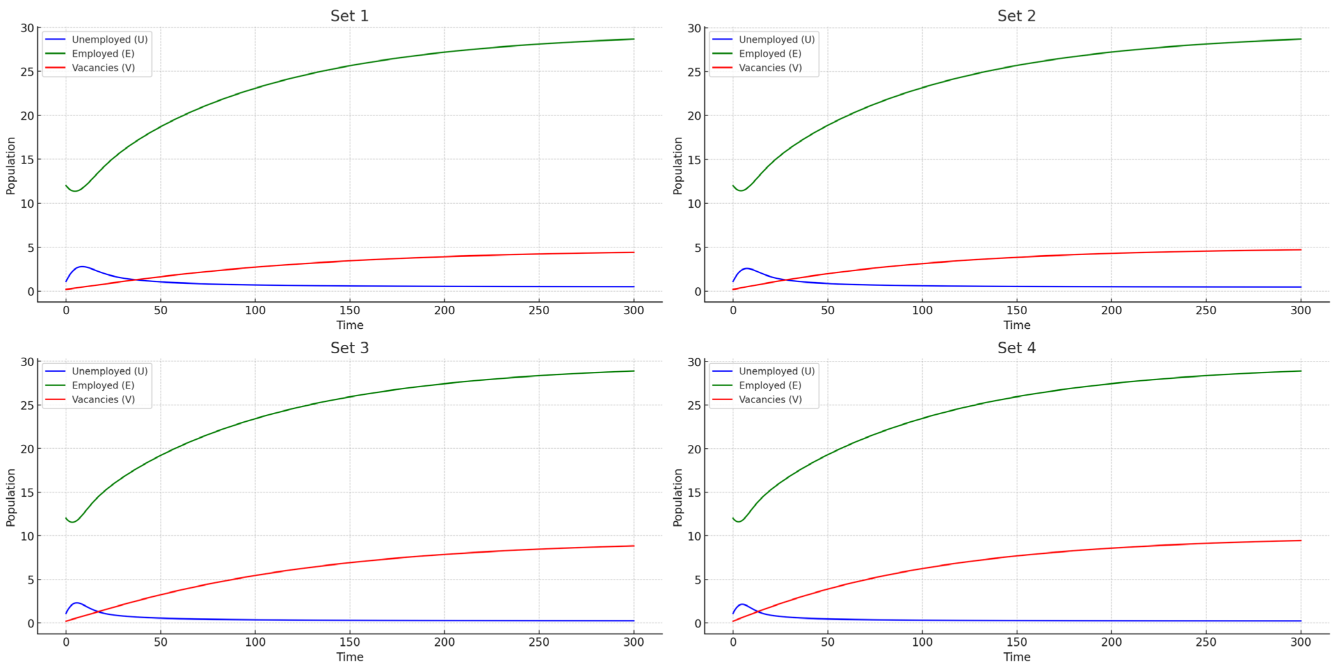

Figure 3: Labor market dynamics for four sets of parameters.

The calculated values of R_0 for each set are as follows:

- Set 1: $R_0 = 195$
- Set 2: $R_0 = 390$
- Set 3: $R_0 = 390$
- Set 4: $R_0 = 780$

The equilibrium points for each set are:

- Set 1: $(U, E, V) = (0.48, 29.52, 4.74)$
- Set 2: $(U, E, V) = (0.46, 29.54, 4.99)$
- Set 3: $(U, E, V) = (0.24, 29.76, 9.49)$
- Set 4: $(U, E, V) = (0.23, 29.77, 9.97)$

The results indicate a noticeable sensitivity of the labor market dynamics to changes in s and b_1. For higher values of s and b_1, the equilibrium points shift towards a larger number of vacancies, implying a more vibrant labor market. The R_0 values above 1 for all sets confirm global asymptotic stability, suggesting that the system will converge to these equilibrium points for the different initial conditions.

7. Conclusion

The mathematical model presented in this study for analyzing unemployment dynamics introduces non-linear functions that describe the matching process and job creation. The primary achievements of this work can be summarized as follows:

Generalized model with Non-linear functions: The inclusion of non-linear functions for the matching process and job creation rate offers a more nuanced representation of the unemployment system. The chosen forms for $C(E)$ and $m(U)$ as non-linear are grounded in empirical data and governmental impacts on job creation, distinguishing this model from traditional bilinear forms.

Existence and uniqueness of equilibriums: Conditions were derived that yield both trivial and non-trivial equilibriums. The parameters of the system were expressed in a manner that ensures the unique existence of these equilibriums.

Global asymptotic stability: Through the application of geometric approach methods, the global asymptotic stability of the non-trivial equilibrium was established. Such stability means the system will converge to a stable state over time, regardless of initial conditions. This finding has been empirically reinforced through numerical simulations under various parameter settings, solidifying its practical relevance. This is invaluable for predicting long-term trends and for the crafting of policies that aim for sustainable employment levels. Within the mathematical framework, the basic reproduction number $R_0 > 1$ emerges not merely as a threshold condition but as a key determinant of global asymptotic stability. Our results delineate two distinct behavioral regimes separated by this number. Specifically, when $R_0 < 1$, the trivial equilibrium point P_0 is globally asymptotically stable, signifying a labor market that will naturally gravitate towards this equilibrium even when perturbed. Conversely, when $R_0 > 1$, this system reveals a non-trivial positive equilibrium that is also globally asymptotically stable, indicating a fundamentally different state of the labor market that is nonetheless equally stable in the long run. This nuanced understanding of R_0 goes beyond its conventional interpretation as a simple bifurcation parameter. It allows us to predict, with mathematical certainty, the long-term behavior of labor markets under different conditions.

Superiority of Non-linear job creation function: Our analysis distinctly illustrates the advantages of incorporating a non-linear function for job creation, especially when contrasted with traditional linear models. The non-linear formulation is adept at capturing the complex dynamics of labor markets, particularly in scenarios influenced by policy interventions or external shocks. This stands in marked contrast to linear models, which, while computationally simpler, often fail to capture the true complexity of labor markets. The validity of our non-linear approach is further substantiated by numerical simulations conducted using real-world parameter values, thereby reinforcing the theoretical insights and underlining the model's empirical relevance.

Relevance and future directions: The model's flexibility and alignment with observed data make it applicable to diverse scenarios. The mathematical rigor and generalization contribute to the understanding of unemployment dynamics and can guide future research and policymaking.

This study represents a significant step forward in the mathematical modeling of unemployment. By embracing non-linearity that reflect real-world complexities, it offers a comprehensive framework that balances theoretical soundness, practical relevance, and mathematical rigor.

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