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Estimate the shape parameter for the Kumaraswamy distribution via some estimation methods

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Abstract

This paper shows how to estimate one of the two shape parameters of Kumaraswamy distribution (KD) using two estimation methods. The first one is the rank set sampling (RSS) estimation method and the second one is the Bayes estimation method. The rank set sampling was employed as a non-Bayes estimator. In addition, Bayes estimators were used based on asymmetric loss function (LINEX) by utilizing four kinds of informative prior one single prior (Gamma) and three double prior (Gamma-exponential, Gamma-chi-squared, and Chi-squared-exponential). The study was conducted of these estimators using a Monte Carlo simulation study and the shape parameter estimates were compared depending on the mean squared error (MSE). Furthermore, simulation results indicate that the performance of non-Bayes estimators for some cases is better than Bayes estimators, and Bayes estimators under LINEX loss function corresponding to double informative priors (gamma–exponential and gamma–chi-squared) are the best prior distribution for all cases and all sample sizes. Finally, the program (MATLAB 2015) was used to get the mathematical outcomes.

Key words and phrases. Kumaraswamy distribution, Shape parameter, Rank set, LINEX, Double priors.

1. Introduction

Kumaraswamy distribution is used in practical areas for modeling data, like medication, engineering, economics, and physics and it is applicable to many natural phenomena, such as the

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height of individuals, atmospheric temperatures, daily stream flow, etc. A two shape parameters Kumaraswamy's distribution has been proposed by poondi Kumaraswamy (1980) [1] for bounded lower and upper variables. In probability and statistics, Kumaraswamy's double-bound distribution is a group for continuous probability distributions bounded over the interval (0, 1) that differ in the values of the positive shape parameters φ and υ. KD utilizing various estimations are presented by many authors. Al-Noor and Ibraheem (2016) [2] utilized the maximum likelihood (ML), Bayes, and Bayes empirical estimation methods to obtain an estimate of the unknown shape parameter for KD within complete samples and take the other shape parameter is known. Sultana and et al (2018) [3] Estimated and examined parameters for the KD using hybrid censoring systems. Bantan and et al (2019) [4] putted truncated-inverted KD, and Ghosh (2019) [5] found weighted KD for both multivariate and bivariate variables. Abraheem and et al (2020) [6] used ML and Bayes techniques to obtain an estimate for the shape parameter of the KD under asymmetric loss function based on three kinds for informative priors (one double and two single), also find the approximate value of this parameter using methods expansion. Mohamoud and et al (2022) [7] used non-Bayes and Bayes estimation methods to estimate the unknown shape parameter, as well as the approximate methods (Newton and False Position) to find the approximate value of this parameter. The ML is gotten as a non-Bayes estimator, and the Bayes estimator based on asymmetric loss function (NLINEX and De-groot) based on four informative priors (one single and three double).

This work presents two methods of estimations for one of the shape parameters φ of the KD depending on complete data and assuming the other parameter υ is known. These methods are the RSS estimator as the traditional method and the Bayes method which is derived by using informative priors provided by (gamma-exponential, gamma-chi-square, chi-square-exponential, and gamma) based on the asymmetric loss function LINEX.

2. Some Properties for KD

The function for probability density (pdf) and the function for cumulative distribution (cdf) for KD are obtained from [8] as follows:

$$f(t;\varphi,\upsilon) = \varphi\upsilon t^{(\upsilon-1)}(1 - t^{\upsilon})^{\varphi-1}; \qquad 0 < t < 1 \tag{1}$$

$$F(t;\varphi,\upsilon) = 1 - (1 - t^{\varphi})^{\varphi}; \qquad 0 < t < 1 \tag{2}$$

where the shape parameters $\varphi,v > 0$.

The respective reliability function $R(t)$ and failure rate function $h(t)$ for the KD are defined as follows:

$$R(t) = 1 - F(t;\varphi,\upsilon) = (1 - t^{\upsilon})^{\varphi}; \qquad 0 < t < 1 \tag{3}$$

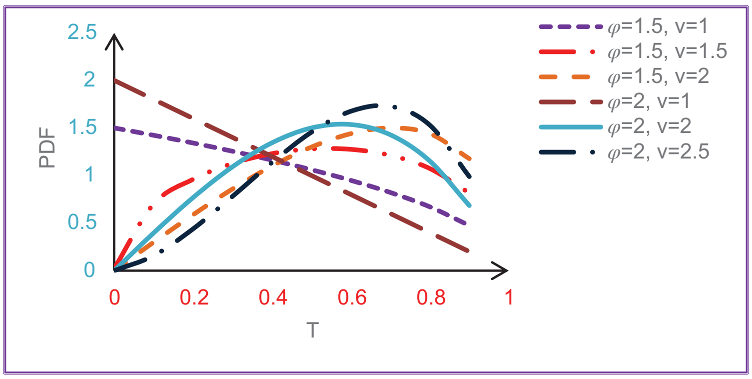

Figure 1: The probability density function for KD with different value of shape parameters.

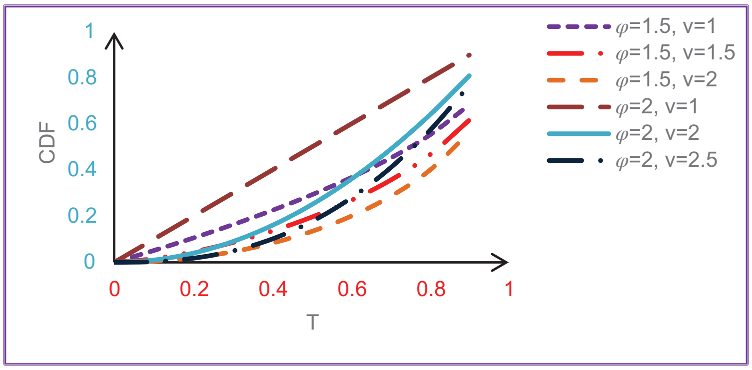

Figure 2: The cumulative distribution function for KD with different value of shape parameters.

$$h(t) = \frac{f(t)}{R(t)} = \frac{\varphi \upsilon t^{\upsilon-1}}{(1-t^{\upsilon})}; \qquad 0 < t < 1 \tag{4}$$

The methodology of the research can be illustrated by the following flowchart.

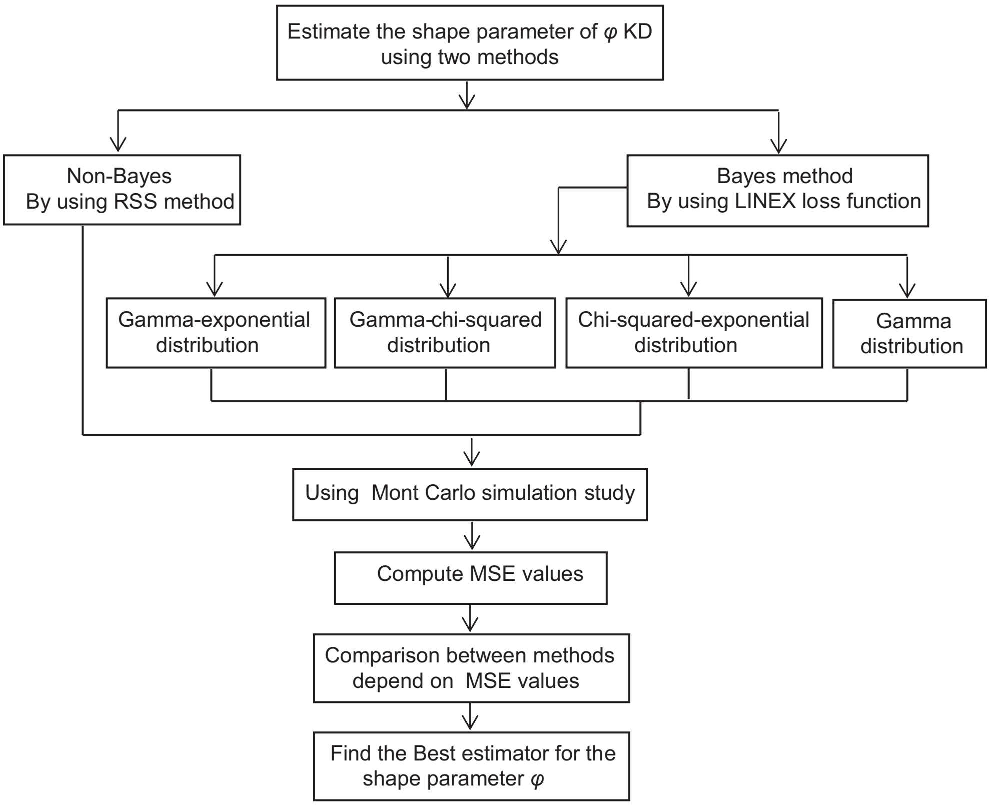

Figure 3: Flowchart for the methodology of the research.

3. Rank Set Sampling Estimation

One of the important topics in statistic is estimator parameter. It has taken a large place in statistical studies. One of the most important ways to estimate the parameters is the rank set sampling (RSS) method [9] [10].

The probability density function of KD which obtained by increasing ordering random sampling ($t_{(1)}$, $t_{(2)}$, …, $t_{(n)}$) is:

$$f(t_{(i)}) = \frac{n!}{(i-1)!(n-1)!}[F(t_{(i)})]^{i-1}[1-F(t_{(i)})]^{n-i} f(t_{(i)}) \tag{5}$$

Let $k = \frac{n!}{(i-1)!(n-1)!}$ with substitution equation (1) and equation (2) in equation (5), gets:

$$f(t_{(i)})_{\mathrm{KD}} = k[1-(1-t_{(i)}^{\upsilon})^{\varphi}]^{i-1}[1-(1-(1-t_{(i)}^{\upsilon})^{\varphi})]^{n-i}[\varphi\upsilon t_{(i)}^{\upsilon-1}(1-t_{(i)}^{\upsilon})^{\varphi-1}]$$

so

$$f(t_{(i)})_{\mathrm{KD}} = k\varphi\upsilon t_{(i)}^{\upsilon-1}[1-(1-t_{(i)}^{\upsilon})^{\varphi}]^{i-1}[1-t_{(i)}^{\upsilon}]^{\varphi(n-i+1)-1} \tag{6}$$

The complete data likelihood function $L_{KD}^{RSS}(\varphi,\upsilon \mid \underline{t})$ for a given order sample ($t_{(1)}$, $t_{(2)}$, …, $t_{(n)}$) can be expressed by:

$$L_{KD}^{RSS}(\varphi,\upsilon \mid \underline{t}) = k^n \varphi^n \upsilon^n \prod_{i=1}^{n} t_{(i)}^{\upsilon-1} \prod_{i=1}^{n}\left[1-\left(1-t_{(i)}^{\upsilon}\right)^{\varphi}\right]^{i-1} \prod_{i=1}^{n}\left[1-t_{(i)}^{\upsilon}\right]^{\varphi(n-i+1)-1} \tag{7}$$

The natural log-likelihood function is:

$$\ell_{KD}^{RSS} = \ln\left(L_{KD}^{RSS}(\varphi,\upsilon \mid \underline{t})\right)$$

That is

$$\ell_{KD}^{RSS} = n\ \ln(k) + n\ \ln(\varphi) + n\ln(\upsilon) + (\upsilon-1)\sum_{i=1}^{n}\ln(t_{(i)}) + \sum_{i=1}^{n}(i-1)\ln\left(1-\left(1-t_{(i)}^{\upsilon}\right)^{\varphi}\right)$$
$$+\sum_{i=1}^{n}(\varphi(n-i+1)-1)\ \ln\left(1-t_{(i)}^{\upsilon}\right) \tag{8}$$

Derived equation (8) with respect to unknown parameter φ and setting it equal to zero yields:

$$\frac{\partial \ell_{KD}^{RSS}}{\partial \varphi} = \frac{n}{\varphi} - \sum_{i=1}^{n}\frac{(i-1)\left(1-t_{(i)}^{\upsilon}\right)^{\varphi}\ \ln\left(1-t_{(i)}^{\upsilon}\right)}{1-\left(1-t_{(i)}^{\upsilon}\right)^{\varphi}} + \sum_{i=1}^{n}(n-i+1)\ln\left(1-t_{(i)}^{\upsilon}\right) = 0. \tag{9}$$

The Rank set sampling estimators denoted by $\hat{\varphi}_{KD}^{RSS}$, it can be obtained by solving the equation (9), but equation (9) is nonlinear equation. Therefore, iterative technique (Newton-Raphson method) is applied to solve this equation and find $\hat{\varphi}_{KD}^{RSS}$.

4. Standard Bayes Estimation Method

Bayes estimators are based on the posterior pdf of the unknown parameter given both data and some prior density for this parameter.

The standard Bayes estimator method consists of the following steps:

- Given the random variables t_1, t_2, …, t_n, find a conditional density function for the parameter φ:

$$P_0(\varphi \mid t) = \frac{L(\varphi \mid \underline{t})\, g(\varphi)}{\int_{\varphi} L(\varphi \mid \underline{t})\, g(\varphi) d\varphi} \tag{10}$$

The result of joining the likelihood function $L(\varphi|\underline{t})$ and density function for the prior distribution $P_0(\varphi)$ is referred to as the posterior density function (PDF) for the parameter φ [11].

- Using loss function $L(\hat{\varphi},\varphi)$ that is defined to be real function satisfies the following conditions:

   i. $L(\hat{\varphi},\ \varphi) \geq 0$ for all estimator $\hat{\varphi}$ and for all parameter φ.

   ii. $L(\hat{\varphi},\ \varphi) = 0$ for $\hat{\varphi} = \varphi$.

- In this paper, considering the linear exponential loss function (LINEX) as an asymmetric loss function, it can be described as follows [12]:

$$L_{c_1}(\hat{\varphi},\ \varphi) = e^{c_1\Delta} - c_1\Delta - 1, c_1 \neq 0 \tag{11}$$

where $\Delta = \hat{\varphi} - \varphi$ and $\hat{\varphi}$ is an estimate of the parameter φ.

- Find the risk function for the parameter $\hat{\varphi}$:

$$\text{Risk }(\hat{\varphi}) = E_{P_0}[L(\hat{\varphi},\varphi)] = \int_{\varphi} L(\hat{\varphi},\varphi)\mathrm{P}_0(\varphi \mid \underline{t})\, d\varphi \tag{12}$$

The value of $\hat{\varphi}$ that minimizes the loss function is called the standard Bayesian estimate. The Bayes estimator for φ can is acquired depending on (LINEX) signified by $\hat{\varphi}_{CL}$ as follows:

$$\hat{\varphi}_{CL} = \frac{1}{2c_1} \ln\left[\frac{E_P(e^{c_1\varphi}|\underline{t})}{E_P(e^{-c_1\varphi}|\underline{t})}\right] \tag{13}$$

The first prior-distribution for the shape parameter φ is gamma-distribution of hyper-parameters α and β for pdf introduced in [13]:

$$P_1(\varphi) = d_1 G_1(\varphi) \tag{14}$$

where

$$d_1 = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \text{ and } G_1(\varphi) = \varphi^{\alpha-1} e^{-\varphi\beta}, \quad \varphi > 0, \quad \alpha,\beta > 0$$

The first PDF for the parameter φ for the KD is:

$$P_{01}(\varphi \mid \underline{t}) = \frac{\varphi^{n+\alpha-1} e^{-\varphi(\beta-T)} (\beta - T)^{n+\alpha}}{\Gamma(n+\alpha)} \tag{15}$$

where $T = \sum_{i=1}^{n} \ln(1 - t_i^{v})$

The second prior-distribution is the exponential-distribution together of the hyper-parameter **c** for the pdf defined as:

$$P_2(\varphi) = d_2 G_2(\varphi) \tag{16}$$

where

$$d_2 = C \text{ and } G_2(\varphi) = e^{-\varphi C};\ \varphi > 0,\ c > 0$$

The second PDF for the shape parameter φ of the KD is:

$$P_{02}(\varphi \mid \underline{t}) = \frac{\varphi^{n} e^{-\varphi(c-T)} (c - T)^{n+1}}{\Gamma(n+1)} \tag{17}$$

The third prior-distribution is the double prior distribution (gamma and exponential) of the parameter φ is obtaining by combining equation (14) and equation (16) as follows [14]:

$$P_3(\varphi) \propto G_1(\varphi)G_2(\varphi)$$

or

$$P_3(\varphi) = k_1\varphi^{\alpha-1}e^{-\varphi(\beta+c)},\ \varphi > 0,\ \alpha,\ \beta,\ c > 0 \tag{18}$$

where

$$k_1 = \frac{(\beta+c)^{\alpha}}{\Gamma(\alpha)}$$

And the third PDF of φ for the KD depend on gamma-exponential is:

$$p_{03}(\varphi \mid \underline{t}) = \frac{\varphi^{n+\alpha-1}\, e^{-\varphi(B+c-T)}(B+c-T)^{n+\alpha}}{\Gamma(n+\alpha)} \tag{19}$$

From equation (19), we conclude: $(\varphi \mid \underline{t}) \sim \text{Gamma}(\alpha_1, \beta_1)$ where $\alpha_1 = (n+\alpha)$, and $\beta_1 = (\beta + c - T)$.

- The chi-squared prior distribution of φ together with hyper-parameter c_2 is defined by [11] as the following:

$$P(\varphi \mid \underline{t}) = d_4 G_4(\varphi) \tag{20}$$

where

$$d_4 = \frac{1}{2^{\frac{c_2}{2}}\,\Gamma\left(\frac{c_2}{2}\right)} \text{ and } G_4(\varphi) = \varphi^{\left(\frac{c_2}{2}-1\right)} e^{-\frac{\varphi}{2}}, \varphi > 0, c_2 > 0$$

The double prior distribution (gamma-chi-square) of φ can be obtained by combining equation (14) and equation (20) as in [15] as follows:

$$P_5(\varphi) \propto G_1(\varphi)G_4(\varphi)$$

or

$$P_5(\varphi) = k_2\varphi^{\left(\alpha+\frac{c_2}{2}-2\right)} e^{-\varphi\left(\beta+\frac{1}{2}\right)}, \varphi > 0, \alpha, \beta,\ c_2 > 0 \tag{21}$$

where

$$k_2 = \frac{\left(\beta+\frac{1}{2}\right)^{\left(\alpha+\frac{c_2}{2}-1\right)}}{\Gamma\left(\alpha+\frac{c_2}{2}-1\right)}$$

The PDF of the parameter φ for the KD based on $P_5(\varphi)$ for given data $\underline{t}$ is:

$$P_{05}(\varphi \mid \underline{t}) = \frac{\varphi^{\left(n+\alpha+\frac{c_2}{2}-2\right)} e^{-\varphi(B+0.5-T)} \left(B+0.5-T\right)^{\left(n+\alpha+\frac{c_2}{2}-1\right)}}{\Gamma\left(n+\alpha+\frac{c_2}{2}-1\right)} \tag{22}$$

From equation (22), we conclude: $(\varphi \mid \underline{t}) \sim \text{Gamma}(\alpha_2, \beta_2)$ where $\alpha_2 = (n + \alpha + 0.5c_2 - 1)$ and $\beta_2 = (\beta + 0.5 - T)$.

- Chi-Squared-Exponential priors of the parameter φ can be obtained by combining equation (16) and equation (20) as follows:

$$\mathrm{P}_6(\varphi) \propto \mathrm{G}_2(\varphi)\mathrm{G}_4(\varphi)$$

or

$$P_6(\varphi) = k_3 \varphi^{\left(\frac{c_2}{2}-1\right)} e^{-\varphi\left(c+\frac{1}{2}\right)}, \quad \varphi > 0, c, c_2 > 0 \tag{23}$$

where

$$k_3 = \frac{\left(c + \frac{1}{2}\right)^{\left(\frac{c_2}{2}\right)}}{\Gamma\left(\frac{c_2}{2}\right)}$$

The PDF of the parameter φ for the KD based on $\mathrm{P}_6(\varphi)$ for given data $\underline{t}$ is:

$$P_{06}(\varphi \mid \underline{t}) = \frac{\varphi^{\left(n+\frac{c_2}{2}-1\right)} e^{-\varphi(c+0.5-T)} \left(c + 0.5 - T\right)^{\left(n+\frac{c_2}{2}\right)}}{\Gamma\left(n + \frac{c_2}{2}\right)} \tag{24}$$

From equation (24), we conclude: $(\varphi \mid \underline{t}) \sim \text{Gamma}(\alpha_3, \beta_3)$ where $\alpha_3 = (n + 0.5c_2)$ and $\beta_3 = (c + 0.5 - T)$.

5. Bayes Estimators under (LINEX) Based Loss Function

In this section, Bayes estimators are obtained of the parameter φ of the KD corresponding to various posterior distributions:

- Corresponding to $P_{01}(\varphi \mid \underline{t})$
  By using equation (12) and equation (15), gets:

$$\begin{aligned}
E_{P_{01}}[(e^{c_1\varphi} \mid \underline{t})] &= \int_{\varphi} e^{c_1\varphi}\, \mathrm{P}_{01}(\varphi \mid \underline{t})\, d\varphi \\
&= \int_0^\infty e^{c_1\varphi} \frac{\varphi^{n+\alpha-1} e^{-\varphi(\beta-T)} \left(\beta - T\right)^{n+\alpha}}{\Gamma(n+\alpha)}\, d\varphi \\
&= \frac{\left(\beta - T\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \varphi^{n+\alpha-1} e^{-\varphi(\beta - T - c_1)}\, d\varphi
\end{aligned}$$

Let $y = \varphi(\beta - T - c_1)$ then $\varphi = \dfrac{y}{(\beta - T - c_1)}$ and $d\varphi = \dfrac{dy}{(\beta - T - c_1)}$

$$E_{P_{01}}[(e^{c_1\varphi} \mid \underline{t})] = \frac{(\beta - T)^{n+\alpha}}{\Gamma(n+\alpha)(\beta - T - c_1)^{n+\alpha}} \int_0^\infty \mathrm{y}^{n+\alpha-1} e^{-\mathrm{y}}\ d\mathrm{y}$$

But $\int_0^\infty \mathrm{y}^{n+\alpha-1} e^{-\mathrm{y}}\ d\mathrm{y} = \Gamma(n+\alpha)$

So $$E_{P_{01}}[(e^{c_1\varphi} \mid \underline{t})] = \frac{\left(\beta - T\right)^{n+\alpha}}{\left(\beta - T - c_1\right)^{n+\alpha}} \tag{25}$$

By the same way, have:

$$E_{P_{01}}[(e^{-c_1\varphi} \mid \underline{t})] = \frac{\left(\beta - T\right)^{n+\alpha}}{\left(\beta - T + c_1\right)^{n+\alpha}} \tag{26}$$

Substituting equation (25) and equation (26) in equation (13) to find the Bayes estimators of the parameter φ based on compounded LINEX loss function depend on the gamma prior informative.

$$\hat{\varphi}_{BCLg} = \frac{n+\alpha}{2c_1} \ln\left[\frac{\beta - T + c_1}{\beta - T - c_1}\right] \tag{27}$$

- Corresponding to $P_{03}(\varphi \mid \underline{t})$
  Under the gamma-exponential prior distribution, the Bayes estimator for the parameter φ depending on compounded LINEX loss function corresponding to $P_{03}(\varphi \mid \underline{t})$ can be obtained by using equation (12) and equation (19) and by the same above way, gets:

$$\hat{\varphi}_{BCLge} = \frac{n+\alpha}{2c_1} \ln\left[\frac{\beta + c - T + c_1}{\beta + c - T - c_1}\right] \tag{28}$$

- Corresponding to $P_{05}(\varphi \mid \underline{t})$
  The Bayes estimator for the parameter φ depending on compounded LINEX loss function corresponding to $P_{05}(\varphi \mid \underline{t})$ under gamma-chi-squared prior distribution can be found by using equation (12) and equation (22) and by the same way, gets:

$$\hat{\varphi}_{BCLgch} = \frac{n+\alpha+0.5c_2-1}{2c_1} \ln\left[\frac{\beta + 0.5 - T + c_1}{\beta + 0.5 - T - c_1}\right] \tag{29}$$

- Corresponding to $\mathrm{P}_{06}(\varphi \mid \underline{t})$
  By the same way, the Bayes estimator of the parameter φ depending on compounded LINEX loss function corresponding to $\mathrm{P}_{06}(\varphi \mid \underline{t})$ under chi-squared-exponential prior distribution can be found by using equation (12) and equation (24), gets:

$$\hat{\varphi}_{BCLche} = \frac{n+0.5c_2}{2c_1} \ln\left[\frac{c + 0.5 - T + c_1}{c + 0.5 - T - c_1}\right] \tag{30}$$

6. Simulation Study

Simulation methods are widely used in many branches of statistics. They can be used to evaluate the behavior of models as well as for some random variables. The stages of the process can be summarized as follows:

Stage 1:

- In Table 1, default values for the constants and parameters for the simulation experiments are summed up.

Table 1: The values are used in simulation experiments.

Sample sizes	n	10, 25, 50, 100
Shape parameter	φ	1.5, 2, 2.5
Hyper parameters	α	2
	β	1
	c	0.5
	c_2	2.5
LINEX value	c_1	0.1
Sample Replicate Number	L	1000

- In order to examine the effect of the shape parameters of the KD on the estimates, nine different cases were taken in the following form, when $\upsilon > \varphi$, $\upsilon = \varphi$ and $\upsilon < \varphi$.

Stage 2:
At this stage, generating random samples U_i (i = 1, 2, …, n) are distributed according to a continuous uniform distribution on the unit interval (0, 1). They can be made utilizing the inverse transformation technique of the cdf as follows:

$$U = F(t) \tag{31}$$

$$t = F^{-1}(U) \tag{32}$$

Now, substituting equation (2) in equation (31), yields:

$$U_i = F(t_i) = 1 - (1 - t^{\upsilon})^{\varphi};\ 0 < t < 1;\ \ \varphi, \upsilon > 0$$

Simplify this equation, gets:

$$t_i = [1 - (1 - U_i)^{1/\varphi}]^{1/\upsilon};\ \ i = 1,\ldots,n \tag{33}$$

Stage 3:
The MSE was used to compare how well different estimation techniques perform in finding an estimate for the shape parameter φ.

$$MSE(\hat{\varphi}) = \frac{1}{L}\sum_{j=1}^{L}(\hat{\varphi}_j - \varphi)^2 \tag{34}$$

where
L : Sample replicated number.
$\hat{\varphi}$: The estimate for the φ at j^{th} repeat.

7. Discussion of the Simulation Study

The simulation results of MSE related to the parameter φ for the KD distribution using non-Bayes (RSS) and Bayes methods are shown in Table 2 with different cases.

Table 2: Values MSE for RSS and Bayes estimators of shape parameter φ for KD.

n	**RSS Method** $\hat{\varphi}_{KD}^{RSS}$	**Prior**	**Bayes Method** $\hat{\varphi}_{BCL}$
		Case (I): $\varphi = 1.5$, $\upsilon = 1$	
10	*0.1760*	*Gamma-Exponential*	*0.1849*
		Gamma-Chi-squared	*0.1988*
		Chi-squared-Exponential	*0.2229*
		Gamma	*0.2842*
		Best Prior	*Gamma-Exponential*
25	*0.0724*	*Gamma-Exponential*	*0.0835*
		Gamma-Chi-squared	*0.0862*
		Chi-squared-Exponential	*0.0893*
		Gamma	*0.0996*
		Best Prior	*Gamma-Exponential*
50	*0.0358*	*Gamma-Exponential*	*0.0433*
		Gamma-Chi-squared	*0.0442*
		Chi-squared-Exponential	*0.0448*
		Gamma	*0.0477*
		Best Prior	*Gamma-Exponential*
100	*0.0189*	*Gamma-Exponential*	*0.0202*
		Gamma-Chi-squared	*0.0204*
		Chi-squared-Exponential	*0.0205*
		Gamma	*0.0211*
		Best Prior	*Gamma-Exponential*
		Case (II): $\varphi = 1.5$, $\upsilon = 1.5$	
10	*0.1663*	*Gamma-Exponential*	*0.1606*
		Gamma-Chi-squared	*0.1721*
		Chi-squared-Exponential	*0.1897*
		Gamma	*0.2409*
		Best Prior	*Gamma-Exponential*
25	*0.0739*	*Gamma-Exponential*	*0.0882*
		Gamma-Chi-squared	*0.0913*
		Chi-squared-Exponential	*0.0946*
		Gamma	*0.1060*
		Best Prior	*Gamma-Exponential*
50	*0.0376*	*Gamma-Exponential*	*0.0441*
		Gamma-Chi-squared	*0.0448*
		Chi-squared-Exponential	*0.0456*
		Gamma	*0.0479*
		Best Prior	*Gamma-Exponential*
100	*0.0194*	*Gamma-Exponential*	*0.0218*
		Gamma-Chi-squared	*0.0221*
		Chi-squared-Exponential	*0.0222*
		Gamma	*0.0230*
		Best Prior	*Gamma-Exponential*
		Case (III): $\varphi = 1.5$, $\upsilon = 2$	
10	*0.17426*	*Gamma-Exponential*	*0.17429*
		Gamma-Chi-squared	*0.1865*
		Chi-squared-Exponential	*0.2086*
		Gamma	*0.2629*
		Best Prior	*Gamma-Exponential*
25	*0.0725*	*Gamma-Exponential*	*0.0866*
		Gamma-Chi-squared	*0.0895*
		Chi-squared-Exponential	*0.0929*
		Gamma	*0.1040*
		Best Prior	*Gamma-Exponential*

(*continues*)

Table 2: Continued

n	RSS Method $\hat{\varphi}_{KD}^{RSS}$	Prior	Bayes Method $\hat{\varphi}_{BCL}$
50	*0.0353*	*Gamma-Exponential*	*0.0422*
		Gamma-Chi-squared	*0.0429*
		Chi-squared-Exponential	*0.0436*
		Gamma	*0.0462*
		Best Prior	*Gamma-Exponential*
100	*0.0193*	*Gamma-Exponential*	*0.0220*
		Gamma-Chi-squared	*0.0221*
		Chi-squared-Exponential	*0.0223*
		Gamma	*0.0229*
		Best Prior	*Gamma-Exponential*
		Case (IV): $\varphi = 2, \upsilon = 1$	
10	*0.1865*	*Gamma-Exponential*	*0.2456*
		Gamma-Chi-squared	*0.2560*
		Chi-squared-Exponential	*0.3161*
		Gamma	*0.3876*
		Best Prior	*Gamma-Exponential*
25	*0.0891*	*Gamma-Exponential*	*0.1302*
		Gamma-Chi-squared	*0.1323*
		Chi-squared-Exponential	*0.1442*
		Gamma	*0.1561*
		Best Prior	*Gamma-Exponential*
50	*0.0591*	*Gamma-Exponential*	*0.0710*
		Gamma-Chi-squared	*0.0715*
		Chi-squared-Exponential	*0.0745*
		Gamma	*0.0772*
		Best Prior	*Gamma-Exponential*
100	*0.0391*	*Gamma-Exponential*	*0.0359*
		Gamma-Chi-squared	*0.0362*
		Chi-squared-Exponential	*0.0369*
		Gamma	*0.0380*
		Best Prior	*Gamma-Exponential*
		Case (V): $\varphi = 2, \upsilon = 2$	
10	*0.1679*	*Gamma-Exponential*	*0.2276*
		Gamma-Chi-squared	*0.2358*
		Chi-squared-Exponential	*0.2903*
		Gamma	*0.3528*
		Best Prior	*Gamma-Exponential*
25	*0.0918*	*Gamma-Exponential*	*0.1420*
		Gamma-Chi-squared	*0.1448*
		Chi-squared-Exponential	*0.1581*
		Gamma	*0.1724*
		Best Prior	*Gamma-Exponential*
50	*0.0563*	*Gamma-Exponential*	*0.0686*
		Gamma-Chi-squared	*0.0691*
		Chi-squared-Exponential	*0.0720*
		Gamma	*0.0749*
		Best Prior	*Gamma-Exponential*
100	*0.0401*	*Gamma-Exponential*	*0.0349*
		Gamma-Chi-squared	*0.0350*
		Chi-squared-Exponential	*0.0357*
		Gamma	*0.0365*
		Best Prior	*Gamma-Exponential*

(*continues*)

Table 2: Continued

n	RSS Method $\hat{\varphi}_{KD}^{RSS}$	Prior	Bayes Method $\hat{\varphi}_{BCL}$
		Case (VI): φ =2, υ = 2.5	
10	*0.1772*	*Gamma-Exponential*	*0.2304*
		Gamma-Chi-squared	*0.2382*
		Chi-squared-Exponential	*0.2924*
		Gamma	*0.3532*
		Best Prior	*Gamma-Exponential*
25	*0.0942*	*Gamma-Exponential*	*0.1382*
		Gamma-Chi-squared	*0.1403*
		Chi-squared-Exponential	*0.1528*
		Gamma	*0.1647*
		Best Prior	*Gamma-Exponential*
50	*0.0567*	*Gamma-Exponential*	*0.0725*
		Gamma-Chi-squared	*0.0732*
		Chi-squared-Exponential	*0.0763*
		Gamma	*0.0799*
		Best Prior	*Gamma-Exponential*
100	*0.0407*	*Gamma-Exponential*	*0.0383*
		Gamma-Chi-squared	*0.0385*
		Chi-squared-Exponential	*0.0393*
		Gamma	*0.0403*
		Best Prior	*Gamma-Exponential*
		Case (VII): φ = 2.5, υ = 1	
10	*0.2457*	*Gamma-Exponential*	*0.3242*
		Gamma-Chi-squared	*0.3215*
		Chi-squared-Exponential	*0.4094*
		Gamma	*0.4665*
		Best Prior	*Gamma-Chi-squared*
25	*0.1859*	*Gamma-Exponential*	*0.1839*
		Gamma-Chi-squared	*0.1847*
		Chi-squared-Exponential	*0.2060*
		Gamma	*0.2202*
		Best Prior	*Gamma-Exponential*
50	*0.1680*	*Gamma-Exponential*	*0.1095*
		Gamma-Chi-squared	*0.1096*
		Chi-squared-Exponential	*0.1157*
		Gamma	*0.1188*
		Best Prior	*Gamma-Exponential*
100	*0.1541*	*Gamma-Exponential*	*0.05745*
		Gamma-Chi-squared	*0.05749*
		Chi-squared-Exponential	*0.0591*
		Gamma	*0.0599*
		Best Prior	*Gamma-Exponential*
		Case (VIII): φ =2.5, υ = 2.5	
10	*0.2603*	*Gamma-Exponential*	*0.3226*
		Gamma-Chi-squared	*0.3190*
		Chi-squared-Exponential	*0.4022*
		Gamma	*0.4553*
		Best Prior	*Gamma-Chi-squared*
25	*0.1902*	*Gamma-Exponential*	*0.1850*
		Gamma-Chi-squared	*0.1853*
		Chi-squared-Exponential	*0.2062*
		Gamma	*0.2185*
		Best Prior	*Gamma-Exponential*

(*continues*)

Table 2: Continued

n	RSS Method $\hat{\varphi}_{KD}^{RSS}$	Prior	Bayes Method $\hat{\varphi}_{BCL}$
50	*0.1683*	*Gamma-Exponential*	*0.1077*
		Gamma-Chi-squared	*0.1078*
		Chi-squared-Exponential	*0.1138*
		Gamma	*0.1171*
		Best Prior	*Gamma-Exponential*
100	*0.1499*	*Gamma-Exponential*	*0.0549*
		Gamma-Chi-squared	*0.0551*
		Chi-squared-Exponential	*0.0567*
		Gamma	*0.0578*
		Best Prior	*Gamma-Exponential*
		Case (IX): φ =2.5, υ = 3	
10	*0.2659*	*Gamma-Exponential*	*0.3332*
		Gamma-Chi-squared	*0.3286*
		Chi-squared-Exponential	*0.4130*
		Gamma	*0.4619*
		Best Prior	*Gamma-Chi-squared*
25	*0.1834*	*Gamma-Exponential*	*0.1694*
		Gamma-Chi-squared	*0.1690*
		Chi-squared-Exponential	*0.1874*
		Gamma	*0.1971*
		Best Prior	*Gamma-Exponential*
50	*0.1751*	*Gamma-Exponential*	*0.11756*
		Gamma-Chi-squared	*0.11761*
		Chi-squared-Exponential	*0.1240*
		Gamma	*0.1270*
		Best Prior	*Gamma-Exponential*
100	*0.1515*	*Gamma-Exponential*	*0.0563*
		Gamma-Chi-squared	*0.0565*
		Chi-squared-Exponential	*0.0582*
		Gamma	*0.0594*
		Best Prior	*Gamma-Exponential*

- Table 2 includes nine various cases with MSE values for the shape parameter φ estimator using RSS and Bayes estimation methods. The following is clear from it:

1. For all cases, when n=10, RSS estimators is best from Bayes estimators.
2. For all cases, when (n = 10, 25, 50, 100), the best prior for Bayes estimators depending on LINEX loss function is (**gamma-exponential**) except case(VII) to case(IX) with (n = 10) is (**gamma–chi–squared**).
3. From case (I) to case (III), MSE values for the RSS estimators are less than Bayes estimators for all sample size.
4. From case (IV) to case (VI), when (n = 100), MSE values for the Bayes estimators are less than RSS estimators.
5. From case (VII) to case (IX), when (n = 50, 100) the MSE values associated with every one of the priors for the Bayes estimators as well as when (n = 25) the MSE values associated with the two priors for the Bayes estimators relying upon LINEX loss function (gamma–chi-squared and gamma–exponential) are less than RSS estimators.

8. Conclusion

The outcomes obtained in the current work are summed up depending on simulation results of estimating the shape parameter φ of the KD distribution with the assumption the other parameter υ is known based on complete data, the most conclusions can be drawn as follows:

- The MSE values of double prior distribution depending on LINEX loss function are less than single prior distribution for all cases.
- The LINEX loss function with c_1=0.1 is the best loss function for the Bayes estimators correspondent to gamma–exponential and gamma–chi-squared priors for all cases and all sample sizes.
- For all cases when (n=10), the MSE values for the RSS estimates are less than Bayes estimates.
- When increasing the value of the shape parameter, the MSE values for the RSS and Bayes estimates increase for all cases and all sample sizes.

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