Results in Nonlinear Analysis 7 (2024) No. 1, 14–23
https://doi.org/10.31838/rna/2024.07.01.002
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Vieta-Lucas spectral collocation method for solving fractional order volterra integro-differential equations

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Abstract

The shifted Vieta-Lucas polynomial approach is taken into account for the numerical solution of linear and nonlinear fractional-order integro-differential equations of the Volterra type. Fractional derivatives are described in the Caputo sense. The suggested method reduces the complexity of these problems to the linear or nonlinear solution of algebraic equations. The convergence of the recommended strategy is studied in detail. The computing efficiency of this approach is then illustrated with certain numerical examples, and a comparison with prior research is made.

Key words and phrases: fractional order Volterra integro-differential equations, Caputo type fractional derivative, Vieta-Lucas spectral collocation method, Residual error function.

Mathematics Subject Classification: 2010: 65N20; 41A30

1. Introduction

The properties of derivatives and integrals of non-integer orders are explored in the area of mathematics known as fractional calculus (see [1]). Many of the fundamental properties of the differentiation of integer order and the integration of n-fold are preserved because it is an extension of classical calculus. In 1823, Abel applied a derivative of order 1/2 to the integral equation solution

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Received September 20, 2023; Accepted October 18, 2023; Online November 24, 2023

of the Tautochrone problem, which was the first application of fractional calculus (see [2, 3]). A number of scientific disciplines, such as economics [4], medicine [5], viscoelastic dynamics [6], solid mechanics [7], and fluid-dynamic traffic models [8], have also lately been linked to the discovery of fractional differential equations (FDEs). One of the most important and effective ways for simulating differential equations of many types is spectral methods ([9–12]). One of these methods' most important qualities is its ability to produce accurate results with relatively little flaws.

For instance, the orthogonality condition of Vieta Lucas polynomials is utilized to approximate the functions of the period [a,b]. In these techniques, which strongly rely on polynomials, (see [13–15]).

There are several advantages to employing Vieta-Lucas polynomials:

Vieta-Lucas polynomials exhibit a multitude of intriguing and beneficial properties. Utilizing Vieta-Lucas polynomials as fundamental functions yields highly precise solutions. The utilization of Vieta-Lucas polynomials in research contributions is comparatively limited in comparison to other polynomial types. By selecting the modified set of shifted Vieta-Lucas polynomials as the basis functions and retaining only a few terms of the modes, it becomes feasible to generate highly accurate approximations with reduced computational effort. Furthermore, the associated errors are minimal.

The structure of this study is as follows. The definitions of the fractional derivatives and shifting Vieta-Lucas polynomials are briefly discussed in Section 2 as well as some preliminary remarks. We demonstrate the numerical application of the suggested method and applications in Sections 3 and 4. Section 5 provides the conclusion.

2. Preliminaries and notations

2.1 Some definitions of fractional derivatives

Definition 1.

The fractional derivative of order 0 < *nuleq* < 1 in the Caputo sense is provided for $p(\eta) \in \mathrm{H}_1(0,b)$ by:

$$^{C}D^{\nu} p(\eta) = \frac{1}{\Gamma(1-\nu)} \int_0^{\eta} \frac{p'(\tau)}{(\eta-\tau)^{\upsilon}} d\tau \qquad t > 0.$$

Definition 2.

$$D^{\alpha}\eta^{m} = \begin{cases} 0, & m \in \{0,1,2,\ldots,\lceil \alpha \rceil - 1\}, \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\eta^{m-\alpha}, & m \in \mathbb{N} \wedge m \geq \lceil \alpha \rceil, \end{cases}$$

2.2 The shifting Vieta-Lucas polynomials: some concepts

In this section, we give the definitions of the shifted Vieta-Lucas polynomials (VLPs), their notations, and their properties [16]. The majority of our studies have concentrated on an orthogonal polynomial class. The recurrence relations and analytical equations of these polynomials can be used to construct a family of orthogonal polynomials called Vieta-Lucas polynomials.

Vieta–Lucas Polynomials $\Psi_j(z)$ of degree $j \in \mathbb{N}_0$ is defined by [16]:

$$\Psi_j(t) = 2\cos(j\,\psi), \quad \psi = \arccos(0.5\,z), \quad \psi \in [0, \pi], \quad -2 \leq z \leq 2.$$

The existence of the following recurrence relation for $\Psi_j(z)$ can be easily demonstrated:

$$\Psi_j(z) = z\Psi_{j-1}(z) - \Psi_{j-2}(z), \quad j = 2,3,\ldots, \quad \Psi_0(z) = 2, \quad \Psi_1(z) = z.$$

Using VLPs and $z = 4$, *beta* – 2, a new class of orthogonal polynomials on the interval [0, 1] is produced, and it will be indicated by the symbol $\Phi_m(\beta)$, as in:

$$\Phi_j(\beta) = \mathrm{VL}_j(4\,\beta - 2).$$

$\Phi_j(\beta)$ have the following recurrence relation:

$$\Phi_{j+1}(\beta) = (4\beta - 2)\,\Phi_{j-1}(\beta) - \Phi_{j-2}(\beta), \qquad j = 2,\ 3,\ \ldots,$$

where, $\Phi_0(\beta) = 2$, $\Phi_1(\beta) = 4\beta - 2$. Also, we find $\Phi_j(0) = 2(-1)^j$ and $\Phi_j(1) = 2$, $j = 0,1,2,\ldots$.

The analytical formula for $\Phi_j(\beta)$ is:

$$\Phi_k(\beta) = 2j\sum_{j=0}^{k}(-1)^j \frac{4^{k-j}\Gamma(2k-j)}{\Gamma(j+1)\Gamma(2k-2j+1)}\beta^{k-j}, \qquad k = 2,3,\ldots.$$

The polynomials $\Phi_k(\beta)$ are orthogonal polynomials on [0,1] w.r.t. $\frac{1}{\sqrt{\beta-\beta^2}}$, and so we have:

$$\left\langle \Phi_k(\beta), \Phi_j(\beta) \right\rangle = \int_0^1 \frac{\Phi_k(\beta)\Phi_j(\beta)}{\sqrt{\beta-\beta^2}}d\beta = \begin{cases} 0, & k \neq j \neq 0, \\ 4\pi, & k = j = 0, \\ 2\pi, & k = j \neq 0. \end{cases}$$

Let $\Phi(\beta) \in L^2[0,1]$, then using $\Phi_k(\beta)$, we have:

$$\phi(\beta) = \sum_{j=0}^{\infty} c_j \Phi_j(\beta), \tag{1}$$

where c_j must be evaluated in order to transform $\Phi(\beta)$ into the terms of $\Phi^s_m(\beta)$. We can write: by taking into account only the first $m + 1$ words (1).

$$\phi_m(\beta) = \sum_{j=0}^{m} c_j \Phi_j(\beta), \tag{2}$$

It is possible to calculate c_j, $j = 0,1,2,\ldots$, and m from:

$$c_j = \frac{1}{\delta_j}\int_0^1 \frac{\phi(\beta)\Phi_j(\beta)}{\sqrt{\beta-\beta^2}}d\beta, \qquad \delta_j = \begin{cases} 4\pi, & j = 0, \\ 2\pi, & j = 1,2,\ldots,m. \end{cases} \tag{3}$$

2.3 Error Analysis

Lemma 1. *With the weight function* $\frac{1}{\sqrt{\beta-\beta_2}}$ *and* $\Phi(\beta) \leq \varepsilon$, *the series* (2) *uniformly converges to the function* $\Phi(\beta)$ *as* $m \to \infty$ *for some constant* ε. *Additional estimates that have been satisfied include*:

1. *The coefficients' series in Equation (2) are bounded, that is*

$$\left|c_j\right| \leq \frac{\varepsilon}{4j\left(j^2-1\right)}, \qquad \forall\, j > 2.$$

2. *The error estimate norm is subject to the following inequality* ($L^2_{\tilde{\mathbf{w}}}[0,1]$ *-norm):*

$$\left\|\phi(\beta)-\phi_m(\beta)\right\|_{\tilde{\mathbf{w}}} < \frac{\varepsilon}{12\sqrt{m^3}}.$$

3. *If* $\Phi^{(m)}(\beta) \in C[0,1]$, *the absolute error bound is as follows.*

$$\left\|\phi(\beta)-\phi_m(\beta)\right\| \le \frac{\Delta\Pi^{m+1}}{(m+1)!}\sqrt{\pi}.$$

Here, $\Delta = \max_{\beta\in[0,1]}\Phi^{(m+1)}(\beta)$ *and* $\Pi = \max\{1-\beta_0, \beta_0\}$.

For more details on these polynomials and the convergence analysis for the approximation (2), see reference [17].

2.4 The Main Scheme

Therom 1. *In Eq. (1), the approximate solution of the main problem is given in terms of shifted Vieta-Lucas polynomials. Following that, the fractional-order terms can be changed into the following algebraic equations:*

$$D^{\alpha}\left(\phi_n(\beta)\right) = \sum_{i=\lceil\alpha\rceil}^{n}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i\chi_{i,k}^{(\alpha)}\beta^{i-k-\alpha}, \tag{4}$$

$$\chi_{i,k}^{(\alpha)} = (-1)^k\frac{4^{i-k}2i\Gamma(2i-k)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+1)\Gamma(i-k+1-\alpha)}. \tag{5}$$

3. Numerical Method

In this study, we use a different expansion to get at the FIDEs answer. Compared to the current approaches we have explored, the proposed expansion is either more direct, simpler, or both. We are interested in the numerical analysis of the following nonlinear fractional integro-differential equation:

$$D^{\alpha}\phi(\beta) = G\left(\beta,\phi(\beta),\int_0^{\beta}H(\gamma,\phi(\beta))d\gamma\right), 0<\beta\le 1, 0<\alpha<1 \tag{6}$$

Here, we use the shifted Vieta-Lucas polynomials collocation method to solve the FIDE problem. To do this, with initial and boundary conditions (2), we calculated (1) as

$$\sum_{i=\lceil\alpha\rceil}^{n}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i\chi_{i,k}^{(\alpha)}\beta^{i-k-\alpha} = G\left(\beta,\sum_{j=0}^{\infty}c_j\Phi_j(\beta),\int_0^{\beta}H\left(\gamma,\sum_{j=0}^{\infty}c_j\Phi_j(\gamma)\right)d\gamma\right) \tag{7}$$

At these points, β_s, $s = 0, 1,..., m-\alpha$, we collocate (7).

$$\sum_{i=\lceil\alpha\rceil}^{n}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i\chi_{i,k}^{(\alpha)}\beta_s^{i-k-\alpha} = G\left(\beta_s,\sum_{j=0}^{\infty}c_j\Phi_j(\beta_s),\int_0^{\beta_s}H\left(\gamma,\sum_{j=0}^{\infty}c_j\Phi_j(\gamma)\right)d\gamma\right) \tag{8}$$

The roots of the shifted Vieta–Lucas Polynomials are used to find appropriate collocation locations $\Phi_{m+1-\alpha}$. We utilize the transformation to change the γ-interval $[0, \beta_s]$ into the Δ-interval $[-1, 1]$ in order to employ the Gaussian integration formula for (8)

$$\gamma = \frac{\beta_s}{2}(\delta+1).$$

Equation (8) may be rewritten as follows for $s = 0, 1, ..., m - \alpha$

$$\sum_{i=\lceil\alpha\rceil}^{n}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i \chi_{i,k}^{(\alpha)} \beta_s^{i-k-\alpha} = G\left(\beta_s, \sum_{j=0}^{\infty} c_j \Phi_j(\beta_s), \frac{\beta_s}{2}\int_{-1}^{1} H\left(\frac{\beta_s}{2}(1+\delta), \sum_{j=0}^{\infty} c_j \Phi_j\left(\frac{\beta_s}{2}(1+\delta)\right)\right) d\delta\right) \tag{9}$$

For $s = 0, 1, ..., m - \alpha$ and the Gaussian integration formula, we obtain

$$\sum_{i=\lceil\alpha\rceil}^{n}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i \chi_{i,k}^{(\alpha)} \beta_s^{i-k-\alpha} = G\left(\beta_s, \sum_{j=0}^{\infty} c_j \Phi_j(\beta_s), \frac{\beta_s}{2}\sum_{q=0}^{\infty} w_q H\left(\frac{\beta_s}{2}(1+\delta_q), \sum_{j=0}^{\infty} c_j \Phi_j\left(\frac{\beta_s}{2}(1+\delta_q)\right)\right)\right), \tag{10}$$

where δ_q are the Vieta–Lucas Polynomials's $m + 1$ zeros and w_q are the appropriate weights listed in [18]. The above approximation is based on the exactness of the Gaussian integration formula for polynomials with degrees up to $2m + 1$. Additionally, we can get r equations by inserting (2) in the boundary conditions. Equation (10), when combined with the r equations of the boundary conditions, gives $(m + 1)$ of an algebraic equation system that can be solved using the Newton iteration method for the unknowns c_n, $n = 0, 1,..., m$.

4. Numerical Examples

In this section we will present three examples of fractional integro - differential equation of Volterra type by the proposed Vieta-Lucas polynomials method.

Example 1. Consider the following fractional integro-differential equation [19]

$$D^{\alpha}\phi(\beta) = -\frac{1}{5}\left(\beta^2 e^{\beta}\right)\phi(\beta) + \frac{6\beta^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{1}{2}\beta e^{\beta}\int_0^{\beta} e^{\beta}\gamma\phi(\gamma)d\gamma, \tag{11}$$

subject to the initial condition

$$\phi(0)=0. \tag{12}$$

We use the provided procedure with $m = 6$ and arrive at an approximation of the solution as,

$$\phi_m(\beta) = \sum_{j=0}^{6} c_j \Phi_j(\beta). \tag{13}$$

We construct the following schema using Equation (10).

$$\sum_{i=\lceil\alpha\rceil}^{n}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i \chi_{i,k}^{(\alpha)} \beta_s^{i-k-\alpha} = \varphi + \frac{\beta}{2}e^{\beta}\sum_{q=0}^{m} w(q)\frac{\beta_s}{2}(1+\delta_q)\sum_{j=0}^{6} c_j \Phi_j(j, \frac{1}{2}\frac{\beta_s}{2}(1+\delta_q)), \tag{14}$$

where

$$\varphi = -\frac{1}{5}\left(\beta_s^2 e^{\beta_s}\right)\sum_{j=0}^{6} c_j \Phi_j(j,\beta) + \frac{6\beta^{3-\alpha}}{\Gamma(4-\alpha)},$$

where β_s are roots of the shifted Vieta-Lucas polynomial polynomial and $s = 0, 1, 2, 3, 4, 5$. Additionally, δ_q represents the roots of the Vieta-Lucas polynomial while $w(q)$ are the associated weights.

In view (2), the initial condition (22) can be written as

$$\phi_m(0) = \sum_{j=0}^{m} c_j \Phi_j(0) = \sum_{j=0}^{m} (-1)^j c_j = 0 \tag{15}$$

The set of algebraic equations with the coefficient c_j are represented by equations (14 and 15, combined. For finding the coefficients, we solve this algebraic equations and substitute it into equation (23) and then we can obtain the approximate solution of (11).

We will illustrate the numerical results through some figures. Figure 1(A) represents the comparison of the exact solution with the approximate solution with $\alpha = 0.8$ and $m = 6$. While figure 1(B) represents the absolute error between the two solutions. In fact, here we find that the error is very small of the order of 10^{-10}. But to verify further, since the exact solution in the fractional case does not exist, there must be a way to check the amount of error. In this case we will calculate the residual error function as follows:

$$REF(\beta_s) = \sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=0}^{i-\lceil\alpha\rceil} c_i \chi_{i,k}^{(\alpha)} \beta_s^{i-k-\alpha} - \varphi - \frac{\beta}{2} e^{\beta} \sum_{q=0}^{m} w(q) \frac{\beta_s}{2}(1+\delta_q) \sum_{j=0}^{6} c_j \Phi_j\Big(j, \frac{1}{2}\frac{\beta_s}{2}(1+\delta_q)\Big). \tag{16}$$

The REF in Fig. 1(C) is plotted for the same values as in Figs. 1(A) and 1(B). As a result, we were satisfied the accuracy of the solutions for non-integer cases. It is clear from this figure that the order of the REF is very small; i.e. 10^{-12}. In [19, 20], only the numerical solutions were verified by comparing them only with the exact solutions, and this is not enough, although it gives a good impression. In the following two examples, we will follow the same treatment, so we will only list the examples with illustrations.

Example 2. Consider the following fractional integro-differential equation [20]

$$D^{\alpha}\phi(\beta) = \frac{\Gamma(\lambda+1)}{\Gamma(-\alpha+\lambda+1)}\beta^{\lambda-\alpha} - \frac{(2\lambda+3)}{\lambda^2+3\lambda+2}\beta^{\lambda+2} - \beta^{\lambda} + \int_0^{\beta} (\beta+\gamma)\phi(\gamma)d\gamma, \tag{17}$$

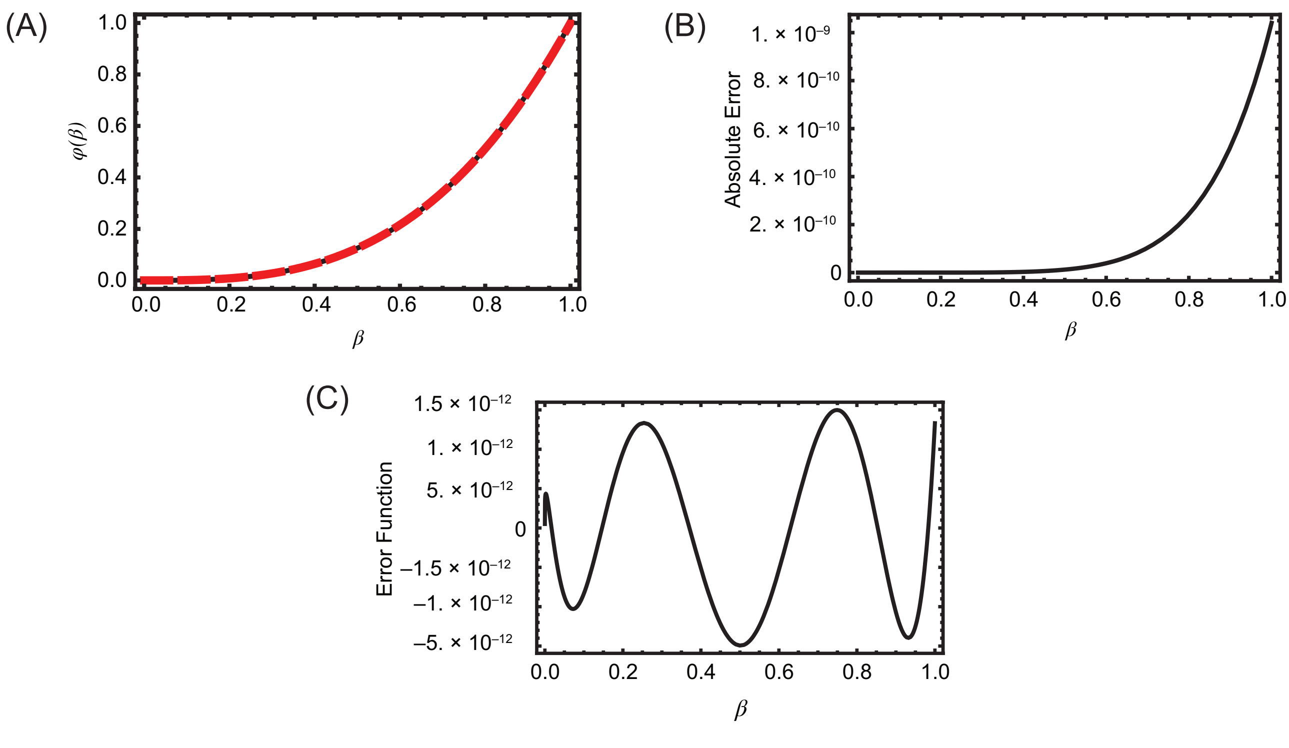

Figure 1. Plot of comparison and absolute error between numerical solution and exact solution for example 1 with $\alpha = 0.8$ and $m = 6$. (C) Residual error function for the same values as in (A)–(B).

subject to the initial condition

$$\phi(0) = 0.$$

Using the proposed approach and the value $m = 6$, we arrive at approximation of the solution as,

$$\phi_m(\beta) = \sum_{j=0}^{6} c_j \Phi_j(\beta).$$

Equation (10) provides us the following schema

$$\sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=0}^{i-\lceil\alpha\rceil} c_i \chi_{i,k}^{(\alpha)} \beta_s^{i-k-\alpha} = \varphi + \frac{\beta}{2} e^{\beta} \sum_{q=0}^{m} w(q) \frac{\beta_s}{2}(1+\delta_q) \sum_{j=0}^{6} c_j \Phi_j(j, \frac{1}{2}\frac{\beta_s}{2}(1+\delta_q)), \tag{20}$$

where

$$\varphi = \frac{\Gamma(\lambda+1)}{\Gamma(-\alpha+\lambda+1)} \beta^{\lambda-\alpha} - \frac{(2\lambda+3)}{\lambda^2+3\lambda+2} \beta^{\lambda+2} - \beta^{\lambda}.$$

The approximate solution to the equation (17) can be found by taking the same procedures as in example 1.

We show in Figure 2 (A)–(B) the comparison and absolute error between numerical solution and exact solution for example 2 with $\alpha = 0.8$, $\lambda = 2$,$ and $m = 6$. Also in Figure 2 (C) the residual error function for the same values as in Figure 2 (A)–(B).

Example 3. Consider the following fractional integro-differential equation [21]

$$D^{\alpha}\phi(\beta) = \frac{2.5}{\Gamma(0.8)} \beta^{0.8} - \frac{1}{252} \beta^9 + \int_0^{\beta} (\beta-\gamma)^2 \phi^3(\gamma) d\gamma, \tag{21}$$

subject to the boundary conditions

$$\phi(0) = 0, \quad \phi(1) = 1 \tag{22}$$

The exact solution in the case of $\alpha = 2$ is $\phi(\beta) = \beta^2$

Using the proposed approach and the value $m = 6$, we arrive at approximation of the solution as,

$$\phi_m(\beta) = \sum_{j=0}^{6} c_j \Phi_j(\beta). \tag{23}$$

Equation (10) provides us the following schema

$$\sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=0}^{i-\lceil\alpha\rceil} c_i \chi_{i,k}^{(\alpha)} \beta_s^{i-k-\alpha} = \varphi + \frac{\beta}{2} e^{\beta} \sum_{q=0}^{m} w(q) \frac{\beta_s}{2}(1+\delta_q) \sum_{j=0}^{6} c_j \Phi_j(j, \frac{1}{2}\frac{\beta_s}{2}(1+\delta_q)), \tag{24}$$

where

$$\varphi = \frac{\Gamma(\lambda+1)}{\Gamma(-\alpha+\lambda+1)} \beta^{\lambda-\alpha} - \frac{(2\lambda+3)}{\lambda^2+3\lambda+2} \beta^{\lambda+2} - \beta^{\lambda}.$$

The approximate solution to the equation (17) can be found by taking the same procedures as in example 1.

We show in Figure 3 (A)–(B) the comparison and absolute error between numerical solution and exact solution for example 3 with $\alpha = 0.8$ and $m = 6$ and in Figure 3(C) the residual error function for the same values as in Figure 3 (A)–(B).

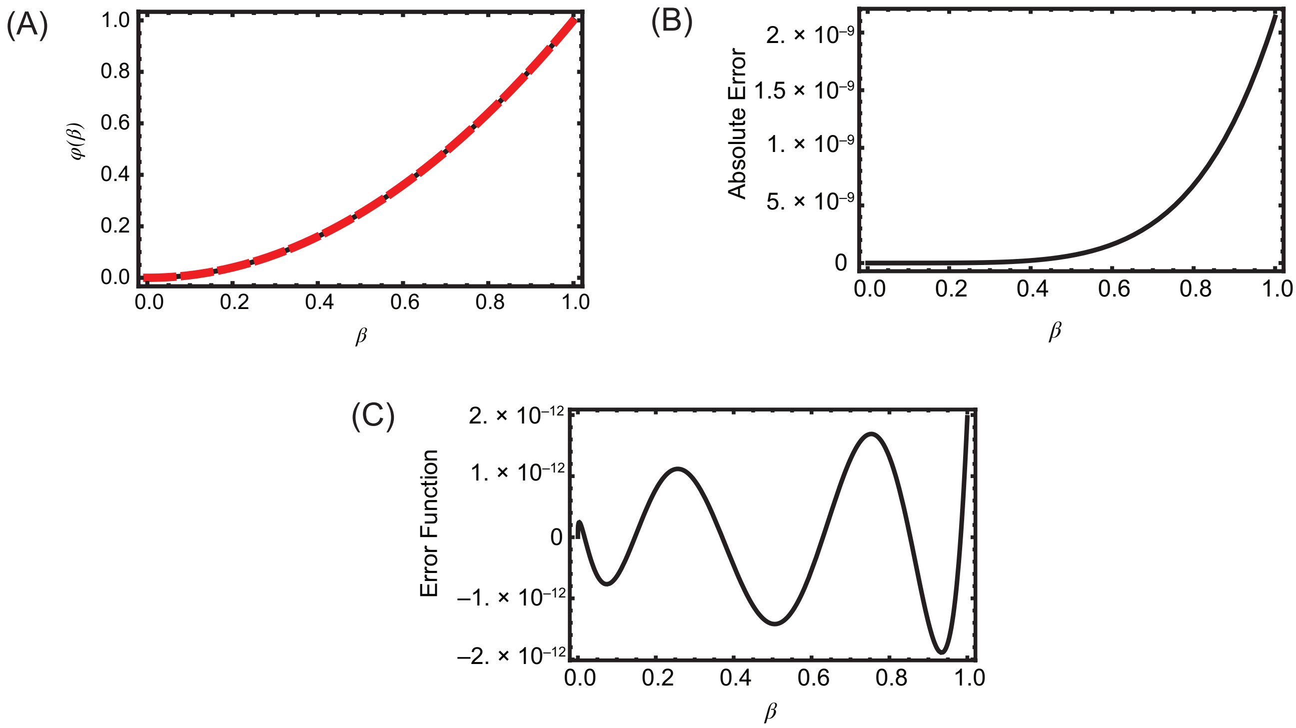

Figure 2. (A)–(B) Plot of comparison and absolute error between numerical solution and exact solution for example 2 with $\alpha = 0.8$, $\lambda = 2$, and $m = 6$. (C) Residual error function for the same values as in (A)–(B).

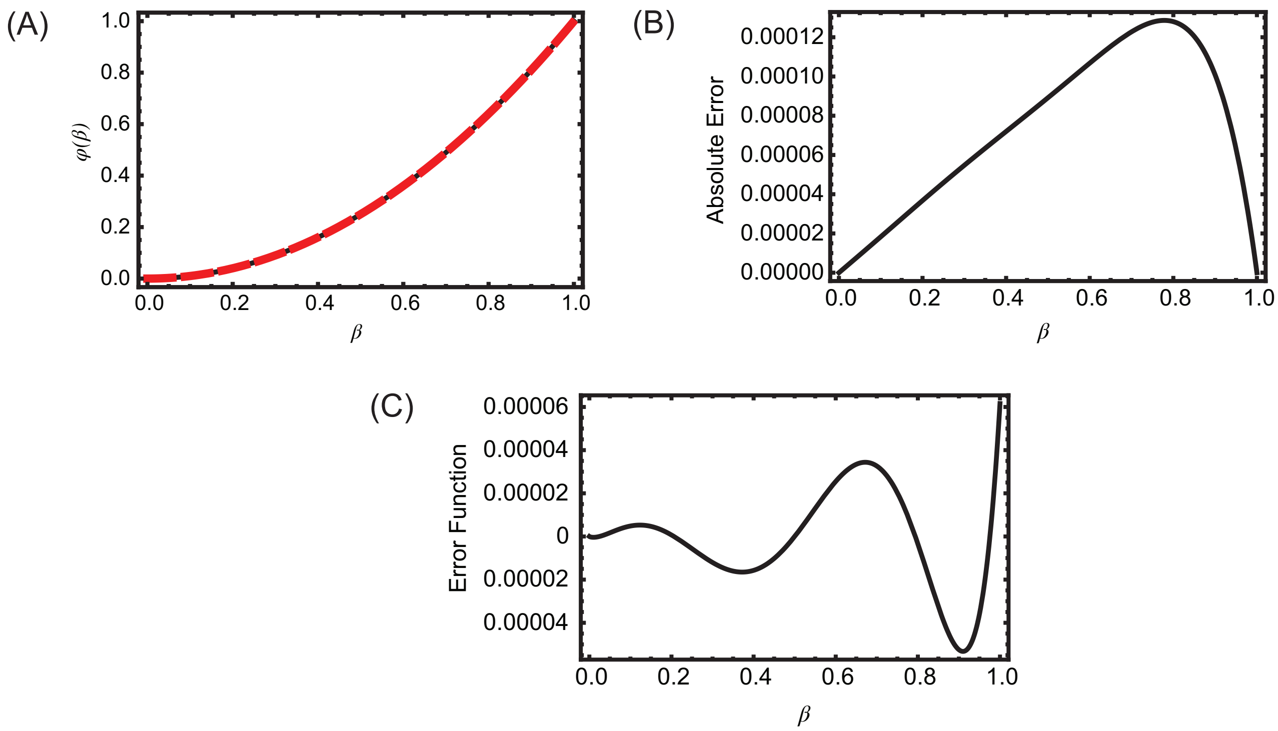

Figure 3. (A)–(B) Plot of comparison and absolute error between numerical solution and exact solution for example 3 with $\alpha = 0.8$ and $m = 6$. (C) Residual error function for the same values as in (A)–(B).

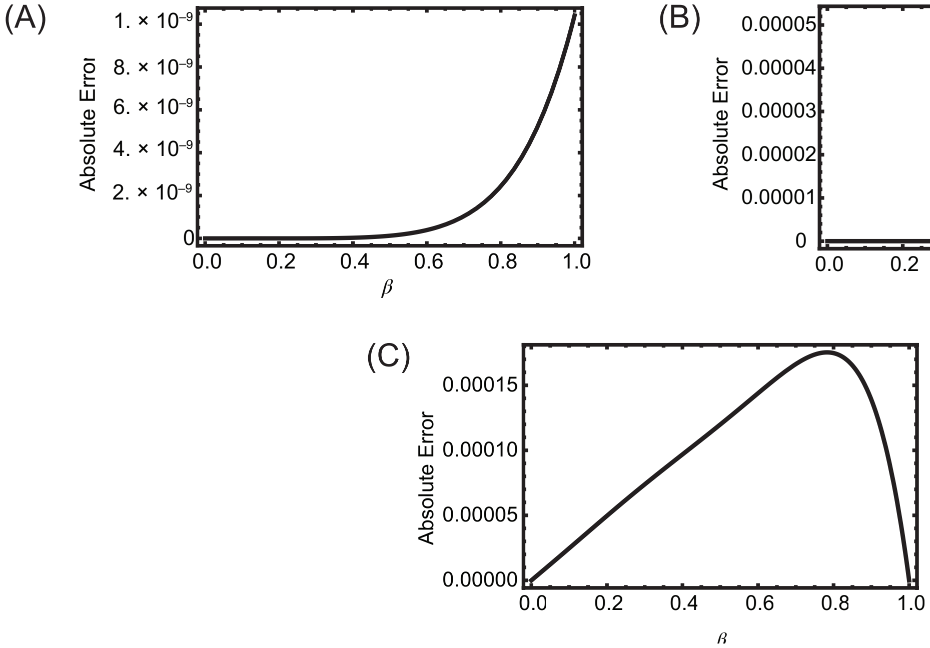

Figure 4. (A)–(B) Plot the absolute error between numerical solution for example 1–3 via our approach and [22] with $\alpha = 0.8$ and $m = 6$.

The results of this study are contrasted with those from another approximative analytical method in Figure 4. Using the method described in [22], the absolute error was determined for the three examples. These figures show that the error is really minor. Every time we use a larger value for m as well as more iterative approximations, the result can be decreased. The values are taken as shown in the figures 1–3, respectively.

5. Conclusions

In this article, fractional integro-differential equations were solved using the Chebyshev spectral method via Caputo fractional derivative. The properties of Vieta--Lucas polynomials were coupled with the Gaussian integration method for reducing the fractional integro-differential equations to algebraic equations. The resulting equations were then solved using well-known techniques like Newton's. The numerical results work is completed using the Mathematica program. We recommend focusing on utilizing fractional space-time derivatives in our upcoming work. Furthermore, we will convert the fractional time derivative into a discrete equation through non-standard finite-difference methods. To simplify complex models into a set of solvable differential equations, we can also employ another special functions. (see, for example, [23–26]).

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