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# Some methods to approximate and estimate the reliability function of inverse Rayleigh distribution

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**Abstract**

This paper presents new work using an approximate method to find the reliability function for inverse Rayleigh distribution and compares it with two statistical estimation methods. In the approximate method, the reliability function is expanded using Bernstein polynomials to find the approximate value for it. As for statistical estimation methods, the first one is the maximum likelihood estimation by finding the scale parameter estimator to estimate the reliability function. The second one is the Bayes estimator is created under the NLINEX loss function to get the reliability function with the least loss where this estimator is determined utilizing chi-squared informative prior distribution. The simulation technique is used to obtain the results of all methods and compare them depending on the integrated mean squared error (IMSE) to determine which of these methods is best. Finally, to determine theoretical results MATLAB 2015 is used.

*Key words and phrases:* Inverse Rayleigh, Maximum-likelihood, NLINEX loss function, Chi-squared, Bernstein polynomials

*Mathematics Subject Classification (2010):* (65C05, 65C10, 65C20 & 68U20).

## 1. Introduction

The inverse Rayleigh (IR) distribution plays a significant in numerous applications, statistics, and operations research, especially in agriculture, biology, and other sciences.

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Ardianti and Sutarman obtained some classical and Bayesian estimators of parameter and R(t) for IR distribution [1]. Mohammed et al. presented maximum likelihood and Bayes estimation methods with numerical methods using power functions to find the reliability function of Burr type XII distribution [2]. Ali et al. used the Monte-Carlo simulation technique to look at the exhibition of three types of standard Bayes estimators for parameter and R(t) of IR distribution [3]. Yunus et al. considered the method of maximum likelihood to estimate the parameter of IR distribution by using the expectation-maximization algorithm [4]. Al-Obedy et al. used non-Bayes, Bayes, and numerical methods (Bernstein polynomial) to find the failure rate function for basic Gompertz distribution [5]. Abu Awwad used maximum likelihood and Bayesian approaches under different loss functions to estimate the Rayleigh model's parameters [6]. Mahmoud et al. introduced the non-Bayes, Bayes, and expansion methods for the reliability function of the Kumaraswamy distribution under different loss functions [7].

This research aims to present new work using an approximate method to see the efficiency of this method in finding the approximate value of the reliability function IR distribution by constructing a new polynomial based on the Bernstein polynomials in addition to comparing it with two statistical estimation methods, the first one is non-Bayes estimator using maximum likelihood and the second one is Bayes estimator under the NLINEX loss function corresponding informative prior chi-squared depending on the integrated mean squared error (IMSE).

## 2. Statistical Properties of Inverse Rayleigh Distribution

In this section, the most important statistical properties of inverse Rayleigh distribution with one scale parameter $\varnothing$ are presented in [8] as follows:

1- The probability density function for IR:

$$f(t:\varnothing)=\frac{2\varnothing}{t^3}e^{\left(-\frac{\varnothing}{t^2}\right)} \qquad t,\ \varnothing>0 \tag{1}$$

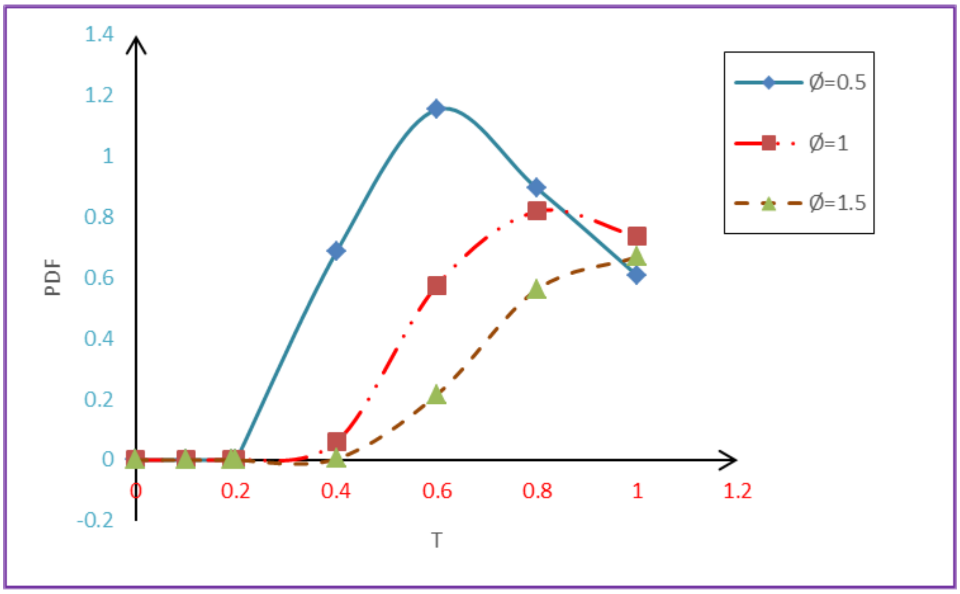


Figure 1: The probability density function for IR with different value of $\varnothing$ = 0.5, 1, and 1.5.

2- The cumulative distribution function for IR is:

$$F(t:\varnothing)=e^{\left(-\frac{\varnothing}{t^2}\right)} \qquad t,\ \varnothing>0 \tag{2}$$

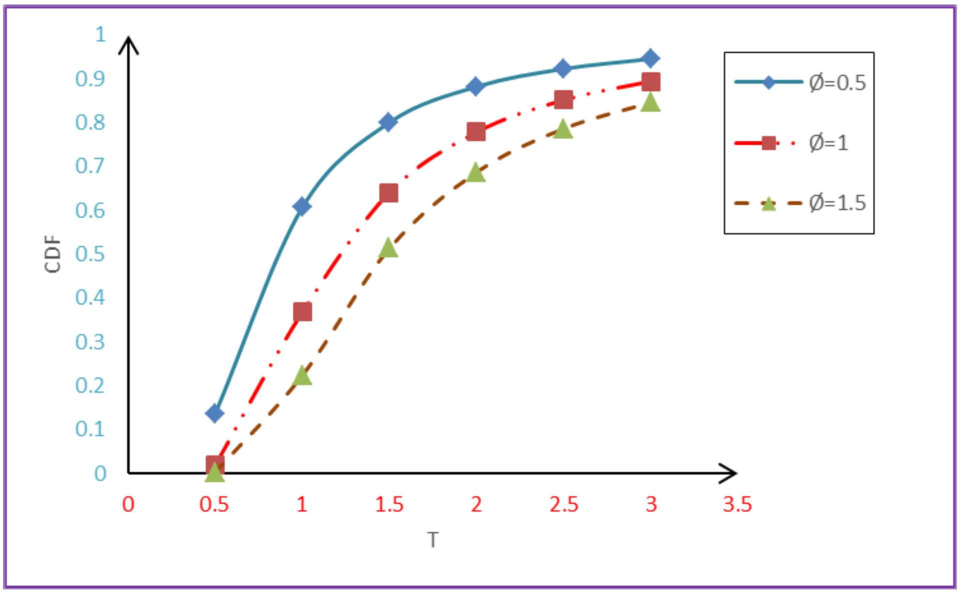


Figure 2: The cumulative distribution function for IR with different value of ∅ = 0.5, 1, and 1.5.

3- The reliability function of this distribution is yielded by:

$$R(t) = 1 - F(t : \varnothing) = 1 - e^{\left(-\frac{\varnothing}{t^2}\right)} \qquad t, \varnothing > 0 \tag{3}$$

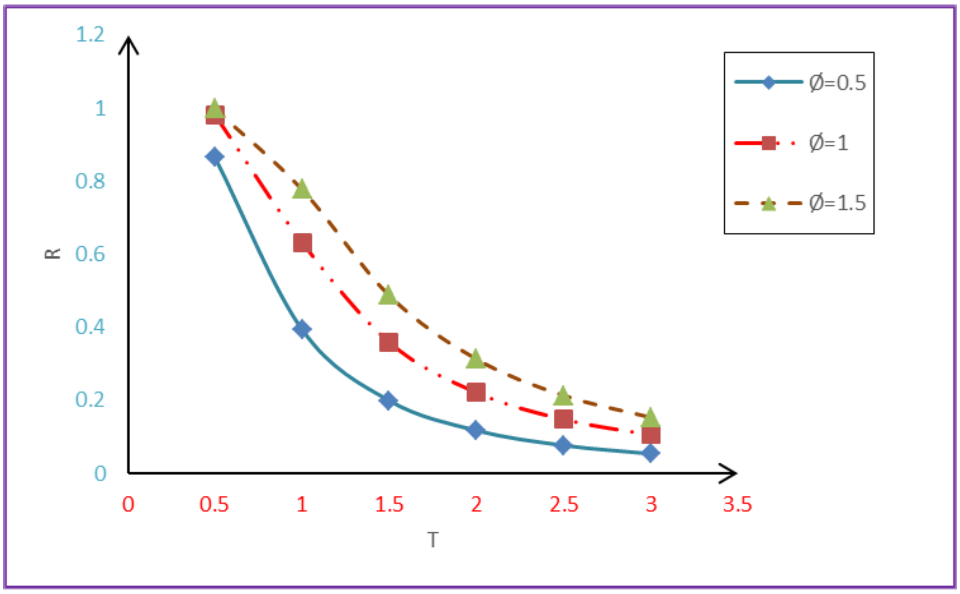


Figure 3: The Reliability function for IR with different value of ∅ = 0.5, 1, and 1.5.

## 3. Approximate Method

In this section, construct polynomials using Bernstein polynomials (BPs) is applied to obtain the approximate value of *R (t)*.

Firstly, write *R (t)* as a series of functions *q(t)* presented in [9, 10]:

$$R(t) = \sum_{k=0}^{m} c_k q_k(t) \tag{4}$$

where $c_k$ are unknown coefficients that must be determined and $q_k(t)$ are the functions that will generally be taken.

Here, take $q_k(t)$ as BPs in [11, 12], that is:

$$B_k^m(t)=\binom{m}{k}(1-t)^{m-k}t^k, \qquad k=0,1,\ldots,m \tag{5}$$

where $m$ is a degree of polynomials.
Substituting equations (3) and (5) in equation (4) to have:

$$1-e^{\left(-\frac{\varnothing}{t^2}\right)}=\sum\nolimits_{k=0}^{m} c_k\binom{m}{k}(1-t_i)^{m-k}t^k \tag{6}$$

Now, choose arbitrary points $t_0$ , $t_1$, …, $t_m$ to obtain a linear system of ($m$) equations and ($m$) unknown $c_k$, $k$ = 0, 1, …, $m$ as follows:

$$1-e^{\left(-\frac{\varnothing}{t_i^2}\right)}=\sum\nolimits_{k=0}^{m} c_k\binom{m}{k}(1-t_i)^{m-k}t_i^k \qquad i=0,1,\ldots,m$$

Finally, solve this system for ($c_k$) utilizing Gauss elimination to find the approximate value of $R(t)$, and write it as $\hat{R}(t)_{ABP}$.

## 4. Maximum Likelihood Estimation Method

Let the lifetime of the random sample is $t=(t_1, t_2, \ldots, t_n)$ and the size n drawn independently of the IR distribution defined in equation (1). The likelihood function is defined as [2, 13]:

$$\mathrm{L}\ (\varnothing\,|\,t)=\prod\nolimits_{i=1}^{n} f(t_i\,|\,\varnothing)=(2\varnothing)^n\prod\nolimits_{i=1}^{n} t_i^{-3}e^{-\varnothing\sum_{i=1}^{n} t_i^{-2}} \tag{7}$$

Taking the logarithm of both sides of equation (7), to get:

$$\mathrm{In\ (L)} = \mathrm{n\ ln\ (2)} + \mathrm{n}\ln(\varnothing)-3\sum\nolimits_{i=1}^{n}\ln(t_i)-\varnothing\sum\nolimits_{i=1}^{n} t_i^{-2} \tag{8}$$

Next, taking the derivative for equation (8) concerning the parameter $\varnothing$ equal to zero, to obtain the following formula:

$$\hat{\varnothing}_{ML}=\frac{n}{\sum_{i=1}^{n} t_i^{-2}} \tag{9}$$

Based on, the invariance for the maximum likelihood estimator (MLE). $R\ (t)$ will be as follows:

$$\hat{R}(t)_{ML}=1-e^{\left(-\frac{\hat{\varnothing}_{ML}}{t^2}\right)} \tag{10}$$

where $\hat{\varnothing}_{ML}$ as in equation (9).

## 5. Bayesian Estimation Method

Bayesian estimation method (BEM) indicates a prior distribution of parameters. Consider chi-squared informative prior for $\varnothing$ two hyper-parameters '$a$' and '$b$' having density function as follows [8, 14]:

$$\mathrm{P}(\varnothing)=\frac{b^{\frac{a}{2}}}{2^{\frac{a}{2}}\Gamma\left(\frac{a}{2}\right)}\varnothing^{\frac{a}{2}-1}e^{-\frac{b\varnothing}{2}}; \qquad \varnothing, a, b>0 \tag{11}$$

The posterior density function for $\varnothing$ of IR has been obtained by joining equation (7) with equation (11) as:

$$\pi(\varnothing\,|\,\underline{t})=\frac{L(\varnothing\,|\,\underline{t})P(\varnothing)}{\int_{\varnothing} L(\varnothing\,|\,\underline{t})P(\varnothing)d\varnothing}$$

$$\pi(\varnothing\,|\,\underline{t})=\frac{(2\varnothing)^n\prod_{i=1}^{n} t_i^{-3}e^{-\varnothing\sum_{i=1}^{n} t_i^{-2}}\dfrac{b^{\frac{a}{2}}}{2^{\frac{a}{2}}\Gamma\left(\frac{a}{2}\right)}\varnothing^{\frac{a}{2}-1}e^{-\frac{b\varnothing}{2}}}{\int_0^{\infty}(2\varnothing)^n\prod_{i=1}^{n} t_i^{-3}e^{-\varnothing\sum_{i=1}^{n} t_i^{-2}}\dfrac{b^{\frac{a}{2}}}{2^{\frac{a}{2}}\Gamma\left(\frac{a}{2}\right)}\varnothing^{\frac{a}{2}-1}e^{-\frac{b\varnothing}{2}}d\varnothing}$$

$$= \frac{\varnothing^{\frac{a}{2}+n-1} e^{-\varnothing\left(\frac{b}{2}+\sum_{i=1}^{n} t_i^{-2}\right)}}{\int_0^\infty \varnothing^{\frac{a}{2}+n-1} e^{-\varnothing\left(\frac{b}{2}+\sum_{i=1}^{n} t_i^{-2}\right)} d\varnothing} = \frac{\varnothing^{\frac{a}{2}+n-1} e^{-\varnothing S}}{\int_0^\infty \varnothing^{\frac{a}{2}+n-1} e^{-\varnothing S} d\varnothing}$$

where $S = \frac{b}{2} + \sum_{i=1}^{n} t_i^{-2}$

Using the transformation, $y = \varnothing S$ that is $\varnothing = \frac{y}{s}$ then $d\varnothing = \frac{dy}{s}$, can obtain the following formula:

$$\pi(\varnothing \mid \underline{t}) = \frac{S^{\frac{a}{2}+n} \varnothing^{\frac{a}{2}+n-1} e^{-\varnothing S}}{\Gamma\left(\frac{a}{2}+n\right)}; \qquad S = \frac{b}{2} + \sum_{i=1}^{n} t_i^{-2} \tag{12}$$

Some Bayesian estimators are obtained based on the non-linear exponential NLINEX loss function as (asymmetric loss function) was submitted by [15], which is a linear combination of LINEX with SE loss function [16].

**Theorem 5.1.** The Bayes estimator of parameter $\varnothing$, of NLINEX loss function [15]:

$$(c+2)\hat{\varnothing}_{NL} = c\hat{\varnothing}_L + 2E(\varnothing) \tag{13}$$

where

$$\hat{\varnothing}_L = -\frac{1}{c}\ln\left[E\left(e^{-c\varnothing} \mid \underline{t}\right)\right] \tag{14}$$

And the Bayes estimator of the reliability function $R(t)$ is:

$$(c+2)\hat{R}(t)_{NL} = c\,\hat{R}(t)_L + 2E_\pi[R(t)]$$

So

$$\hat{R}(t)_{NL} = \frac{c\,\hat{R}(t)_L + 2E_\pi[R(t)]}{c+2} \tag{15}$$

where $c \neq -2$ and $E_\pi(.)$ stands for posterior expectation.

$$E_\pi[R(t) \mid \underline{t}] = \int_\varnothing R(t)\pi(\varnothing \mid \underline{t}) d\varnothing$$

From equation (12) is conclude that,

$$E_\pi[R(t) \mid \underline{t}] = 1 - \left(\frac{s}{s+t^{-2}}\right)^{\frac{a}{2}+n}; \quad S = \frac{b}{2} + \sum_{i=1}^{n} t_i^{-2} \tag{16}$$

Now to find $\hat{\varnothing}_{BL}$ based on LINEX loss function comparing to $\pi(\varnothing \mid \underline{t})$ can be realized as [17]:

$$E\left(e^{-c\varnothing} \mid \underline{t}\right) = \int_\varnothing e^{-c\varnothing}\pi(\varnothing \mid \underline{t}) d\varnothing$$

$$= \int_0^\infty e^{-c\varnothing} \frac{S^{\frac{a}{2}+n} \varnothing^{\frac{a}{2}+n-1} e^{-\varnothing S}}{\Gamma\left(\frac{a}{2}+n\right)} d\varnothing$$

$$= \frac{S^{\frac{a}{2}+n}}{\Gamma\left(\frac{a}{2}+n\right)} \int_0^\infty \varnothing^{\frac{a}{2}+n-1} e^{-\varnothing(S+c)} d\varnothing$$

By using the transformation, $y = \varnothing(S + c)$ and $\varnothing = \frac{y}{s+c}$ then $d\varnothing = \frac{dy}{s+c}$, get:

$$E\left(e^{-c\varnothing} \mid \underline{t}\right) = \frac{S^{\frac{a}{2}+n}}{\Gamma\left(\frac{a}{2}+n\right)} \int_0^\infty \frac{y^{\frac{a}{2}+n-1}}{(s+c)^{\frac{a}{2}+n}} e^{-y} dy = \left(\frac{s}{s+c}\right)^{\frac{a}{2}+n}$$

Then $$\ln\left[E\left(e^{-c\varnothing}\mid \underline{t}\right)\right]=\ln\left(\frac{s}{s+c}\right)^{\frac{a}{2}+n}$$

From equation (14) gets:

$$\hat{\varnothing}_{BL}=-\frac{1}{c}\ln\left(\frac{s}{s+c}\right)^{\frac{a}{2}+n}$$

where $S=\frac{b}{2}+\sum_{i=1}^{n}t_i^{-2}$

The reliability estimator based on the LINEX loss function will be [10]:

$$\hat{R}(t)_{BL}=1-\exp\left(-\frac{\hat{\varnothing}_{BL}}{t^2}\right)=1-\exp\left[-\left(\frac{-\frac{1}{c}\ln\left(\frac{S}{S+c}\right)^{\frac{a}{2}+n}}{t^2}\right)\right]$$

Simplify the above equation to obtain:

$$\hat{R}(t)_{BL}=1-\left(\frac{S}{S+c}\right)^{\frac{a+2n}{2ct^2}} \tag{17}$$

Substituting equations (16) and (17) in equation (15), to obtain the Bayes estimator of *R(t)* based on NLINEX and chi-squared informative prior as:

$$\hat{R}(t)_{BNL}=\frac{c-c\left(\frac{S}{S+c}\right)^{\frac{a+2n}{2ct^2}}+2\left(1-\left(\frac{S}{S+t^{-2}}\right)^{\frac{a}{2}+n}\right)}{c+2} \tag{18}$$

## 6. Simulation Study and Results

- The simulation study of Monte Carlo be directed to look at the estimators obtained in the previous sections of *R (t)* for IR depending on integrated mean square error (IMSE):

$$IMSE\left(\hat{R}(t)\right)=\frac{1}{L}\sum_{i=1}^{L}\left[\frac{1}{n_t}\sum_{j=1}^{n_t}\left(\hat{R}_i(t_j)-R(t_j)\right)^2\right] \tag{19}$$

where
L: Number of samples replicated.
$n_t$: The number of times selected is (4) where ( t = 0.5, 1, 1.5, 2 ).
$\hat{R}_i(t_j)$: The estimate of *R (t)* at the $j^{th}$-time and $i^{th}$-replicate.

- In (Table 1), the values of parameters and constants imposed for the simulation experiments are defined.

Table 1: The values of parameters and constants imposed for the simulation experiments

| | | |
|---|---|---|
| **Sample sizes** | *n* | 10, 25, 50 |
| **Scale parameter** | ∅ | 1, 1.5, 0.5 |
| **Hyper Parameter** | *a* | 2 |
| **Chi-squared** | *b* | 3 |
| **Sample Replicate Number** | *L* | 1000 |

- In this study, $\varnothing = 1$ was taken, and then results were found and discussed for the smaller and larger cases ($\varnothing = 1.5$ and 0.5).
- Generate random samples, assume that U is a variable for uniform distribution at (0, 1), then the data can be generated by inverse transform method for cdf that means:

Let $U = F(t)$ (20)

Then $t = F^{-1}(U)$

Now, substitute equation (2) in equation (20) to get:

$$U_i = F(t_i) = e^{\left(-\frac{\varnothing}{t_i^2}\right)} \qquad t, \varnothing > 0 \qquad i = 1, \ldots, n \tag{21}$$

From equation (21) obtain the following equation:

$$t_i = \left[\frac{\varnothing}{\ln\left(\frac{1}{U_i}\right)}\right]^{\left(\frac{1}{2}\right)} \qquad i = 1, \ldots, n \qquad t, \varnothing > 0 \tag{22}$$

The simulation study can be illustrated through the flowchart in Figure 4 shown below:

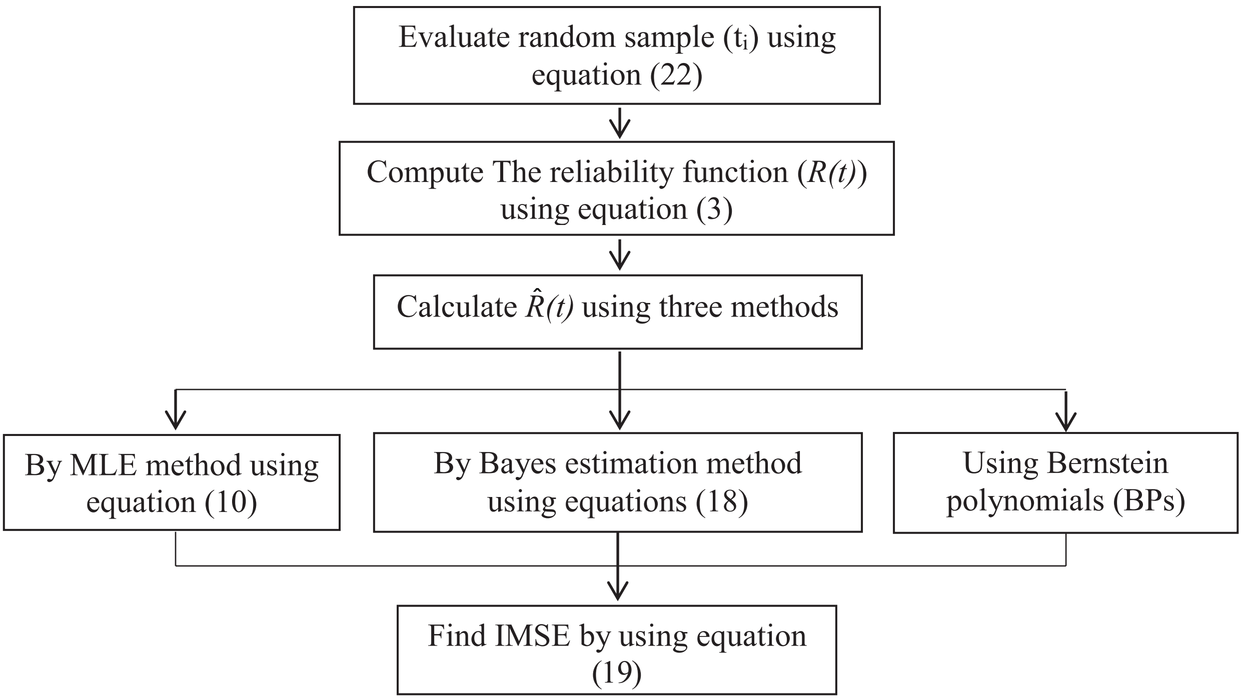


Figure 4: Flowchart of the simulation study.

- The simulation program has been composed by utilizing (MATLAB2015) to find all accounts and theoretical results. Summarize the outcomes of the Monte Carlo simulation in Table 2.

Table 2: IMSE values for approximate and estimator values for reliability function ($R(t)$) of inverse Rayleigh distribution with a = 2 and b = 3

**Case (I):** $\varnothing = 1$

| $n$ | Non-Bayes Method $\hat{R}(t)_{ML}$ | Bayes Method NLNEX $\hat{R}(t)_{BNL}$ c = 0.7 | Approximate Method $\hat{R}(t)_{ABP}$ |
|---|---|---|---|
| 10 | 0.0057 | 0.0035 | 2.9420e-04 |
| 25 | 0.0025 | 0.0020 | 9.7895e-05 |
| 50 | 0.0012 | 0.0010 | 5.0619e-04 |

| Case (II): $\varnothing = 1.5$ | | | |
|---|---|---|---|
| $n$ | Non-Bayes Method $\hat{R}(t)_{ML}$ | Bayes Method NLNEX $\hat{R}(t)_{BNL}$ c = 0.7 | Approximate Method $\hat{R}(t)_{ABP}$ |
| 10 | 0.0065 | 0.0062 | 5.5259e-04 |
| 25 | 0.0025 | 0.0024 | 2.2908e-04 |
| 50 | 0.0015 | 0.0014 | 1.8779e-04 |
| **Case (III):** $\varnothing = 0.5$ | | | |
| $n$ | Non-Bayes Method $\hat{R}(t)_{ML}$ | Bayes Method NLNEX $\hat{R}(t)_{BNL}$ c = 0.7 | Approximate Method $\hat{R}(t)_{ABP}$ |
| 10 | 0.0096 | 0.0073 | 5.3104e-04 |
| 25 | 0.0027 | 0.0024 | 4.9330e-04 |
| 50 | 0.0015 | 0.0014 | 4.9042e-04 |

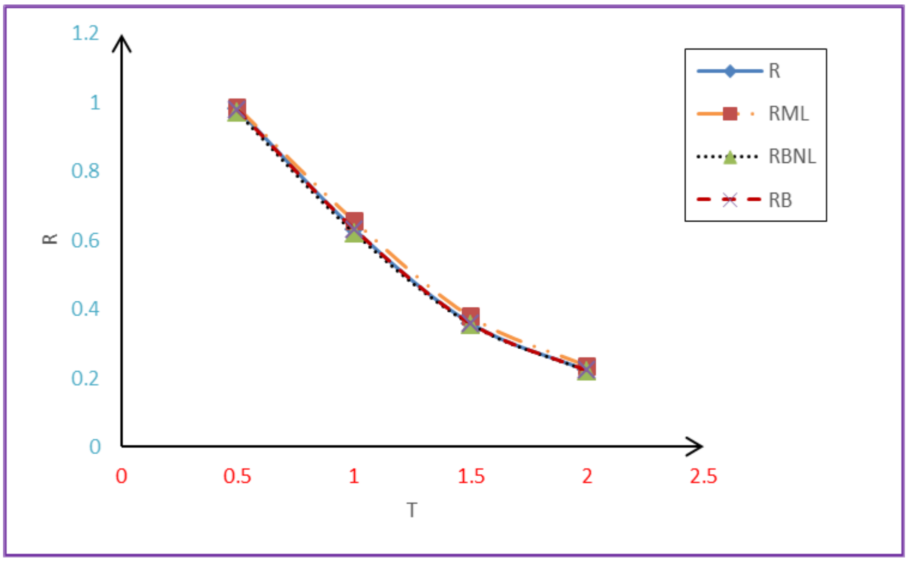


Figure 5: Approximate and estimator values for reliability function ($R(t)$) of inverse Rayleigh distribution with n = 10, $\varnothing$ = 1, a = 2 and b = 3 using different methods.

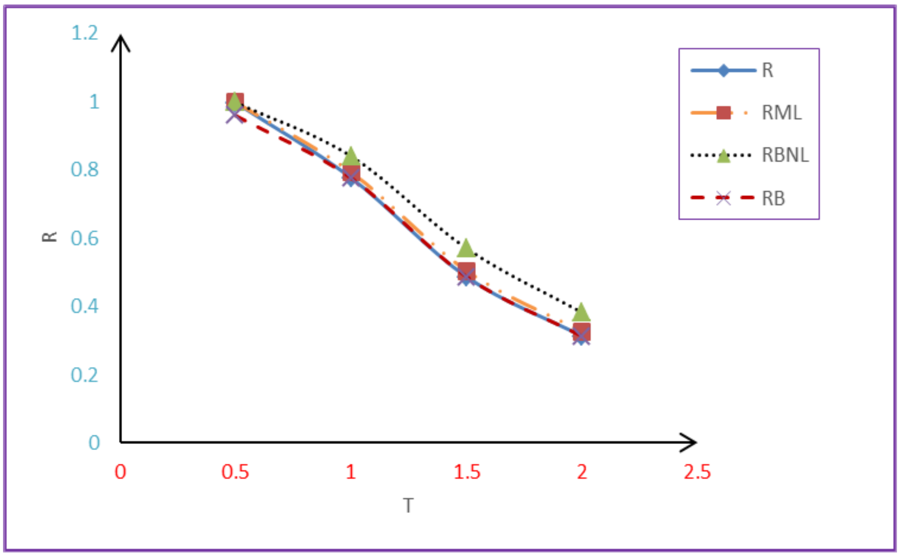


Figure 6: Approximate and estimator values for reliability function ($R(t)$) of inverse Rayleigh distribution with n = 10, $\varnothing$ = 1.5, a = 2 and b = 3 using different methods.

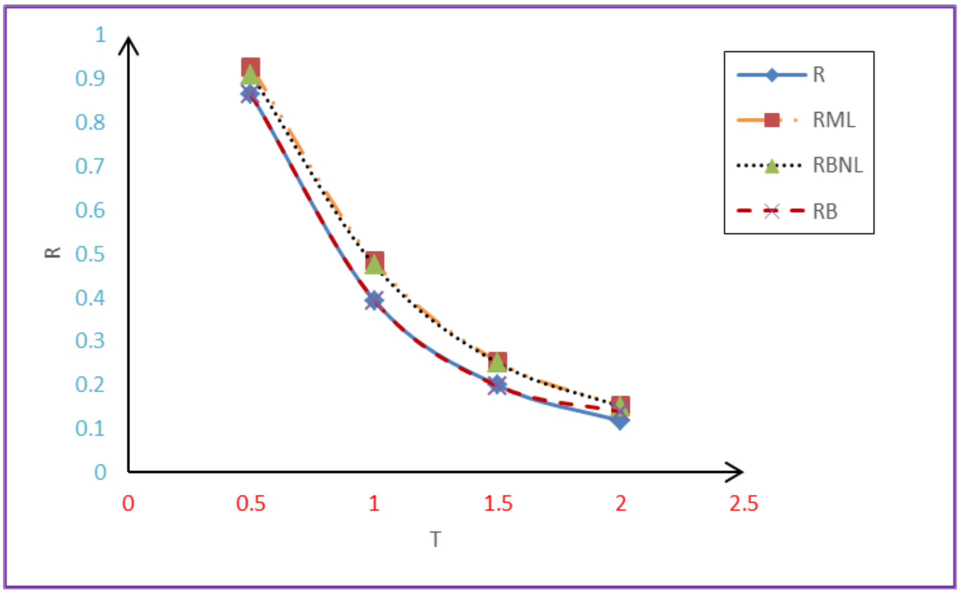


Figure 7: Approximate and estimator values for reliability function (*R(t)*) of inverse Rayleigh distribution with n = 10, ∅ = 0.5, a = 2 and b = 3 using different methods.

## 7. Discussion of Results

Here, the most important results obtained from the simulation study will be presented and discussed, as follows:

1. When ∅ = (1, 1.5, 0.5) and for all sample sizes, the $\hat{R}(t)_{BNL}$ (Bayes estimator of reliability function based on NLINEX) is the best from the $\hat{R}(t)_{ML}$ (non-Bayes estimator of reliability function).
2. The values of IMSE related to the approximate values are less than Bayes $\hat{R}(t)_{BNL}$ and non-Bayes $\hat{R}(t)_{ML}$ estimate values for all sample sizes and all cases.
3. The values of IMSE when ∅ = 1 for all sample sizes and all methods are better than from ∅ = (1.5, 0.5).
4. All methods to find the IMSE values when ∅ = 1.5 are better than from ∅ = 0.5 for all sample sizes.
5. In all cases, the IMSE values decreases as the sample size increases.

## 8. Conclusions

In this paper, an approximate method based on the Bernstein polynomial was used to find the approximate value of the reliability function for the inverse Rayleigh distribution, as well as non-Bayes and Bayes estimation methods are used to estimate the reliability function for IR distribution. Also, a simulation study was performed to produce different sample sizes. The estimator values and approximate value of the reliability function were compared depending on the Integrated mean squared error. It should be noted that the approximate values are better than non-Bayes and Bayes estimators, and also the Bayes estimator is better than the non-Bayes estimator of the reliability function for the inverse Rayleigh distribution because it contains the least error for all sample sizes. The results of simulation experiments showed the efficiency of the approximate method in finding the approximate value of the reliability function for IR distribution, compared with non-Bayes and Bayes estimators.

We hope that this work will open the way for more research in the future to find statistical estimates of the parameters and functions associated with statistical distributions using other different numerical methods for example (The power function and Boubaker polynomials).

In a forthcoming document, the authors will show further progress on estimating functions associated with other distributions using the same numerical method as used in the paper or using other numerical methods (Newton's method or False position method) to get the approximate value of parameters or (The power function and Boubaker polynomials) to obtain the approximate value of functions associated with the above distribution.

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