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Variance estimator of repeated measurements model by iterated bootstrap with an application to oil industry pollutants in basrah

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Abstract

The aim of this research deals with the study of the estimators of the variance compounds by the Bootstrap approximate method of the one-way Repeated Measurements (RM) model and the calculation of the amount of bias in the estimators of the variance components. The model contains two fixed factors (one factor within units and one factor between units) and their interactions and two random factors. As an applied aspect, a study was undertaken to measure the air pollutants (CO, CO2 and CH4) in the Al-Shuaiba region – Basrah in Iraq to study the variation in pollutant concentrations in two randomly selected stations from the region with five sections with two directions for each station during the summer and winter seasons 2019–2020. The SPSS statistical analysis program was used to analyze the study data and calculate the amount of bias in the estimators of the variance components.

Key words and phrases: Variance estimator Repeated Measurement Model Bias estimator Iterated Bootstrap Oil pollutants Basrah.

Mathematics Subject Classification (2010): 62J10, 62J12, 62H12

1. Introduction

In many scientific and industrial research settings, data are often collected in a repeated measurements fashion, where multiple observations are taken on the same subjects or units over time or under different conditions. Examples of such studies can be found in medical trials, agricultural experiments, and manufacturing processes, among others. Analyzing repeated measurements data

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requires specialized statistical methods that can effectively account for the within-subject correlation and variability. Repeated Measurements Models (RMMs) are a class of statistical models designed to handle the complexities associated with repeated measurements data. These models consider the correlation structure within subjects and provide valuable insights into the underlying processes over time or under varying conditions. RMMs offer several advantages, including increased statistical power, improved efficiency in estimating parameters, and the ability to address missing data appropriately. A crucial aspect of analyzing repeated measurements data is estimating variance components, which quantify the extent of variability within and between subjects. Accurate estimation of variance components is essential for making meaningful inferences and drawing reliable conclusions from the data. However, deriving exact variance component estimators can be challenging, especially for complex RMMs with nonlinear relationships and heterogeneous variances.

The Bootstrap is a powerful resampling technique that allows us to approximate the sampling distribution of the estimators without making strong parametric assumptions. By repeatedly sampling from the observed data with replacement, the bootstrap generates an empirical distribution from which variance component estimates and their confidence intervals can be derived.

Basrah, a prominent city located in southern Iraq, has long been a hub of economic activity due to its rich oil reserves. However, this economic boon has come at a considerable environmental cost, with the oil industries in Basra contributing significantly to air pollution. The rapid expansion of oil extraction, refining, and related activities has led to a complex web of pollution sources that have dire consequences for both the environment and public health. The aim of this research is to calculate the variance component estimators for a one-way Repeated Measurements RM model with two random effects and calculate the amount of bias using the bootstrap approximation method and study of air pollutants in the province of Basra as an applied aspect.

2. Repeated Measurements (RM) Model

In a one-way Repeated Measurements (RM) design, there are numerous effects for the between-units factors. In a randomized one-way Repeated Measurements (RM) experiment, the experimental units are randomized to one between-units factor (a group with n_2 levels), one within-units factor (time with n_3 levels), random effect to experimental unit i within treatment group j and random effect to experimental unit i within treatment group k. For the one-way RM, designed with one between-unit factor, we use the following linear model and parameterization:

$$t_{ijk} = \xi + \varphi_j + \Psi_k + (\varphi\Psi)_{jk} + \omega_{1i(j)} + \omega_{2i(k)} + e_{ijk} \tag{1}$$

where

$i = 1, 2, \ldots, n_1$ is a unite of experimentation index. $j = 1, 2, \ldots, n_2$ is an indicator of the between units factor's levels (Group). $k = 1, 2, \ldots, n_3$ is an indicator of the within unite factor's levels (Time). t_{ijk} is the measurement of unit's response over time in a group. e_{ijk} is The random error. ξ is The overall mean. In the Table 1 classifies the effects of the math model (1) by type.

Table 1: Classifies the effects of one-way repeated measurements model.

The factor	The type	The definition	The condition
φ_i	Fixed	Is treatment between unite factor (group)	$\sum_{j=i}^{n_2} \varphi_i = 0$
Ψ_k	Fixed	Is treatment within unite factor (Time)	$\sum_{k=1}^{n_3} \Psi_k = 0$
$(\varphi\Psi)_{ij}$	Fixed	Is the effect (between X within) unites	$\sum_{j=1}^{n_2} \varphi_j \Psi_k = 0$ $\sum_{k=1}^{n_3} \varphi_j \Psi_k = 0$
$\omega_{1i(j)}$	Random	Is the random effect to unit *i* within Group (*j*)	$\omega_{1ij} \sim N(0, \sigma^2_{\omega_1})$
$\omega_{2i(k)}$	Random	Is the random effect to unit *i* within time (*k*)	$\omega_{2ij} \sim N(0, \sigma^2_{\omega_2})$
e_{ijk}	Random	Is the random error	$e_{ijk} \sim N(0, \sigma^2_e)$

The one-way RM model (1) can be written in its observations as follows:

$$
\begin{bmatrix} t_{111} \\ \vdots \\ t_{11n_3} \\ \vdots \\ t_{1n_21} \\ \vdots \\ t_{1n_2n_3} \\ \vdots \\ t_{n_111} \\ \vdots \\ t_{n_11n_3} \\ \vdots \\ t_{n_1n_21} \\ \vdots \\ t_{n_1n_2n_3} \end{bmatrix}
=
\begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \xi +
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & & & & & \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{matrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \vdots \\ \varphi_{n_2} \end{matrix}
+
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & & & & & \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & & & & & \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & & & & & \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{matrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \vdots \\ \Psi_{n_3} \end{matrix}
$$

$$
+
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{matrix} (\varphi\Psi)_{11} \\ (\varphi\Psi)_{12} \\ (\varphi\Psi)_{13} \\ (\varphi\Psi)_{14} \\ \vdots \\ (\varphi\Psi)_{1n3} \\ \vdots \\ (\varphi\Psi)_{n_21} \\ (\varphi\Psi)_{n_22} \\ (\varphi\Psi)_{n_23} \\ (\varphi\Psi)_{n_24} \\ \vdots \\ (\varphi\Psi)_{n_2n_3} \end{matrix}
$$

$$
+
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{matrix} (\omega_1)_{11} \\ (\omega_1)_{12} \\ (\omega_1)_{13} \\ (\omega_1)_{14} \\ \vdots \\ (\omega_1)_{1n2} \\ \vdots \\ (\omega_1)_{n_11} \\ (\omega_1)_{n_12} \\ (\omega_1)_{n_13} \\ (\omega_1)_{n_14} \\ \vdots \\ (\omega_1)_{n_1n_2} \end{matrix}
$$

$$
+\begin{bmatrix}
1&0&0&0&\cdots&0&\cdots&0&0&0&0&\cdots&0\\
0&1&0&0&\cdots&0&\cdots&0&0&0&0&\cdots&0\\
0&0&1&0&\cdots&0&\cdots&0&0&0&0&\cdots&0\\
0&0&0&1&\cdots&0&\cdots&0&0&0&0&\cdots&0\\
\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
0&0&0&0&\cdots&1&\cdots&0&0&0&0&\cdots&0\\
\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
0&0&0&0&\cdots&0&\cdots&1&0&0&0&\cdots&0\\
0&0&0&0&\cdots&0&\cdots&0&1&0&0&\cdots&0\\
0&0&0&0&\cdots&0&\cdots&0&0&1&0&\cdots&0\\
0&0&0&0&\cdots&0&\cdots&0&0&0&1&\cdots&0\\
\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots\\
0&0&0&0&\cdots&0&\cdots&0&0&0&0&\cdots&1
\end{bmatrix}
\begin{matrix}
(\omega_1)_{11}\\ (\omega_1)_{12}\\ (\omega_1)_{13}\\ (\omega_1)_{14}\\ \vdots\\ (\omega_1)_{1n2}\\ \vdots\\ (\omega_1)_{n_1 1}\\ (\omega_1)_{n_1 2}\\ (\omega_1)_{n_1 3}\\ (\omega_1)_{n_1 4}\\ \vdots\\ (\omega_1)_{n_1 n_2}
\end{matrix}
+\begin{bmatrix}
e_{111}\\ \vdots\\ e_{11n_3}\\ \vdots\\ e_{1n_2 1}\\ \vdots\\ e_{1n_2n_3}\\ \vdots\\ e_{n_1 11}\\ \vdots\\ e_{n_1 1n_3}\\ \vdots\\ e_{n_1n_2 1}\\ \vdots\\ e_{n_1n_2n_3}
\end{bmatrix}
$$

$$
\begin{aligned}
&=[1_{n_1}\otimes 1_{n_2}\otimes 1_{n_3}]\xi+\left[1_{n_1}\otimes I_{n_2}\otimes 1_{n_3}\right]\varphi+\left[1_{n_1}\otimes 1_{n_2}\otimes I_{n_3}\right]\Psi\\
&+\left[1_{n_1}\otimes I_{n_2}\otimes I_{n_3}\right](\varphi\Psi)+\left[I_{n_1}\otimes I_{n_2}\otimes 1_{n_3}\right]\omega_1\\
&+\left[I_{n_1}\otimes 1_{n_2}\otimes I_{n_3}\right]\omega_2+[I_{n_1}\otimes I_{n_2}\otimes I_{n_3}]e
\end{aligned}
\tag{2}
$$

Where I denoted the identity matrix, 1 denoted the vector one's, $\otimes$ be the kronecker product operation of two matrices. To formulate our study model more appropriately, we used the matrix formula, first we write the model (1) in the form below:

$$
t_{ij}=\alpha_j+1_{n_3}b_{i(j)}+1_{n_2}c_i+e_{ij} \tag{3}
$$

where $t_{ij}=\left[t_{ij1,},\ldots,t_{ijn_3}\right]'$ is a vector of response, $\alpha_j=\left[\alpha_{j1,},\ldots,\alpha_{jn_3}\right]'$ is a vector of the fixed treatments $(\alpha_{ij}=\xi+\varphi_j+\Psi_k+(\varphi\Psi)_{jk})$, $b_{i(j)}=\omega_{1i(j)}$ is the random effect of unit i within a group (j), $c_i=\left[c_{i(1),},\ldots,c_{i(n_3)}\right]'$ is the random effect to unit i within time$(\omega_{2i(k)})$ and $\boldsymbol{e_{ij}}=\left[e_{ij1,},\ldots,e_{ijn_3}\right]'$ is a vector of random error. Let б$_{ij}$ represent the indicators of a group such that б$_{ij}$$=\begin{cases}1 & \text{if unit i is from group j}\\ 0 & \text{o.w}\end{cases}$ j = 1,…, n_2. By setting $1_i'=\left[\alpha_{i1,},\ldots,\alpha_{in_2}\right]'$, we can rewrite model (4) as

$$
\tau_i=X_i\alpha+Z_iW\varepsilon_i \tag{4}
$$

where $t_i' = [t_{i1,\ldots,} \; t_{in_3}]$, $X_i = I_{n_3} \otimes б_i$, $\alpha' = \left[\alpha_{11,},\ldots,\alpha_{n_2 n_3}\right]$, $Z_i = 1_{n_3} \otimes 1_{n_2}$, $W = [\omega_{1i(j)} \;, \omega_{2i(k)} \;]$, $\varepsilon_i' = [\varepsilon_{\,i1,\ldots,} \; \mu_{in_3}]$ and $\alpha = \mathrm{Vec}\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n_3} \\ \vdots & \cdots & \vdots \\ \alpha_{n_2 1} & \cdots & \alpha_{n_2 n_3} \end{bmatrix}$. The final formulation of the study model is an alternative form of the model (4) as shown in the relationship (5), let $\tau' = [\, t_1',\ldots,t_{n_1}' \,]$, $X' = [\, X_1',\ldots,X_{n_1}' \,]$, $W = [\omega_{1i(j)} \;, \omega_{2i(k)}, \varepsilon_i \;]$ and $Z = [z_1',\ldots,z_{n_1}']$ then

$$\tau = X\alpha + ZW \tag{5}$$

In our One-Way RM model, we have three random effects with its variance as follows:

$\mathrm{Var}\left(\omega_{1i(j)}\right) = \sigma^2_{\omega_1}$, $\mathrm{Var}\left(\omega_{2i(k)}\right) = \sigma^2_{\omega_2}$, $\mathrm{Var}\left(e_{ijk}\right) = \sigma^2_e$, Cov ($\omega_{1i(j)}$, $\omega_{2i(k)}$) = 0, Cov ($\omega_{1i(j)}$, e_{ijk}) = 0 and Cov ($\omega_{2i(j)}$, e_{ijk}) = 0. The variance matrix of τ is denoted as $\prod$, and the elements of $\prod$ depended on terms of (2) such that:

$$\begin{aligned} \prod_{\omega_1} &= var([I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}]\omega_1) = \left[I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}\right]\sigma^2_{\omega_1} I_{n_1}\left[I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}\right]' \\ &= \left[I_{n_1} \otimes I_{n_2} \otimes 1_{n_3}\right]\sigma^2_{\omega_1}[I_{n_1}{}' \otimes I_{n_2}{}' \otimes I_{n_3}{}'] = \sigma^2_{\omega_1}\left[I_{n_1} \otimes I_{n_2} \otimes J_{n_3}\right] \\ \prod_{\omega_2} &= var(\left[I_{n_1} \otimes 1_{n_2} \otimes I_{n_3}\right]\omega_2) = \left[I_{n_1} \otimes 1_{n_2} \otimes I_{n_3}\right]\sigma^2_{\omega_2} I_{n_1}\left[I_{n_1} \otimes 1_{n_2} \otimes I_{n_3}\right]' \\ &= \left[I_{n_1} \otimes 1_{n_2} \otimes I_{n_3}\right]\sigma^2_{\omega_2}[I_{n_1}{}' \otimes 1_{n_2}{}' \otimes I_{n_3}{}'] = \sigma^2_{\omega_2}\left[I_{n_1} \otimes J_{n_2} \otimes I_{n_3}\right], \end{aligned}$$

where J denoted the matrix of ones.

$$\begin{aligned} \prod_e &= [I_{n_1} \otimes I_{n_2} \otimes I_{n_3}]\sigma^2_e I_{n_1}[I_{n_1} \otimes I_{n_2} \otimes I_{n_3}]' = [I_{n_1} \otimes I_{n_2} \otimes I_{n_3}]\sigma^2_e[I_{n_1}{}' \otimes I_{n_2}{}' \otimes I_{n_3}{}'] \\ &= \sigma^2_e[I_{n_1} \otimes I_{n_2} \otimes I_{n_3}] \end{aligned}$$

Then the variance matrix $\prod = \begin{bmatrix} \Pi_{\omega_1} + \Pi_{\omega_2} + \Pi_e & 0 & \cdots & 0 \\ 0 & \Pi_{\omega_1} + \Pi_{\omega_2} + \Pi_e & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \Pi_{\omega_1} + \Pi_{\omega_2} + \Pi_e \end{bmatrix}$

3. Estimation of Variance Components for the One-way RM Model

One of the classic methods used in estimation is the maximum likelihood method, after applying the algorithm of this method to our study model we obtained the estimators of the variance components $\hat{\Theta} = (\widehat{\sigma^2_{\omega_1}}, \widehat{\sigma^2_{\omega_2}}, \widehat{\sigma^2_e})$ is defined by the formula (6) and $\Theta = (\sigma^2_{\omega_1}, \sigma^2_{\omega_2}, \sigma^2_e) = \Theta_r$ i.e Θ_r is γ^{th} component of vector of variance Θ.

$$\hat{\Theta}_{k+1} = \hat{\Theta}_k - \Gamma^{-1}_{\hat{\Theta}_k \hat{\Theta}_k} \Gamma_{\hat{\Theta}_k} \tag{6}$$

Where:

$$\Gamma_{\Theta} = \begin{bmatrix} \frac{1}{2}\left[R'\Pi^{-1}(\Theta)\Pi_0\Pi^{-1}(\Theta)R - t_r\left(\Pi^{-1}(\Theta)\Pi_0\right)\right] \\ \frac{1}{2}\left[R'\Pi^{-1}(\Theta)\Pi_1\Pi^{-1}(\Theta)R - t_r\left(\Pi^{-1}(\Theta)\Pi_1\right)\right] \\ \frac{1}{2}\left[R'\Pi^{-1}(\Theta)\Pi_2\Pi^{-1}(\Theta)R - t_r\left(\Pi^{-1}(\Theta)\Pi_2\right)\right] \end{bmatrix}, R = (\tau - X\alpha).$$

$$\Gamma_{\Theta\Theta} = -\frac{1}{2}[tr(\Pi^{-1}\Pi_s\Pi^{-1}\Pi_r - \Pi^{-1}\Pi_r\Pi^{-1}\Pi_s) + R'(\Pi^{-1}\Pi_r\Pi^{-1}\Pi_s + \Pi^{-1}\Pi_s\Pi^{-1}\Pi_r) - \Pi_{rs}\Pi^{-1})\Pi^{-1}R]$$

where r and $s = \{0, 1, 2\}$.

The maximum likelihood method is considered a good method in most linear models, but after applying it to our study model in estimating the variance components, it was found that the estimators are biased such that from equation (7), the bias amount is Ω.

$$E(\hat{\Theta}_{k+1}) = E(\hat{\Theta}_k) - E(\Gamma_{\hat{\Theta}_k\hat{\Theta}_k}{}^{-1}\Gamma_{\hat{\Theta}_k}) \tag{7}$$

$$= \begin{bmatrix} \widehat{\sigma_e^2} \\ \widehat{\sigma_{\omega_1}^2} \\ \widehat{\sigma_{\omega_2}^2} \end{bmatrix} - \left(\frac{1}{N}\begin{bmatrix} \widehat{\sigma_e^2} \\ \widehat{\sigma_{\omega_1}^2} \\ \widehat{\sigma_{\omega_2}^2} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & \mathrm{E}_{01} & \mathrm{E}_{02} \\ \mathrm{E}_{10} & 0 & \mathrm{E}_{12} \\ \mathrm{E}_{20} & \mathrm{E}_{21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} t_r\left(\Pi^{-1}(\Theta)\Pi_0\right) \\ t_r\left(\Pi^{-1}(\Theta)\Pi_1\right) \\ t_r\left(\Pi^{-1}(\Theta)\Pi_2\right) \end{bmatrix}\right)$$

$$\Omega = \left(\frac{1}{N}\begin{bmatrix} \widehat{\sigma_e^2} \\ \widehat{\sigma_{\omega_1}^2} \\ \widehat{\sigma_{\omega_2}^2} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & \mathrm{E}_{01} & \mathrm{E}_{02} \\ \mathrm{E}_{10} & 0 & \mathrm{E}_{12} \\ \mathrm{E}_{20} & \mathrm{E}_{21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} t_r\left(\Pi^{-1}(\Theta)\Pi_0\right) \\ t_r\left(\Pi^{-1}(\Theta)\Pi_1\right) \\ t_r\left(\Pi^{-1}(\Theta)\Pi_2\right) \end{bmatrix}\right) \tag{8}$$

where:

$$E = \begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}$$

$$E01 = -\frac{1}{2}tr(\Pi^{-1}\Pi_1\Pi^{-1}\Pi_0 - \Pi^{-1}\Pi_0\Pi^{-1}\Pi_1).$$

$$E02 = -\frac{1}{2}tr(\Pi^{-1}\Pi_2\Pi^{-1}\Pi_0 - \Pi^{-1}\Pi_0\Pi^{-1}\Pi_2).$$

$$E10 = -\frac{1}{2}tr(\Pi^{-1}\Pi_0\Pi^{-1}\Pi_1 - \Pi^{-1}\Pi_1\Pi^{-1}\Pi_0).$$

$$E12 = -\frac{1}{2}tr(\Pi^{-1}\Pi_2\Pi^{-1}\Pi_1 - \Pi^{-1}\Pi_1\Pi^{-1}\Pi_2).$$

$$E20 = -\frac{1}{2}tr(\Pi^{-1}\Pi_0\Pi^{-1}\Pi_2 - \Pi^{-1}\Pi_2\Pi^{-1}\Pi_0).$$

$$E21 = -\frac{1}{2}tr(\Pi^{-1}\Pi_1\Pi^{-1}\Pi_2 - \Pi^{-1}\Pi_2\Pi^{-1}\Pi_1).$$

4. Iterated Bootstrap Method

The bootstrapping method is a modern technique of resampling, it can be defined as resampling including n elements pulled randomly from N original data. The bootstrap method is one of the inferential simulation methods that includes estimating the parameters and the bias in Bootstrap sample. Now to applied the bootstrap method, let $\tilde{\Theta}(X_{\varkappa})$ is a solution of $\Gamma_{\Theta}(X_{\varkappa}) = 0$ and $X_{\varkappa}$ are responses simulated by one-way RM model with Θ, From solving the scour function with the bootstrap method (equation (9)) by Newton Raphson numerical method we get the estimator $\tilde{\Theta}^*$

$$\Gamma^*_{\tilde{\Theta}} = \begin{bmatrix} \frac{1}{2}\left[R'\Pi^{-1}(\Theta)\Pi_0\Pi^{-1}(\Theta)R - t_r\left(\Pi^{-1}(\Theta)\Pi_0\right)\right] \\ \frac{1}{2}\left[R'\Pi^{-1}(\Theta)\Pi_1\Pi^{-1}(\Theta)R - t_r\left(\Pi^{-1}(\Theta)\Pi_1\right)\right] \\ \frac{1}{2}\left[R'\Pi^{-1}(\Theta)\Pi_2\Pi^{-1}(\Theta)R - t_r\left(\Pi^{-1}(\Theta)\Pi_2\right)\right] \end{bmatrix} + \frac{1}{2}tr\left[\begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} D^{-1}\sum_{\varkappa=1}^{D}\tilde{\Theta}_0(X_{\varkappa}) - \Theta \\ D^{-1}\sum_{\varkappa=1}^{D}\tilde{\Theta}_1(X_{\varkappa}) - \Theta \\ D^{-1}\sum_{\varkappa=1}^{D}\tilde{\Theta}_2(X_{\varkappa}) - \Theta \end{bmatrix}\right] \quad (9)$$

The estimator $\tilde{\Theta}^*$ is

$$\widetilde{\Theta}^*_{(j+1)} = \begin{bmatrix} \widetilde{\sigma^2_{\omega_1}}^*_{(j+1)} \\ \widetilde{\sigma^2_{\omega_2}}^*_{(j+1)} \\ \widetilde{\sigma^2_{e}}^*_{(j+1)} \end{bmatrix} = \left(2\begin{bmatrix} \widetilde{\sigma^2_{\omega_1}}^*_{(j)} \\ \widetilde{\sigma^2_{\omega_2}}^*_{(j)} \\ \widetilde{\sigma^2_{e}}^*_{(j)} \end{bmatrix} - \begin{bmatrix} \widetilde{\sigma^2_{\omega_1}}^*_{(j)} \\ \widetilde{\sigma^2_{\omega_2}}^*_{(j)} \\ \widetilde{\sigma^2_{e}}^*_{(j)} \end{bmatrix}\right) + \frac{1}{2}tr\left[\begin{bmatrix} 0 & E_{01} & E_{02} \\ E_{10} & 0 & E_{12} \\ E_{20} & E_{21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}\left[R'\,\Pi^{-1}\left(\widetilde{\sigma^2_{\omega_1}}^*_{(j)}\right)\Pi_0\,\Pi^{-1}\left(\widetilde{\sigma^2_{\omega_1}}^*_{(j)}\right)R - t_r\left(\Pi^{-1}\left(\widetilde{\sigma^2_{\omega_1}}^*_{(j)}\right)\Pi_0\right)\right] \\ \frac{1}{2}\left[R'\,\Pi^{-1}\left(\widetilde{\sigma^2_{\omega_2}}^*_{(j)}\right)\Pi_1\,\Pi^{-1}\left(\widetilde{\sigma^2_{\omega_2}}^*_{(j)}\right)R - t_r\left(\Pi^{-1}\left(\widetilde{\sigma^2_{\omega_2}}^*_{(j)}\right)\Pi_1\right)\right] \\ \frac{1}{2}\left[R'\,\Pi^{-1}\left(\widetilde{\sigma^2_{e}}^*_{(j)}\right)\Pi_2\,\Pi^{-1}\left(\widetilde{\sigma^2_{e}}^*_{(j)}\right)R - t_r\left(\Pi^{-1}\left(\widetilde{\sigma^2_{e}}^*_{(j)}\right)\Pi_2\right)\right] \end{bmatrix}\right] \quad (10)$$

Where jth is iteration and $\overline{\widetilde{\sigma^2_{..}}^*_{(j)}}$ is the mean of the maximum likelihood estimators at each of D bootstrap sample in $\widetilde{\sigma^2_{..}}^*_{(j)}$. The criterion for stopping the recurrence of the bootstrap method in our study is when the following condition is met:

$$\begin{bmatrix} \widetilde{\sigma^2_{\omega_1}}^*_{(j+1)} \\ \widetilde{\sigma^2_{\omega_2}}^*_{(j+1)} \\ \widetilde{\sigma^2_{e}}^*_{(j+1)} \end{bmatrix} - \begin{bmatrix} \widetilde{\sigma^2_{\omega_1}}^*_{(j)} \\ \widetilde{\sigma^2_{\omega_2}}^*_{(j)} \\ \widetilde{\sigma^2_{e}}^*_{(j)} \end{bmatrix} < 10^{-6} \quad (11)$$

We summarize the bootstrap algorithm in estimating the components of variance for the least biased repeated measures model as in the steps below:

1. From the sample $\tau = (t_1, t_2, \ldots, t_n)$, choose a partial random sample by return, with a specific size is $\tau^* = (t_1^*, t_2^*, t_n^*)$ a number of times D = 1000 and 10000.
2. Calculate the variance component estimator for the one-way RM model of the bootstrap sample from the formula (10).
3. Calculate the amount of bias for the estimators of the variance compounds.

5. The Experiment

The experiment data was taken from the University of Basra - College of Education, which represents a study of the effect of three gases CO, CO2, and CH4 as air pollutants in the city of Shuaiba - Basra (Shuaiba Oil Refinery) in two directions inside and outside the refinery. The experiment was studied in two randomly selected stations A and B, where each station was divided into five random section during the summer and winter seasons in 2019 and 2020.

6. The Results and Discussion

The experiment was described According to the mathematical formula of the model (1) such that the seasons of the year (summer and winter) represent a factor within the units (time) and the stations (A, B) represent a factor between the units (group), the direction (inside I and outside II) And the five sites (a, b, c, d, e) represent the random effects of the model, as shown in the table below.

By using the SPSS statistical analysis program. The variance components for random effects were estimated using the maximum likelihood method $\hat{\Theta} = [0.13, 0.003, 1.193]$ but these estimates are considered unreliable because we obtained the matrix (E) is not positive meaning that the maximum likelihood method was not feasible in our study model, so the estimates were calculated using the approximate method (bootstrap $\tilde{\Theta}^*$) by reshaping the sample D = 1000, 10000 at a confidence level (coverage = 95%, 98% and 99%) and then the amount of bias at each random effect as shown in Table 3. As a result, we obtained estimators with a negative bias in all cases of estimation using the

Table 2: The Description of Experiment's Effects

Fixed Effects			Random Effects		
	Ψ_k	Summer		$\omega_{2i(k)}$	Inside I
		Winter			Outside II
	φ_j	Station A		$\omega_{1i(j)}$	Section a
		Station B			Section b
					Section c
					Section d
					Section e

Table 3: The results table

		$\tilde{\Theta}^*_0$		$\tilde{\Theta}^*_1$		$\tilde{\Theta}^*_2$	
D	Coverage	Bias	Std. error	Bias	Std. error	Bias	Std. error
1000	95%	–0.08	0.109	–0.001	0.02	–0.03	0.041
	98%	–0.013	0.111	–0.001	0.001	–0.005	0.04
	99%	–0.07	0.109	–0.001	0.002	–0.001	0.04
10000	95%	–0.008	0.11	–0.001	0.001	–0.002	0.43
	98%	–0.007	0.108	–0.001	0.002	–0.001	0.43
	99%	–0.008	0.108	–0.001	0.001	–0.002	0.43

bootstrap method, and this can be explained that the estimated values are less than the real values, also the results of the analysis indicate that all the studied locations have exceeded the allowable limit for air pollution with gases CO, CO2, and CH4, and that each of the Sections and their directions affected the concentrations of these Gases inside the Shuaiba Oil Refinery in Basrah Governorate.

7. Conclusion

The approximate Bootstrap method is considered good for finding variance component estimators for a one-way Repeated Measurements Model. This method solves one of the most important problems we face in calculating estimators of variance components, which is the bias in the estimators. As another aspect of the study, it was found that there is pollution in the air in the governorate. Basra (Shuaiba) and exceeded the permissible limit.

Reference

[1] Al-Mouel, A. H. S. and Wang, J. L.,One–Way Multivariate Repeated Measurements Analysis of Variance Model. *Applied Mathematics a Journal of Chinese Universities* 19(4), pp. 435–448. (2004).

[2] AL-Mouel, A. H. S and Mustafa, H. I., The Sphericity Test for One-Way Multivariate Repeated Measurements Analysis of Variance Mode. *Journal of Kufa for Mathematics and Computer* 2(2), pp. 107–115. (2014).

[3] Al-Mouel, A.H.S., Multivariate Repeated Measures Models and Comparison of Estimators, Ph.D. Thesis, East China Normal University, China. (2004).

[4] Alsarayfi A., Environmental Impacts Resulting from Al-Shuaiba Refinery (Southern Iraq) on the Surrounding Environment. *Arabian Gulf Journal*, 49(1). (2021).

[5] Jassim N. O. and Al-Mouel, A H S., Lasso Estimation for High-Dimensional Repeated Measurement Model, *AIP Conf. Proc.*, 2292, 020002, pp. 1–10. (2020).

[6] Jiang, J., Linear and Generalized Linear Mixed Models and Their Applications. Springer Series in Statistics. (2007).

[7] Joseph G Ibrahim, Hongtu Zhu, Ramon I Garcia, and Ruixin Guo. Fixed and random effects selection in mixed effects models. *Biometrics*, 67(2), pp. 495–503. (2011).

[8] Kori H. A. and AL-Mouel, A. H. S,. Expected mean square rate estimation of repeated measurements model, *Int. J. Nonlinear Anal. Appl.* 12, (2021).

[9] Kosmidis I, Guolo A and Varin C., Improving the accuracy of likelihood-based inference in meta-analysis and metaregression. *Biometrika*. 104, pp. 489–496. (2017).

[10] Kosmidis, I.,Bias in parametric estimation: reduction and useful sideeffects. *Wiley Interdisciplinary Reviews: Computational Statistics* 6, pp. 185–196. (2014).

[11] Mustafa, H.I. and Al-Mouel, A.H.S. Asymptotic Normality of Estimator Variance Components for One-Way Repeated Measurements Model, *Utilitas Mathematicathis.* 120, pp. 261–268. (2023).

[12] Okamura H., Takasuka A., (2012), A bootstrap method for testing equality of peak months, *Popul Ecol.* 54, pp. 169–176. (2012).

[13] Vonesh, E.F. and Chinchilli, V.M.,Linear and Nonlinear Models for the Analysis of Repeated Measurements, *Marcel Dakker, Inc.*, New York. (1997).

[14] Wand, M., Fisher information for generalised linear mixed models. *Journal of Multivariate Analysis.* 98, pp. 1412–1416. (2007).

[15] Yan-fei Y. Evaluating the quality of statistical data based on the Bootstrap method, *CORD Conference Proceedings* 4, pp. 413. (2011).