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Fixed point theorems in generalized b-Menger spaces

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Abstract

The purpose of this work is to define the generalized b-Menger spaces and prove a fixed point theorem in this new setting. As application, we establish the existence and uniqueness of a solution for Volterra type integral equation. Our results extend and generalize the existing results in literature.

Key words and phrases: b-Menger space, Cauchy sequence, Fixed point.

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1. Introduction and preliminaries

A metric space is a fundamental concept in analysis and topology, which has been introduced by Fréchet [1].

Definition 1.1. A metric (or distance) on a nonempty set Z is a function $\rho : \mathrm{Z} \times \mathrm{Z} \to \mathbb{R}^+$ which satisfies the following conditions, for all $p, q, r \in$ Z$

(1) $\rho(p, q) = 0$ if and only if $p = q$,
(2) $\rho(p, q) = \rho(q, p)$,
(3) $\rho(p, q) \leq \rho(p, r)+\rho(r, q)$.

The pair (Z, ρ) is called a metric space.

There is a generalization of this notion, which were obtained by various alternation of one, two or all three conditions above. For example of these generalizations we can find the notion of b-metric space introduced by Bakhtin [2] and Czerwik [3].

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Definition 1.2. A function ρ: $Z \times Z \to \mathbb{R}^+$ is called a b-metric on Z if it satisfies the following conditions, for all $p, q, r \in Z$:

(1) $\rho(p, q) = 0$ if and only if $p = q$,
(2) $\rho(p, q) = \rho(q, p)$,
(3) There exists a constant $\ell \geq 1$ such that:

$$\rho(p, q) \leq \ell(\rho(p, r)+\rho(r, q)).$$

The triplet (Z, ρ, ℓ) is called a b-metric space. Fixed point theorems in b-metric spaces can be found in [4, 5].

In 2017, Kamran et al. [6] introduced the notion of an extended b-metric space and proved fixed point theorems in these spaces.

Definition 1.3. Let Z be a non empty set and $\alpha : Z \times Z \to [1, \infty)$. A function $\rho : Z \times Z \to \mathbb{R}^+$ is called an extended b-metric on Z if for all $p, q, r \in Z$, we have:

(1) $\rho(p, q) = 0$ if and only if $p = q$,
(2) $\rho(p, q) = \rho(q, p)$,
(3) $\rho(p, q) \leq \alpha(p, q)(\rho(p, r)+\rho(r, q))$.

The pair (Z, ρ_α) is called an extended b-metric space.

Remark 1.1. By setting $\rho = \ell$ with $\ell \geq 1$, the above definition becomes the definition of a b-metric space.

In other hand, Menger [7] in 1942 introduced the notion of probabilistic metric spaces, as a generalization of metric spaces. Some basic results on these spaces were summarized in the book [8]. To generalize the notion of probabilistic metric, Mbarki and Oubrahim [9] defined the probabilistic b-metric spaces by extending the triangle inequality assertion in the definition of probabilistic metric spaces and they studied a fixed point theory in these spaces using the topoligical and geometrical properties. For more details see [10, 11, 12].

We now recall some basic definitions in the theory of b-Menger spaces (See [9] and [10]).

Definition 1.4. A distance distribution function is a nondecreasing function h defined on $\mathbb{R}^+ \cup \{\infty\}$ that satisfies $h(0) = 0$ and $h(\infty) = 1$, and is left continuous on $(0, \infty)$. The set of all distribution function will be denoted by Δ^+ and the set of all h in Δ^+ for $\lim_{t\to\infty} h(t) = 1$ by $\mathcal{D}^+$.

A simple example of distribution function is the unit step function in $\mathcal{D}^+$

$$\epsilon(t) = \begin{cases} 0 & if \quad t \leq a, \\ 1 & if \quad t > a. \end{cases}$$

The first field where triangular norms (t-norm briefly) played a major role was the theory of probabilistic metric spaces (see [13]).

Definition 1.5. A function $N : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if the following conditions are satisfied for any$\lambda, \mu, \nu, \xi \in [0, 1]$:

(1) $N(\mu, 1) = \mu$,
(2) $\mathrm{N}(\mu, \nu) = \mathrm{N}(\nu, \mu)$,
(3) $N(\mu, \nu) \geq N(\lambda, \xi)$, for $\mu \geq \lambda$, $\nu \geq \xi$,
(4) $N(N(\mu, \nu), \xi) = N(\mu, N(\nu, \xi))$.

A t-norm N is said to be continuous if $N(\mu, \nu)$ is continuous at each point $(\mu, \nu) \in [0, 1] \times [0, 1]$. Typical continuous t-norms are

(1) The minimum t-norm $N_M(\mu, \nu) = \min(\mu, \nu)$.
(2) The product t-norm $N_p(\mu, \nu) = \mu\nu$.
(3) The Lukasiewicz t-norm $N_L(\mu, \nu) = \max(\mu + \nu - 1, 0)$.

Remark 1.2. We have $N \leq N_M$ for each t-norm N.

Definition 1.6. [9] A b-Menger space is a quadruple (Z, G, N, ℓ) where Z is a nonempty set, G is a function from $Z \times Z$ into $\mathcal{D}^+$, N is a t-norm and the following conditions are satisfied for all $p, q, r \in Z$ and $a, b > 0$

(1) $G_{p,q} = \epsilon_0$ if and only if $p = q$,
(2) $G_{p,q} = G_{q,p}$,
(3) There exists a constant $\ell \geq 1$ such that $G_{p,q}(\ell(a + b)) \geq N(G_{p,r}(a), G_{r,q}(b))$.

Mbarki and Oubrahim [9] proved that if the t-norm N is continuous, then (Z, G, N, ℓ) is a first countable topological space which means that the family of sets $\{V_p(\gamma)\colon \gamma > 0\}$ is a base of neighborhoods of point $p \in Z$, where

$$V_p(\gamma) = \{q \in Z\colon G_{p,q}(\gamma) > 1 - \gamma\}.$$

Some fundamental roles of the theory of probabilistic metric spaces can be apply in controllability of probabilistic systems, analysing the complexity of algorithms and quantum particle physics (see [14, 15, 16]).

2. Generalized *b*-Menger spaces

Now, we extend the concept of Menger spaces by defining a generalized b-Menger spaces.

Definition 2.1. A generalized b-Menger space is a triplet (Z, G^α, N) where Z is a nonempty set, $\alpha\colon Z \times Z \to [1, \infty)$, G is a function from $Z \times Z$ into Δ^+, N is a continuous t-norm, and the following conditions are satisfied for all $p, q, r \in Z$ and $a, b > 0$

(1) $G_{p,q} = \epsilon_0$ if and only if $p = q$,
(2) $G_{p,q} = G_{q,p}$,
(3) $G_{p,q}(\alpha(p, q)(a + b)) \geq N(G_{p,r}(a), G_{r,q}(b))$.

Remark 2.1. Setting $\alpha = \ell$ with $\ell \geq 1$ then Definition 1.6 of a b-Menger space becomes a special case of the above definition of generalized b-Menger space.

Now we present an example of generalized b-Menger space.

Example 2.1. Let $Z = \{1, 2, 3\}$ and define $\rho : Z \times Z \to \mathbb{R}^+$ by $\rho(p, q) = (p - q)^2$. We consider the mapping $\alpha : Z \times Z \to [1, \infty)$ defined by $\alpha(p, q) = p + q + 1$. Define the mapping $G\colon Z \times Z \to \Delta^+$ by

$$G_{p,q}(\gamma) = \begin{cases} \dfrac{\gamma}{\gamma + \rho(p,q)} & \text{if} \quad \gamma > 0, \\ 0 & \text{if} \quad \gamma = 0. \end{cases}$$

and consider the continuous t-norm N_M.

We prove that (Z, G^α, N_M) is a generalized b-Menger space. The conditions (1) and (2) of Definition 2.1 are trivially satisfied. Now we show the assertion (3) for all $p, q \in Z$. We have

$$G_{p,q}\left(\alpha(p,q)(a+b)\right) = \frac{\alpha(p,q)(a+b)}{\alpha(p,q)(a+b) + \rho(p,q)}.$$

If $p = 1$, $q = 2$, then

$$G_{1,2}\left(\alpha(1,2)(a+b)\right) = \frac{\alpha(1,2)(a+b)}{\alpha(1,2)(a+b)+\rho(1,2)} = 1 - \frac{1}{4(a+b)+1},$$

$$G_{1,3}(a) = 1 - \frac{4}{a+4},$$

and

$$G_{3,2}(b) = 1 - \frac{1}{b+1}.$$

For all a, $b > 0$, we have

$$\begin{aligned} G_{1,2}\left(\alpha(1,2)(a+b)\right) &= 1 - \frac{4}{16a+16b+4} \\ &> 1 - \frac{4}{16a+4} \\ &> 1 - \frac{4}{a+4}. \end{aligned}$$

Thus

$$G_{1,2}(\alpha(1, 2)(a + b)) > G_{1,3}(a).$$

By the same way we have

$$G_{1,2}(\alpha(1, 2)(a + b)) > G_{3,2}(b).$$

Therefore

$$G_{1,2}(\alpha(1, 2)(a + b)) \geq \min\{G_{1,3}(a), G_{3,2}(b)\}.$$

Also we can prove that

$$G_{1,3}(\alpha(1, 3)(a + b)) \geq \min\{G_{1,2}(a), G_{2,3}(b)\},$$

$$G_{2,3}(\alpha(1, 2)(a + b)) \geq \min\{G_{2,1}(a), G_{1,3}(b)\}.$$

Hence for all $p, q, r \in Z$

$$G_{p,q}(\alpha(p,q)(a + b)) \geq N_M\{G_{p,r}(a), G_{r,q}(b)\}.$$

Therefore (Z, G^{α}, N_M) is a generalized b-Menger space.

Lemma 2.1. Let (Z, ρ_α) be an extended b-metric space. Define $G : Z \times Z \to \Delta^+$ by

$$G_{p,q}(\gamma) = \epsilon_{\rho\alpha(p,q)}(\gamma).$$

Then

(1) (Z, G^{α}, N_M) is a generalized b-Menger space.
(2) (Z, G^{α}, N_M) is complete leads (Z, ρ_α) is complete and vice versa.

Proof.

(1) It is easy to check the conditions (1) and (2) of Definition 2.1. So, for condition (3), let $p, q, r \in Z$, let $a, b \in [0, \infty)$.

If $\min(G_{p,r}(a), G_{r,q}(b)) = 0$, then $G_{p,q}(\alpha(p,q)(a + b)) \geq \min(Gp, r(a), G_{r,q}(b))$.
Else if $min(G_{p,r}(a),G_{r,q}(b)) = 1$, then $a > \rho(p, r)$ and $b > \rho(r, q)$. Since (Z, ρ_a) is an extended b-metric space, we have

$$\begin{aligned}\rho(p,q) &\leq \alpha(p,q)[\rho(p,r)+\rho(r,q)] \\ &\leq \alpha(p,q)(a+b).\end{aligned}$$

Then we get $G_{p,q}(\alpha(p, q)(a + b)) = 1$. Thus

$$G_{p,q}(\alpha(p, q)(a + b)) \geq N_M(G_{p,r}(a), G_{r,q}(b)).$$

Then condition (3) holds. So (Z, G^a, N_M) is a generalized b-Menger space.

(2) We have for every $\gamma > 0$,

$$V_p(\gamma) = \{q \in \mathrm{Z}: \rho_a(p, q) < \gamma\}.$$

So (Z, G^a, N_M) is complete if and only if (Z, ρ_a) is complete.

Let $\{z_n\}$ be a sequence in a generalized b-Menger space (Z, G^a, N).

Definition 2.2.

(1) A sequence $\{z_n\}$ in Z is said to be convergent if there exists $z \in \mathrm{Z}$ such that

$$\lim_{n\to\infty} G_{z_n,z}(\gamma) = 1, \ \ \forall \gamma > 0.$$

(2) A sequence $\{z_n\}$ in Z is said to be Cauchy sequence if

$$\lim_{n\to\infty} G_{z_n,x_{n+j}}(\gamma) = 1 \ for\ all\ \gamma > 0\ and\ j > 0.$$

(3) A generalized b-Menger space in which every Cauchy sequence is convergent is called a complete generalized b-Menger space.

3. Fixed point theorem in generalized *b*-Menger *s*pace

In the proof of our main theorem, we use the following lemma.

Lemma 3.1. Let (Z, G^a, N) be a generalized b-Menger with $RanG \subset \mathcal{D}^+$ and let $\{b_n\}$ a sequence in Z. Suppose that there exists $\beta \in (0, 1)$ such that

$$G_{b_n,b_{n+1}}(\beta\gamma) \geq G_{b_0,b_1}\left(\frac{\gamma}{\beta^{n-1}}\right) \tag{3.1}$$

and suppose also that for all $n, j \in \mathcal{N}$, we have

$$\alpha(b_n, b_{n+j}) < \frac{1}{\beta}.$$

Then $\{b_n\}$ is a Cauchy sequence.

Proof. Let $\{b_n\}$ be a sequence in Z satisfying (3.1). We have $\gamma = \frac{\gamma}{j} + \frac{(j-1)\gamma}{j}$ for any $j \in \mathcal{N}$ and using (3) of Definition 2.1 by successive applications we obtain

$$G_{b_n,b_{n+j}}(\gamma) \geq N\left(G_{b_n,b_{n+1}}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})}\right);G_{b_{n+1},b_{n+j}}\left(\frac{(j-1)\gamma}{j\alpha(b_n,b_{n+j})}\right)\right)$$

$$\geq N\left(G_{b_n,b_{n+1}}\left(\frac{t}{j\alpha(b_n,b_{n+j})}\right);N\left(G_{b_{n+1},b_{n+2}}\left(\frac{\gamma}{\gamma\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})}\right);\right.\right.$$

$$\left.\left.G_{b_{n+2},b_{n+j}}\left(\frac{(j-2)\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})}\right)\right)\right)$$

$$= N\left(N\left(G_{b_n,b_{n+1}}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})}\right);G_{b_{n+1},b_{n+2}}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})}\right)\right);\right.$$

$$\left.G_{b_{n+2},b_{n+j}}\left(\frac{(j-2)\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})}\right)\right)$$

$$\geq N\left(N\left(...\left(N\left(G_{b_n,b_{n+1}}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})}\right);G_{b_{n+1},b_{n+2}}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})}\right)\right);\right.\right.\right.$$

$$G_{b_{n+2},b_{n+3}}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})\alpha(b_{n+2},b_{n+j})}\right);...;$$

$$\left.G_{b_{n+j-1},b_{n+j}}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})...\alpha(b_{n+j-1},b_{n+j})}\right)\right).$$

By (3.1) and in view of (3) of Definition 2.1 we get

$$G_{b_n,b_{n+j}}(\gamma) \geq N\left(N\left(...\left(N\left(G_{b_0,b_1}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\beta^n}\right);G_{b_0,b_1}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})\beta^{n+1}}\right)\right);\right.\right.\right.$$

$$G_{b_0,b_1}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})\alpha(b_{n+2},b_{n+j})\beta^{n+2}}\right);...;$$

$$\left.G_{b_0,b_1}\left(\frac{\gamma}{j\alpha(b_n,b_{n+j})\alpha(b_{n+1},b_{n+j})...\alpha(b_{n+j-1},b_{n+j})\beta^{n+j-1}}\right)\right).$$

Since for all $n, j \in \mathcal{N}$ we have $\alpha(b_n, b_{n+j})\beta < 1$ with $\beta \in (0, 1)$ and letting $n \to \infty$, it follows

$$\lim_{n\to\infty} G_{b_n,b_{n+j}}(\gamma) = N(N(...(N(1;1);1)...;1) = 1.$$

So we conclude that $\{b_n\}$ is a Cauchy sequence in Z.

Theorem 3.1. Let (Z, G^a, N) be a complete generalized b-Menger space with $RanG \subset \mathcal{D}^+$. Let $h : Z \to Z$ be a mapping which satisfies the condition

$$G_{hp,hq}(\beta\gamma) \geq G_{p,q}(\gamma) \quad \forall\, p, q \in Z, \tag{3.2}$$

where $\beta \in (0, 1)$. Also suppose that for $b_0 \in Z$, and $n, j \in \mathbb{N}$, we have

$$\alpha(b_n, b_{n+j}) < \frac{1}{\beta},$$

where $b_n = h^n b_0$. Then h has a unique fixed point.

Proof. **Existence**
Let $b_0 \in$ Z be arbitrary, and we consider the sequence $\{b_n\}$ defined by

$$b_n = h(b_{n-1}) = h^n b_0 \quad \text{for each } \; n \in \mathbb{N}.$$

Firstly, we prove that

$$G_{b_n, b_{n+1}}(\beta\gamma) \geq G_{b_0, b_1}\left(\frac{\gamma}{\beta^{n-1}}\right)$$

Let n, $\gamma > 0$, from (3.2) we have

$$\begin{aligned} G_{b_n, b_{n+1}}(\beta\gamma) &= G_{hb_{n-1}, hb_n}(\beta\gamma) \\ &\geq G_{b_{n-1}, b_n}(\gamma) \\ &\geq G_{b_{n-2}, b_{n-1}}\left(\frac{\gamma}{\beta}\right) \\ &\geq G_{b_{n-3}, b_{n-2}}\left(\frac{t}{\beta^2}\right) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ &\geq G_{b_0, b_1}\left(\frac{t}{\beta^{n-1}}\right). \end{aligned}$$

By Lemma 3.1, the sequence $\{b_n\}$ is a Cauchy sequence in Z. Since (Z, G^a, N) is complete, there is some $b \in Z$ such that

$$b_n \to b \quad \text{as} \quad n \to \infty.$$

Now we will prove that b is a fixed point of h. Applying (3.2) and (3) of Definition 2.1 we obtain

$$\begin{aligned} G_{hb,b}(\gamma) &\geq N\left(G_{hb,hb_n}\left(\frac{\gamma}{2\alpha(hb,b)}\right); G_{hb_n,b}\left(\frac{\gamma}{2\alpha(hb,b)}\right)\right) \\ &\geq N\left(G_{b,b_n}\left(\frac{\gamma}{2\beta\alpha(hb,b)}\right); G_{b_{n+1},b}\left(\frac{\gamma}{2\alpha(hb,b)}\right)\right). \end{aligned}$$

Letting $n \to \infty$ we obtain

$$G_{hb,b}(\gamma) \geq 1.$$

Which holds unless $G_{hb,b} = \epsilon_0$, so b is a fixed point of h.

Uniqueness

To prove uniqueness, assume that there exists another fixed point $r \in Z$ of h. Then, let $\gamma > 0$, from 3.2 we get

$$\begin{aligned} G_{b,r}(\gamma) &= G_{hb,hr}(\gamma) \\ &\geq G_{b,r}\left(\frac{\gamma}{\beta}\right) \\ &= G_{hb,hr}\left(\frac{\gamma}{\beta}\right) \\ &\geq G_{b,r}\left(\frac{\gamma}{\beta^2}\right) \geq \ldots \geq G_{b,r}\left(\frac{\gamma}{\beta^n}\right). \end{aligned}$$

Letting $n \to \infty$ we obtain

$$G_{b,r}(\gamma) \geq 1.$$

We conclude that $b = r$. This completes the proof.

Remark 3.1. In the particular case $\alpha(p, q) = \ell \geq 1$ for all $p, q \in Z$, we found the Theorem 4.1 in [9] if we consider $\varphi(\gamma) = \beta\gamma$.

4. Application

Let $Z = \mathcal{C}([0, 1], \mathbb{R}$ be the set of real continuous functions defined on [0, 1]. For $f, g \in Z$, consider $\rho : Z \times Z \to \mathbb{R}^+$ and $\alpha : Z \times Z \to [1, \infty)$ defined by

$$\rho(f,g) = \max_{\gamma\in[0,1]} | f(\gamma) - g(\gamma) |^2$$

and

$$\alpha(f, g) = |f(\gamma)| + |g(\gamma)| + 2.$$

So (Z, ρ_a) is a complete extended b-metric space. Define $G : Z \times Z \to \Delta^+$ by

$$G_{f,g}(\gamma) = \epsilon_{\rho a}(f, g)(\gamma).$$

In view of Lemma 2.1, (Z, G^a, N_M) is a complete generalized b-Menger space.

Let $\Gamma \in \mathcal{C}([0, 1] \times [0, 1] \times \mathbb{R}, \mathbb{R})$ be an operator such that

$$\sup_{\mu,\nu\in[0,1],f\in\mathcal{C}([0,1],\mathbb{R})} | \Gamma(\mu,\nu,f(\nu)) |$$

is finite and there exists $K > 0$ such that $\forall \mu, \nu \in [0, 1]$ and $\forall f, g \in \mathcal{C}([0, 1], \mathbb{R})$ we have

$$| \Gamma(\mu,\nu,hf(\nu)) - \Gamma(\mu,\nu,hg(\nu)) | \leq \frac{K}{\sqrt{2}} | f(\nu) - g(\nu) |,$$

where h: $\mathcal{C}([0, 1], \mathbb{R}) \to \mathcal{C}([0, 1], \mathbb{R})$ is given by

$$hf(\mu) = \omega(\mu) + \int_0^{\mu} \Gamma(\mu,\nu,hf(\nu))d\nu, \quad \omega \in \mathcal{C}([0,1],\mathbb{R}).$$

By Theorem 5.1 in [12] we have for all $f, g \in \mathcal{C}([0, 1], \mathbb{R})$ and $\gamma > 0$

$$G_{hf,hg}(\beta\gamma) \geq G_{f,g}(\gamma),$$

where $\beta = \dfrac{(1-e^{-K})^2}{2} \in (0,1)$. We conclude by Theorem 3.1 that h has a unique fixed point which is the unique solution of the integral equation

$$f(\mu) = \omega(\mu) + \int_0^{\mu} \Gamma(\mu,\nu,f(\nu))d\nu.$$

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