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Sharp coefficient inequalities for a class of analytic functions defined by q-difference operator associated with q-Lemniscate of Bernoulli

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Abstract

Quantum theory has many applications in mathematics, particularly in the study of special functions and quantum physics. In this article, based on the concept of Lemniscate of Bernoulli, we provide q-Lemniscate of Bernoulli $(x^2+y^2)-2(x^2-y^2)=q^2-1$. The q-Lemniscate Bernoulli is used to define a new class of analytic functions using a quantum difference operator, and for this class, an upper bound Fekete-Szegö problems and the second Hankel determinant are studied. Furthermore, several known cases with proven findings are presented. In addition, by using the Ruscheweyh q-differential operator, certain useful applications of the main findings are attained.

Key words and phrases: Analytic functions, q-Differential operator, Hankel determinant, q-Starlike functions, Ruscheweyh q-differential operator, q-Lemniscate of Bernoulli.

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1. Introduction

Basically, quantum calculus is limitless classical calculus. It is sometimes called "h-calculus," where "h" is Plank's constant. Due to its widespread relevance in several subfields of mathematics and physics, quantum calculus has recently drawn the interest of many scholars. Jackson [1, 2] systematically explored q-derivative and q-integral after introducing them. The class of star-like functions was extended using quantum calculus by Ismail et al. [3]. Geometric characteristics of q-operators in certain classes of analytic functions were investigated by Mohammed and Darus [4]. Sahoo and Sharma [5] first defined the class of q-close-to-convex functions connected with q-derivative while complex order q-starlike and q-convex functions were also studied by Seoudy and Aouf [6]. The geometric features of q-hypergeometric series are extensively studied by Agarwal and Sahoo [7]. The q-analogue of the Ruscheweyh differential operator was first described by Kanas and Raducanu [8], who then utilized it to create a new class of uniformly q-starlike functions and investigated several extremely important results in the context of the conic domain. Some of its applications for multivalent functions were described by Arif et al. [9, 10], while Zang et al. [11] examined q-starlike functions connected to the generalized conic domain. Srivastava et al. [12] presented the q-Noor integral operator and examined some of its applications; they also published a series of papers (see [13, 14, 15, 16, 17]) that connected the class of q-starlike functions from various perspectives. Scholars and researchers working in these fields may also find Srivastava's recent survey-cum-expository review the article [18] helpful. You may find the most recent studies on q-calculus in [19, 20, 21, 22].

Consider the function f of the form, and let us say that it belongs to the class $\mathcal{A}$ and every $f \in \mathcal{A}$ in the open unit disc $E = \{\tau : |\tau| < 1\}$ has the following form:

$$f(\tau) = \tau + \sum_{n=2}^{\infty} a_n \tau^n. \tag{1.1}$$

Let $n \in \mathbb{N},\ 0 < q < 1$. The definition of the q-integer is

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \qquad [0]_q = 0 \tag{1.2}$$

Clearly,

$$\lim_{q \to 1-} [n]_q = n.$$

Following the definition of Jackson q-derivative operator (or q-difference operator) [1] of a function $f \in \mathcal{A}$ provided by (1.1) for $0 < q < 1$, we have:

$$\mathcal{D}_q f(\tau) = \begin{cases} \dfrac{f(\tau) - f(q\tau)}{(1-q)\tau} & \text{for} \quad \tau \neq 0 \\ \\ f'(0) & \text{for} \quad \tau = 0. \end{cases} \tag{1.3}$$

For a function $h(\tau) = \tau^n$, we obtain

$$\mathcal{D}_q h(\tau) = \mathcal{D}_q \tau^n = \frac{1-q^n}{1-q} \tau^{n-1} = [n]_q \tau^{n-1} \tag{1.4}$$

and

$$\lim_{q \to 1-} \mathcal{D}_q h(\tau) = \lim_{q \to 1-} \left([n]_q \tau^{n-1} \right) = n\tau^{n-1} = h'(\tau).$$

For $f \in \mathcal{A}$ defined in (1.1) and using (1.3), we conclude that,

$$\mathcal{D}_q f(\tau) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \tau^{n-1} \qquad (\tau \in E), \tag{1.5}$$

where $[n]_q$ is given by (1.2).

Schwarz functions f is said to be subordinate to g, represented in the form $f \prec g$, if and only if $w(0) = 0$ and $|w(\tau)| < 1$, then $f(\tau) = g(w(\tau))$, where $\tau \in E$. The above subordination is equal to $f(0) = g(0)$ and $f(E) \subset g(E)$ if and only if the function g is univalent in E.

Let $\mathcal{P}$ denote the class of analytic function $p \in \mathcal{P}$ with

$$p(\tau) = 1 + \sum_{n=1}^{\infty} p_n \tau^n, \tag{1.6}$$

and $\Re\left(p(\tau)\right) > 0$ in E.

The familier class $\mathcal{S}^*$ of starlike functions is defined as:

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \frac{\tau f'(\tau)}{f(\tau)} \prec \frac{1+\tau}{1-\tau} \right\}.$$

Robertson in [23] introduced the class $\mathcal{S}^*(\beta)$ of starlike of order β, $\beta \leq 1$ as:

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} : \Re\left[\frac{\tau f'(\tau)}{f(\tau)} \right] > \beta \right\}.$$

If $\beta \in 0,1)$, then each function in class $\mathcal{S}^*(\beta)$ is univalent, if $\beta < 0$, it may fails to be univalent.

Subordination is used to create a number of classes analytic functions whose image domains may be understood geometrically. When domains is like a right half plane [24], a circular disc [25], a conic domain [26, 27], a generalized conic domain [28], an oval domain and petal domain [29], a leaf domain [30], and a shell curve [31, 32, 33, 34], we obtain some interesting geometrical classes.

Recently Sokol [35] introduced the class SL^* as follows:

$$SL^* = \left\{ f \in \mathcal{A} : \Re\left[\left[\frac{\tau f'(\tau)}{f(\tau)} \right]^2 - 1 \right] < 1 \right\}.$$

It is intuitively clear that

$$f \in SL^* \Leftrightarrow \frac{\tau f'(\tau)}{f(\tau)} \prec \sqrt{1+\tau}, \tau \in E.$$

Malik et al. [36] developed and considered a new geometrical structure as image domain, and they were inspired by the idea of shell-like curves and the circular disc. Here we get inspiration from [36] and define a q-Lemniscate of Bernoulli and also define a new class of analytic functions using a quantum difference operator, and for this class, an upper bound Fekete-Szegö problems and the second Hankel determinant are studied.

Let $q \in (0,1)$ be given and let us consider the class

$$\mathcal{S}^*(Q_q) = \left\{ f \in \mathcal{A} : \Re\left[\left[\frac{\tau \mathcal{D}_q f(\tau)}{f(\tau)} \right]^2 - 1 \right] < q \right\}. \tag{1.7}$$

It is straightforward to verify that $f \in S^*(Q_q)$ if and only if,

$$\frac{\tau \mathcal{D}_q f(\tau)}{f(\tau)} \prec \sqrt{1+q\tau} \equiv Q_q(\tau), \ \tau \in E. \tag{1.8}$$

For $q \to 1-$, then the class $\mathcal{S}^*(Q_q)$ reduces to the class SL^* studied in [37]. Some properties of this class were investigated in [38, 39] for $q \to 1-$.

Geometrical Interpretation

A function $f \in \mathcal{S}^*(Q_q)$ if and only if $\frac{\tau \mathcal{D}_q f(\tau)}{f(\tau)}$ take all the values in the set

$$Q_q = \left\{ w \in \mathbb{C} : \Re w > 0, \left| w^2 - 1 \right| < q \right\}.$$

The boundary of $\partial\left(Q_q\right)$ is the right half of the q-Lemniscate of Bernoulli $(x^2+y^2)-2(x^2-y^2)=q^2-1$ and for $q \to 1-$, $\mathcal{D}\left(Q_1\right)$ is the Lemniscate of Bernoulli, for detail see [39]. It is important to note that when $q \to 1-$, $\mathcal{S}^*(Q_q) = SL^*$ studied in [38] and [39].

In 1976, the s^{th} Hankel determinant was defined by Noonan and Thomas [40] as:

$$H_s(n) = \begin{vmatrix} a_n & a_{n+1} & .. & .. & a_{n+s-1} \\ a_{n+1} & a_{n+2} & .. & .. & a_{n+s+2} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ a_{n+s-1} & a_{n+s-2} & .. & .. & a_{n+2s-2} \end{vmatrix}, \tag{1.9}$$

where $n \geq 1$ and $s \geq 1$.

(i) Fekete-Szegö functional is obtained when $s = 2$ and $n = 1$.

$$H_2(1) = \left| a_3 - a_2^2 \right|$$

and in its most basic form, this functional may be written as:

$$\left| a_3 - \mu a_2^2 \right|,$$

where, $\mu \in \mathbb{C}$, (see [41]).

(ii) The following version of the second Hankel determinant was given by Janteng [42], and it has been examined by other scholars for several new classes of analytic functions:

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left| a_2 a_4 - a_3^2 \right|.$$

For example, see [43] to learn more about the significance of the Hankel determinant in the study of singularities. The Hankel transform and its properties for integer sequences were discussed by Layman [44]. See [45] for more on how Hankel determinants are used to the study of meromorphic functions, and check out [39] for a variety of their features. Several academics are focusing more on finding sharp bounds for the Hankel determinants of a certain class of functions. Janteng et al. [42] found sharp problems for the second Hankel determinant of the subfamily $(K, \mathcal{S}^*, R)$ of class $\mathcal{S}$

univalent functions. Classes of starlike functions of order β and strongly starlike functions of order β were studied by Cho et al. [46], and the Hankel determinant was developed, along with the bounds for $|\mathcal{H}_{2,2}(\eta)|$ is bounded by $(1-\beta)^2$ and β^2. See [47, 48, 49, 50, 51, 52, 53, 54] for an example of current research on Hankel determinants. Recent work in [37, 35, 55] has explored the third Hankel determinant results for a class of analytic functions associated with the right half of Lemniscate of Bernoulli. In this research, we extend their findings to a more extensive class of q-derivative and right-half-q-Lemniscate of Bernoulli.

2. Introduction

Our primary results will be shown using the following lemmas:

Lemma 1: *[56]. Supoose $p \in \mathcal{P}$ and defined in (1.6). Then*

$$\left|p_2 - up_1^2\right| \leq \begin{cases} -4u+2 & \text{if} \quad u < 0 \\ 2 & \text{if} \quad 0 \leq u \leq 1 \\ 4u-2 & \text{if} \quad u > 1. \end{cases}$$

The equality holds

$$p(\tau) = \frac{1+\tau}{1-\tau}, \text{if } \ u < 0 \text{or } u > 1$$

or one of its relations. The equality holds if and only if

$$p(\tau) = \frac{1+\tau^2}{1-\tau^2}, \text{ if } 0 < u < 1$$

or one of its rotations. For the case where $0 < u < 1$, the above upper bound is sharp, but it may be further improved as shown below:

$$\left|p_2 - up_1^2\right| + u\left|p_1\right|^2 \leq 2, \quad \left(0 < u \leq \frac{1}{2}\right)$$

and

$$\left|p_2 - up_1^2\right| + (1-u)\left|p_1\right|^2 \leq 2 \quad \left(\frac{1}{2} < u \leq 1\right).$$

Lemma 2: [56]. *Let $p \in \mathcal{P}$ and be of the form (1.6), then $\mu \in \mathbb{C}$, we have*

$$\left|p_2 - \mu p_1^2\right| \leq 2\max(1, |2\mu - 1|).$$

This result is sharp for

$$p(\tau) = \frac{1+\tau^2}{1-\tau^2} \text{ and } p(\tau) = \frac{1+\tau}{1-\tau}.$$

Lemma 3: [57] *Suppose $p \in \mathcal{P}$ and defined in (1.6), then*

$$p_2 = \frac{p_1^2}{2} + \frac{\rho(4-p_1^2)}{2}$$

and

$$p_3 = \frac{p_1^3}{4} + \frac{(4-p_1^2)p_1\rho}{2} - \frac{(4-p_1^2)p_1\rho^2}{4} + \frac{(4-p_1^2)(1-|\rho|^2)\tau}{2},$$

for some τ and ρ, we have $|\tau| \le 1$ and $|\rho| \le 1$.

3. Main Result

Theorem 1: *Let $f \in \mathcal{S}^*(Q_q)$, and be of the form (1.1), then for complex μ,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{q}{4(1+q)}\left\{\dfrac{1}{q} - \dfrac{1}{2} - \mu\left(\dfrac{1+q}{q}\right)\right\} & \mu < \sigma_1, \\ \dfrac{1}{2(1+q)} & \text{if} \quad \sigma_1 \le \mu \le \sigma_{2,} \\ \dfrac{q}{4(1+q)}\left\{\mu\left(\dfrac{1+q}{q}\right) + \dfrac{1}{2} - \dfrac{1}{q}\right\} & \text{if} \quad \mu > \sigma_{2,} \end{cases}$$

where

$$\sigma_1 = \frac{4}{(1+q)}\left\{q\left(\frac{1}{4q} - \frac{1}{8}\right) - \frac{1}{2}\right\}$$

and

$$\sigma_2 = \frac{4}{(1+q)}\left\{\frac{1}{2} - q\left(\frac{1}{8} - \frac{1}{4q}\right)\right\}.$$

Proof. Let $f \in \mathcal{A}$ defined by (1.8), belong to the class $\mathcal{S}^*(Q_q)$, then

$$\frac{\tau \mathcal{D}_q f(\tau)}{f(\tau)} \prec Q_q(\tau). \tag{3.1}$$

Define $p \in \mathcal{P}$ as follows:

$$w(\tau) = \frac{p(\tau)-1}{p(\tau)+1},$$

then from (3.1), we have

$$\frac{\tau \mathcal{D}_q f(\tau)}{f(\tau)} = Q_q(w(\tau)) \tag{3.2}$$

Using (1.1), (1.6) and (1.8) and after some simplification, we obtain

$$\begin{aligned} Q_q(w(\tau)) &= 1 + \frac{qp_1}{4}\tau + q\left(\frac{2p_2 - p_1^2}{8} - \frac{qp_1^2}{32}\right)\tau^2 \\ &+ q\left\{\left(\frac{4p_3 - 4p_1p_2 + p_1^3}{16}\right) - q\left(\frac{2p_1p_2 - p_1^3}{32}\right)\right\}\tau^3 + \cdots \end{aligned} \tag{3.3}$$

and

$$\frac{\tau\mathcal{D}_q f(\tau)}{f(\tau)} = 1 + qa_2\tau + \{q(1+q)a_3 - qa_2^2)\}\tau^2 \\ +\{q(1+q+q^2)a_4 - q(2+q)a_2a_3 + qa_2^3\}\tau^3 + \cdots. \tag{3.4}$$

From equations (3.2), (3.3) and (3.4), we obtain

$$a_2 = \frac{p_1}{4}, \tag{3.5}$$

$$a_3 = \frac{1}{(1+q)}\left\{\frac{1}{4}p_2 - \left\{q\left(\frac{1}{32} - \frac{1}{16q}\right) + \frac{1}{8}\right\}p_1^2\right\} \tag{3.6}$$

and

$$a_4 = \frac{1}{(1+q+q^2)}\left[\begin{array}{c} \frac{1}{4}p_3 + \left\{\frac{(2+q)}{16(1+q)} - \frac{q}{16} - \frac{1}{4}\right\}p_1p_2 \\ +\left\{\frac{1}{16} + \frac{q}{32} - \frac{1}{64} - q\left(\frac{q+2}{4q(1+q)}\right)\left(\frac{1}{8} + \frac{q}{32} - \frac{1}{16}\right)\right\}p_1^3 \end{array}\right], \tag{3.7}$$

which together implies,

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{4(1+q)}\left|p_2 - \left\{\frac{1}{2} + q\left(\frac{1}{8} - \frac{1}{4q}\right) + \mu\frac{(1+q)}{4}\right\}p_1^2\right|. \tag{3.8}$$

By using lemma 1 on (3.8), we obtain the required result.

For sharpness consider the functions $f_1 : E \to \mathbb{C}$ such that

$$\frac{\tau\mathcal{D}_q f_1(\tau)}{f_1(\tau)} = \sqrt{1+q\tau}.$$

Now

$$f_1(\tau) = \tau + \frac{q}{2([2]_q - 1)}\tau^2 - \frac{q^2([2]_q - 3)}{8([2]_q - 1)([3]_q - 1)}\tau^3 + \cdots. \tag{3.9}$$

Also consider the function $f_2 : E \to \mathbb{C}$ such that

$$\frac{\tau\mathcal{D}_q f_2(\tau)}{f_2(\tau)} = \sqrt{1+q\tau^2}$$

and

$$f_2(\tau) = \tau + \frac{q}{2([3]_q - 1)}\tau^3 + \cdots. \tag{3.10}$$

This completes the proof.

Corollary 1: [35, 37] *Let* $q \to 1-$ *and* $f \in SL^*$ *be of the form (1.1). Then for complex* μ,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{16}(1-4\mu) & \text{if} \quad \mu < -\frac{3}{4}, \\ \frac{1}{4} & \text{if} \quad -\frac{3}{4} \leq \mu \leq \frac{5}{4}, \\ \frac{1}{16}\{4\mu - 1\} & \text{if} \quad \mu > \frac{5}{4}. \end{cases}$$

Theorem 2: *For complex number* μ. *Suppose that* $f \in S^*(Q_q)$ *and defined by (1.1). Then*

$$\left|a_3 - v a_2^2\right| \leq \frac{1}{2(1+q)} \max\left\{1, \left|q\left(\frac{1}{4} - \frac{1}{2q}\right) + \mu \frac{(1+q)}{2}\right|\right\},$$

where v is given by (3.13).

Proof. From the definition of $S^*(Q_q)$ in connection with (3.8), we have

$$\left|a_3 - v a_2^2\right| = \frac{1}{4(1+q)} \left|p_2 - \left\{\frac{1}{2} + q\left(\frac{1}{8} - \frac{1}{4q}\right) + \mu \frac{(1+q)}{4}\right\} p_1^2\right|, \tag{3.11}$$

$$\left|a_3 - v a_2^2\right| = \frac{1}{4(1+q)} \left|p_2 - v p_1^2\right|, \tag{3.12}$$

where

$$v = \left\{\frac{1}{2} + q\left(\frac{1}{8} - \frac{1}{4q}\right) + \mu \frac{(1+q)}{4}\right\}. \tag{3.13}$$

Applying the Lemma 2 on (3.12), we obtain the required result.

Corollary 2: *[37]. Let* $q \to 1-$ *and* $f \in SL^*$ *defined by (1). Then*

$$\left|a_3 - v a_2^2\right| \leq \frac{1}{4} \max\left\{1, \left|\mu - \frac{1}{4}\right|\right\}.$$

Theorem 3: *Let* $f \in \mathcal{S}^*(Q_q)$ *and be of the from (1.1), then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{q^2}{4\left([3]_q - 1\right)^2}.$$

Result is sharp for the function f_2 given in (3.10).

Proof. From equations (3.5), (3.6) and (3.7), we have

$$a_2 a_4 - a_3^2 = A_q p_1 p_3 + \left(B_q + C_q\right) p_1^2 p_2 - \mathcal{G}_q p_2^2 + \left(E_q - F_q\right) p_1^4, \tag{3.14}$$

where

$$A_q = \frac{1}{16\left(1+q+q^2\right)},$$

$$B_q = \frac{1}{4\left(1+q+q^2\right)}\left\{\frac{(2+q)}{16(1+q)} - \frac{q}{16} - \frac{1}{4}\right\},$$

$$C_q = \frac{1}{(1+q)^2}\left\{\frac{1}{16} + q\left(\frac{1}{64} - \frac{1}{32q}\right)\right\},$$

$$\mathcal{G}_q = \frac{1}{16(1+q)^2},$$

$$E_q = \frac{1}{4\left(1+q+q^2\right)}\left\{\begin{array}{c}\left(\frac{1}{16} + \frac{q}{32} - \frac{1}{64}\right) \\ -\left(\frac{(2+q)}{4(1+q)}\right)\left(\frac{1}{8} + \frac{q}{32} - \frac{1}{16}\right)\end{array}\right\},$$

$$F_q = \frac{1}{(1+q)^2}\left\{\frac{1}{8} + q\left(\frac{1}{32} - \frac{1}{16q}\right)\right\}^2.$$

By using the Lemma 3 and assume that $p > 0$ and taking $p_1 = p \in [0,2]$, we obtain

$$\begin{aligned} a_2a_4 - a_3^2 = {} & A_q\left(p\left(\frac{p^3 + 2(4-p^2)p\rho - (4-p^2)p\rho^2 + 2(4-p^2)(1-|\rho|^2)\tau}{4}\right)\right) \\ & + \left(B_q + C_q\right)p^2\left(\frac{p^2 + \rho(4-p^2)}{2}\right) - \mathcal{G}_q\left(\frac{p^2 + \rho(4-p^2)}{2}\right)^2 \\ & + \left(E_q - F_q\right)p^4. \end{aligned}$$

Some simple computations yields

$$\left|a_2a_4 - a_3^2\right| = \left|\begin{array}{l}\left(\frac{A_q}{4} + \frac{B_q + C_q}{2} - \frac{\mathcal{G}_q}{4} + E_q - F_q\right)p^4 + \left(\frac{A_q}{2} + \frac{B_q + C_q}{2} - \frac{\mathcal{G}_q}{2}\right)p^2(4-p^2)\rho \\ -\left(\frac{A_q}{4}p^2 + \frac{\mathcal{G}_q}{4}(4-p^2)\right)(4-p^2)\rho^2 + \frac{A_q}{2}p(4-p^2)\tau - \frac{A_q}{2}p(4-p^2)|\rho|^2\tau\end{array}\right|.$$

By applying the triangle inequality and replace $|\rho|$ by ξ and $|\tau| \le 1$, we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \le {} & \left\{\begin{array}{l}\left|\left(\frac{A_q}{4} + \frac{B_q + C_q}{2} - \frac{\mathcal{G}_q}{4} + E_q - F_q\right)\right|p^4 + \left|\left(\frac{A_q}{2} + \frac{B_q + C_q}{2} - \frac{\mathcal{G}_q}{2}\right)\right|p^2(4-p^2)\xi \\ + \left|\left(\frac{A_q}{4}p^2 + \frac{\mathcal{G}_q}{4}(4-p^2)\right)\right|(4-p^2)\xi^2 + \left|\frac{A_q}{2}\right|p(4-p^2) + \left|\frac{A_q}{2}\right|p(4-p^2)\xi^2\end{array}\right\} \\ = {} & F(p,\xi) \ \text{(say)}. \end{aligned} \tag{3.15}$$

Now

$$\frac{\partial F(p,\xi)}{\partial \xi}=\left[\begin{array}{c}\left|\left(\frac{A_q}{2}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{2}\right)\right|p^2(4-p^2)\\ \\ +2\left|\left(\frac{A_q}{4}p^2+\frac{\mathcal{G}_q}{4}(4-p^2)\right)\right|(4-p^2)\xi+2\left|\frac{A_q}{2}\right|p(4-p^2)\xi\end{array}\right].$$

Clearly $\frac{\partial F(p,\xi)}{\partial \xi}>0$, proving that, inside the range $[0,1]$, $F(p,\xi)$ is an increasing function, therefore maximum attains at $\xi=1$ and Max $F(p,\xi)=F(p,1)=G(p)$ with

$$\begin{aligned}G(p)=&\left|\left(\frac{A_q}{4}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{4}+E_q-F_q\right)\right|p^4\\ &+\left\{\left|\left(\frac{A_q}{2}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{2}\right)\right|+\left|\frac{A_q}{4}\right|\right\}p^2(4-p^2)\\ &+\left|\frac{\mathcal{G}_q}{4}\right|(4-p^2)^2+2\left|\frac{A_q}{2}\right|p(4-p^2).\end{aligned}$$

Therefore,

$$\begin{aligned}G'(p)=&4\left|\left(\frac{A_q}{4}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{4}+E_q-F_q\right)\right|p^3\\ &-2\left\{\left|\left(\frac{A_q}{2}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{2}\right)\right|+\left|\frac{A_q}{4}\right|\right\}p^3\\ &+2\left\{\left|\left(\frac{A_q}{2}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{2}\right)\right|+\left|\frac{A_q}{4}\right|\right\}p(4-p^2)\\ &-4\left|\frac{\mathcal{G}_q}{4}\right|p(4-p^2)-4\left|\frac{A_q}{2}\right|p^2+2\left|\frac{A_q}{2}\right|(4-p^2).\end{aligned}$$

Implies that

$$\begin{aligned}G''(p)=&12\left|\left(\frac{A_q}{4}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{4}+E_q-F_q\right)\right|p^2\\ &-6\left\{\left|\left(\frac{A_q}{2}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{2}\right)\right|+\left|\frac{A_q}{4}\right|\right\}p^2\\ &+2\left\{\left|\left(\frac{A_q}{2}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{2}\right)\right|+\left|\frac{A_q}{4}\right|\right\}(4-p^2)\\ &-4\left\{\left|\left(\frac{A_q}{2}+\frac{B_q+C_q}{2}-\frac{\mathcal{G}_q}{2}\right)\right|+\left|\frac{A_q}{4}\right|\right\}p^2\\ &-4\left|\frac{\mathcal{G}_q}{4}\right|(4-p^2)+8\left|\frac{\mathcal{G}_q}{4}\right|p^2-8\left|\frac{A_q}{2}\right|p-4\left|\frac{A_q}{2}\right|p.\end{aligned}$$

It is clear that $G''(0)<0,$ so G has Max at $p=0$ and

$$\begin{aligned} G(0) &= 4\mathcal{G}_q \\ &= \frac{1}{4\left(1+q\right)^2}. \end{aligned}$$

Hence from (3.15), we have

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{4\left(1+q\right)^2}.$$

Corollary 3: [37, 35]. *For* $q\to 1-$, $f\in SL^*$ *and be of the from (1.1), then*

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{16}.$$

Theorem 4: *Let* $f\in S^*(Q_q)$ *and be of the from (1.1), then*

$$\left|a_2a_3-a_4\right|\le\frac{1}{2\left(1+q+q^2\right)}.$$

Proof. From equations (3.5), (3.6) and (3.7), we have

$$a_2a_3-a_4=K_qp_1p_2-L_qp_3-\left(M_q+N_q\right)p_1^3,$$

where

$$\begin{aligned} K_q &= \frac{1}{16(1+q)}-\frac{1}{\left(1+q+q^2\right)}\left\{\frac{\left(2+q\right)}{16\left(1+q\right)}-\frac{q}{16}-\frac{1}{4}\right\}, \\ L_q &= \frac{1}{4\left(1+q+q^2\right)}, \\ M_q &= \frac{1}{4\left(1+q\right)}\left\{\frac{1}{8}+q\left(\frac{1}{32}-\frac{1}{16q}\right)\right\}, \\ N_q &= \frac{1}{\left(1+q+q^2\right)}\left\{\begin{array}{c} \frac{1}{16}+\frac{q}{32}-\frac{1}{16} \\ -\frac{\left(2+q\right)}{4\left(1+q\right)}\left(\frac{1}{8}+\frac{q}{32}-\frac{1}{16}\right) \end{array}\right\}. \end{aligned}$$

By using Lemma 3 and assume that $p>0$ and taking $p_1=p\in 0,2]$, we obtain

$$\begin{aligned} a_2a_3-a_4 = {} & K_q\left(\frac{p^2+\rho(4-p^2)}{2}\right)p \\ & -L_q\left(\frac{p^3+2(4-p^2)p\rho-(4-p^2)p\rho^2+2(4-p^2)(1-\left|\rho\right|^2)\tau}{4}\right) \\ & -\left(M_q+N_q\right)p^3. \end{aligned}$$

By applying the triangle inequality and replace $|\rho|$ by ξ and $|\tau|\leq 1$, we have

$$\left|a_2a_4-a_3^2\right|\leq\left\{\begin{array}{c}\left|K_q-M_q-N_q\right|p^3+\dfrac{\left|K_q\right|}{2}p(4-p^2)\xi+\dfrac{\left|L_q\right|}{4}p^2+\dfrac{\left|L_q\right|}{4}p(4-p^2)\xi^2\\ +\dfrac{\left|L_q\right|}{2}p(4-p^2)\xi+\dfrac{\left|L_q\right|}{2}(4-p^2)\left(1-\left|\xi^2\right|\right)\end{array}\right\}\tag{3.16}$$
$$=T(p,\xi)\ \text{(say)}.$$

Now

$$\frac{\partial T(p,\xi)}{\partial\xi}=\frac{\left|K_q\right|}{2}p(4-p^2)+\frac{\left|L_q\right|}{2}p(4-p^2)\xi+\frac{\left|L_q\right|}{2}p(4-p^2)-\left|L_q\right|(4-p^2)\xi.$$

Clearly, $\dfrac{\partial T(p,\xi)}{\partial\xi}>0$, proving that, inside the range $[0,1]$, $T(p,\xi)$ is an increasing function, therefore maximum occurs at $\xi=0$ and max $T(p,\xi)=(p,0)=G_1(p)$ with

$$G_1(p)=\left|K_q-M_q-N_q\right|p^3+\frac{\left|L_q\right|}{4}p^2+\frac{\left|L_q\right|}{2}(4-p^2).$$

Therefore,

$$G_1'(p)=3\left|K_q-M_q-N_q\right|p^2-\frac{\left|L_q\right|}{2}p,$$
$$G_1''(p)=6\left|K_q-M_q-N_q\right|p-\frac{\left|L_q\right|}{2}.$$

It is clear that $G_1^{''}(0)<0$, so $G_1(p)$ has Max at $p=0$, we have

$$G_1(0)=\frac{1}{2\left(1+q+q^2\right)}.$$

Hence from (3.16), we obtain

$$\left|a_2a_3-a_4\right|\leq\frac{1}{2\left(1+q+q^2\right)}.$$

For sharpness, consider the function $f_3:E\to\mathbb{C}$ such that

$$\frac{\tau\mathcal{D}_qf_2(\tau)}{f_2(\tau)}=\sqrt{1+q\tau^3}$$

and

$$f_3(\tau)=\tau+\frac{q}{2\left([4]_q-1\right)}\tau^4+\cdots.\tag{3.17}$$

Now

$$a_2 = a_3 = 0,\ \ a_4 = \frac{q}{2\left([4]_q - 1\right)} = \frac{1}{2\left(1+q+q^2\right)}.$$

Hence we have the required result.

Corollary 4: *[37, 35]. For $q \to 1-$ and for $f \in SL^*$ be of the from (1.1), then*

$$\left|a_2a_3 - a_4\right| \le \frac{1}{6}.$$

Theorem 5: *Let $f \in S^*(Q_q)$ and defined by (1.1), then for $n \ge 2$*

$$\left(\left[n\right]_q - 1\right)^2 \left|a_n\right|^2 \le \sum_{k=1}^{n-1} \left|a_k\right|^2 \left\{\left(\delta\left[k\right]_q - 1\right)^2 - \left(\left[k\right]_q - 1\right)^2\right\}, \tag{3.18}$$

where, $\delta = \sqrt{2} - 1.$

Proof. First note that from (1.8), we have

$$\frac{\tau\mathcal{D}_q f(\tau)}{f(\tau)} \prec Q_q(\tau) \prec \frac{1+\tau}{1+\delta\tau},$$

or

$$\frac{\tau\mathcal{D}_q f(\tau)}{f(\tau)} \prec \sqrt{1+q\tau} \prec \frac{1+\tau}{1+\delta\tau}$$

and so

$$\frac{\tau\mathcal{D}_q f(\tau)}{f(\tau)} = \frac{1+w(\tau)}{1+\delta w(\tau)},$$

where, for $\tau \in E$, $w(0) = 0$ and $|w(\tau)| < 1$, and

$$w(\tau) = \sum_{k=1}^{\infty} p_k \tau^k. \tag{3.19}$$

Thus, we obtain

$$\tau\mathcal{D}_q f(\tau) - f(\tau) = w(\tau)\left\{f(\tau) - \delta\tau\mathcal{D}_q f(\tau)\right\}.$$

From (1.1) and (3.19), we have

$$\sum_{k=1}^{\infty}\left(\left[k\right]_q - 1\right)a_k\tau^k = w(\tau)\sum_{k=1}^{\infty}\left(1 - \delta\left[k\right]_q\right)a_k\tau^k, a_1 = 1.$$

Now write

$$\sum_{k=1}^{n}\left(\left[k\right]_q - 1\right)a_k\tau^k + \sum_{k=n+1}^{\infty}\left(\left[k\right]_q - 1\right)a_k\tau^k = w(\tau)\left\{\sum_{k=1}^{n-1}\left(1 - \delta\left[k\right]_q\right)a_k\tau^k + \sum_{k=n}^{\infty}\left(1 - \delta\left[k\right]_q\right)a_k\tau^k\right\},$$

it may also be expressed as:

$$\sum_{k=1}^{n}\left(\left[k\right]_q-1\right)a_k\tau^k+\sum_{k=n+1}^{\infty}\left(\left[k\right]_q-1\right)a_k\tau^k-w(\tau)\sum_{k=n}^{\infty}\left(1-\delta\left[k\right]_q\right)a_k\tau^k=w(\tau)\sum_{k=1}^{n-1}\left(1-\delta\left[k\right]_q\right)a_k\tau^k.$$

We now write using Clunie and Keogh's technique [58]:

$$\sum_{k=1}^{n}\left(\left[k\right]_q-1\right)a_k\tau^k+\sum_{k=n+1}^{\infty}b_k\tau^k=w(\tau)\sum_{k=1}^{n-1}\left(1-\delta\left[k\right]_q\right)a_k\tau^k, \tag{3.20}$$

for some b_k, with $n+1\le k<\infty$. This allows us to write down an expression for b_k in terms of the coefficients a_k and p_k :

$$b_k=\left(\left[k\right]_q-1\right)a_k-\sum_{j=1}^{k-n}\left(1-\delta\left[k\right]_q\right)c_j a_{k-j}.$$

Taking modulus of (3.20), we have

$$\left|\sum_{k=1}^{n}\left(\left[k\right]_q-1\right)a_k\tau^k+\sum_{k=n+1}^{\infty}b_k\tau^k\right|^2=\left|w(\tau)\sum_{k=1}^{n-1}\left(1-\delta\left[k\right]_q\right)a_k\tau^k\right|^2\le\left|\sum_{k=1}^{n-1}\left(1-\delta\left[k\right]_q\right)a_k\tau^k\right|^2,$$

where

$$\sum_{k=1}^{n}\left(\left[k\right]_q-1\right)a_k\tau^k+\sum_{k=n+1}^{\infty}b_k\tau^k=\sum_{k=1}^{\infty}d_k\tau^k,$$

is an analytic function in the unit disc. Parseval's Theorem (see, for instance [59]) gives

$$\int_0^{2\pi}\left|\sum_{k=1}^{\infty}d_k(re^{i\theta})^k\right|^2 d\theta=2\pi\sum_{k=1}^{\infty}\left|d_k\right|^2 r^{2k},$$

$0<r<1$ is true for every r. When we do an integration from 0 to 2π with regard to, we get

$$\sum_{k=1}^{n}\left(\left[k\right]_q-1\right)^2\left|a_k\right|^2 r^{2k}+\sum_{k=n+1}^{\infty}\left|b_k\right|^2 r^{2k}\le\sum_{k=1}^{n-1}\left(1-\delta\left[k\right]_q\right)^2\left|a_k\right|^2 r^{2k}.$$

Therefore,

$$\sum_{k=1}^{n}\left(\left[k\right]_q-1\right)^2\left|a_k\right|^2 r^{2k}\le\sum_{k=1}^{n-1}\left(1-\delta\left[k\right]_q\right)^2\left|a_k\right|^2 r^{2k}.$$

Letting $r\to1$ gives

$$\sum_{k=1}^{n}\left(\left[k\right]_q-1\right)^2\left|a_k\right|^2\le\sum_{k=1}^{n-1}\left(1-\delta\left[k\right]_q\right)^2\left|a_k\right|^2$$

and this leads to

$$\left(\left[n\right]_q-1\right)^2\left|a_n\right|^2\le\sum_{k=1}^{n-1}\left|a_k\right|^2\left\{\left(\delta\left[k\right]_q-1\right)^2-\left(\left[k\right]_q-1\right)^2\right\},$$

which completes the result.

Corollary 5: *Let $f \in \mathcal{S}^*(Q_q)$ and be of the from (1.1), then for $n \geq 2$*

$$|a_n| \leq \frac{2-\sqrt{2}}{[n]_q - 1}, \tag{3.21}$$

Proof. From (3.18), we have

$$\begin{aligned}\left([n]_q - 1\right)^2 |a_n|^2 &\leq \sum_{k=1}^{n-1} |a_k|^2 \left\{\left(\delta [k]_q - 1\right)^2 - \left([k]_q - 1\right)^2\right\} \\ &= (\delta - 1)^2 - \sum_{k=2}^{n-1} |a_k|^2 \{\}\left([k]_q - 1\right)^2 - \left(\delta [k]_q - 1\right)^2 \\ &\leq (\delta - 1)^2 \\ &= \left(\sqrt{2} - 2\right)^2,\end{aligned}$$

which leads to the desired result (3.21).

Corollary 6: *[35]. For $q \rightharpoonup 1-$, $f \in S_L^*$ and be of the from (1.1), then for $n \geq 2$*

$$(n-1)^2 |a_n|^2 \leq \sum_{k=1}^{n-1} |a_k|^2 \left\{(\delta k - 1)^2 - (k-1)^2\right\},$$

where $\delta = \sqrt{2} - 1$.

Corollary 7: *[35]. For $q \rightharpoonup 1-$, $f \in S_L^*$ and be of the from (1.1), then for $n \geq 2$*

$$|a_n| \leq \frac{2-\sqrt{2}}{n-1}, \tag{3.22}$$

where $\delta = \sqrt{2} - 1$.

3.1 Applications of Ruscheweyh q-differential operator

The Ruscheweyh q-differential operator was defined by Kanas and Raducanu [8] using the q-difference operator.

For $f \in \mathcal{A}$,

$$\mathcal{R}_q^\lambda f(\tau) = f(\tau) * F_{q,\lambda+1}(\tau) \qquad (\lambda > -1, \tau \in E), \tag{3.23}$$

where

$$F_{q,\lambda+1}(\tau) = \tau + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]_q! \Gamma_q(1+\lambda)} \tau^n = \tau + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{q,n-1}}{[n-1]_q!} \tau^n. \tag{3.24}$$

We note that

$$\lim_{q \to 1^-} F_{q,\lambda+1}(\tau) = \frac{\tau}{(1-\tau)^{\lambda+1}}, \qquad \lim_{q \to 1^-} \mathcal{R}_q^\lambda f(\tau) = f(\tau) * \frac{\tau}{(1-\tau)^{\lambda+1}}.$$

Making use of (3.23) and (3.24), we have

$$\mathcal{R}_q^\lambda f(\tau) = \tau + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]_q! \Gamma_q(1+\lambda)} a_n \tau^n = \tau + \sum_{n=2}^{\infty} \Psi_n a_n \tau^n \qquad (\tau \in E), \tag{3.25}$$

where

$$\Psi_n = \frac{\Gamma_q(n+\lambda)}{[n-1]_q!\Gamma_q(1+\lambda)}. \tag{3.26}$$

From (3.25), we get

$$\mathcal{R}_q^0 f(\tau) = f(\tau), \mathcal{R}_q^1 f(\tau) = \tau\partial_q f(\tau).$$

In general

$$\mathcal{R}_q^m f(\tau) = \frac{\tau\partial_q^m(\tau^{m-1}f(\tau))}{[m]_q!} \quad (m \in \mathbb{N}).$$

Also, we have

$$\mathcal{L} = \mathcal{D}_q(\mathcal{R}_q^\lambda f(\tau)) = 1 + \sum_{n=2}^{\infty}[n]_q \Psi_n a_n \tau^{n-1}. \tag{3.27}$$

Now, applying Theorem 1 to the function $\mathcal{L}$, defined by Eq (3.27), the results are as follows::

Theorem 6: *Let $\mathcal{L}$ belongs to the class $S^*(Q_q)$ and be given by the equation (3.27), then for a complex μ*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{q^2}{4([3]_q \Psi_3 - 1)}\left\{\dfrac{1}{\left([2]_q \Psi_2 - 1\right)} - \dfrac{1}{2} - \mu\dfrac{\left([3]_q \Psi_3 - 1\right)}{\left([2]_q \Psi_2 - 1\right)^2}\right\} & \mu < \sigma_1, \\ \dfrac{q}{2\left([3]_q \Psi_3 - 1\right)} & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ \dfrac{q^2}{4([3]_q \Psi_3 - 1)}\left\{\mu\dfrac{\left([3]_q \Psi_3 - 1\right)}{\left([2]_q \Psi_2 - 1\right)^2} + \dfrac{1}{2} - \dfrac{1}{\left([2]_q \Psi_2 - 1\right)}\right\} & \text{if} \quad \mu > \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{4\left([2]_q \Psi_2 - 1\right)^2}{q\left([3]_q \Psi_3 - 1\right)}\left\{q\left(\frac{1}{4\left([2]_q \Psi_2 - 1\right)} - \frac{1}{8}\right) - \frac{1}{2}\right\},$$

$$\sigma_2 = \frac{4\left([2]_q \Psi_2 - 1\right)^2}{q\left([3]_q \Psi_3 - 1\right)}\left\{\frac{1}{2} - q\left(\frac{1}{8} - \frac{1}{4\left([2]_q \Psi_2 - 1\right)}\right)\right\}.$$

Next, applying Theorem 2 to the function $\mathcal{L}$, defined by Eq (36), the results are as follows:

Theorem 7: *Let $\mathcal{L}$ belongs to the class $S^*(Q_q)$ and be given by the equation (3.27), then for a complex υ_1*

$$\left|a_3 - \upsilon_1 a_2^2\right| \le \frac{q}{2\left([3]_q \Psi_3 - 1\right)} \max\left\{1, \left|q\left(\frac{1}{4} - \frac{1}{2\left([2]_q \Psi_2 - 1\right)}\right) + \mu\frac{q\left([3]_q \Psi_3 - 1\right)}{2\left([2]_q \Psi_2 - 1\right)^2}\right|\right\},$$

where

$$v_1 = \left\{ \frac{1}{2} + q\left(\frac{1}{8} - \frac{1}{4\left(\left[2\right]_q \Psi_2 - 1\right)}\right) + \mu \frac{q\left(\left[3\right]_q \Psi_3 - 1\right)}{4\left(\left[2\right]_q \Psi_2 - 1\right)^2} \right\}.$$

Again, applying Theorem 3 to the function $\mathcal{L}$, defined by Eq (3.27), the results are as follows:

Theorem 8: *Let $\mathcal{L}$ belongs to the class $S^*(Q_q)$ and be given by the equation (3.27), then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{q^2}{4\left(\left[3\right]_q \Psi_3 - 1\right)^2}.$$

Applying Theorem 4 to the function $\mathcal{L}$, defined by Eq (3.27), the results are as follows:

Theorem 9: *Let $\mathcal{L}$ belongs to the class $S^*(Q_q)$ and be given by the equation (3.27), then*

$$\left|a_2a_3 - a_4\right| \leq \frac{q}{2\left(\left[4\right]_q \Psi_4 - 1\right)}.$$

Finally applying Theorem 5 to the function $\mathcal{L}$, defined by Eq (3.27), the results are as follows:

Theorem 10: *Let $\mathcal{L}$ belongs to the class $S^*(Q_q)$ and be given by the equation (3.27), then, for $n \geq 2$*

$$\left(\left[n\right]_q \Psi_n - 1\right)^2 \left|a_n\right|^2 \leq \sum_{k=1}^{n-1} \left|a_k\right|^2 \left\{\left(\delta \Psi_k \left[k\right]_q - 1\right)^2 - \left(\Psi_k \left[k\right]_q - 1\right)^2\right\},$$

where $\delta = \sqrt{2} - 1$.

4. Conclusion

In our investigation, by using q-differential operator we introduced new class $S^*(Q_q)$ and investigated upper bound for second Hankel determinant of analytic functions related with q-Lemniscate of Bernoulli $(x^2 + y^2) - 2(x^2 - y^2) = q^2 - 1$. Also we investigated Fekete-Szegö problems and other coefficient of the analytic functions belonging to the class $S^*(Q_q)$. Further we discussed some special cases of our results. Finally, we addressed several novel applications of our major discoveries using the Ruscheweyh q-differential operator.

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Not applicable.

Data availability statement

No data, models, or code were generated or used during the study.

Conflict of interest

The authors declare that they have no conflict of interest.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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