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New analogues of Hilbert integral inequality of three variables

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Abstract

In this paper, we apply Hölder inequality, use the special functions Gamma and Beta functions as tools, and use special substitutions to evaluate the integrals that appears in the main results to give a new form for Hilbert Integral Inequality for three variables. The reverse form and equivalent forms of the inequality in theorem 1 are also obtained. The constant we obtained is the best constant.

Key words and phrases: Hilbert's integral inequality, Hölder inequality, Equivalent Forms.

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1. Introduction

For $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and for the $+ve$ functions f and g that satisfy $\int_0^\infty f^p(x)dx \in (0,\infty)$, $\int_0^\infty g^q(x)dx \in (0,\infty)$, the following inequality holds

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}} \tag{1.1}$$

where the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible.

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Also, the discrete form of inequality (1.1) for two *+ve* sequences of real numbers $\langle a_m \rangle$ and $\langle b_n \rangle$ is given as:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1.2}$$

From the end of 19th until the beginning of the 21st century, many authors have followed D. Hilbert to reprove, apply or generalize his great work in inequalities, (see [1–8]).

In recent years, Authors have continued to forge ahead in using Hilbert inequalities to extend, improve, and generalize the results concerning Hilbert inequalities in discrete, half-discrete and integral form for three variables, in [9], Tserendorj Batbold and Laith E.Azar gave the following:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{f(x,y)g(z)}{(x+y+z)^{\lambda}}dxdydz < C\left(\int_0^{\infty}\int_0^{\infty}(x+y)^{2p-\lambda-p\gamma-2}f^p(x,y)dxdy\right)^{\frac{1}{p}} \times\left(\int_0^{\infty} z^{q+q\xi-\lambda-1}g^q(z)dz\right)^{\frac{1}{q}} \tag{1.3}$$

where $\lambda > 0$, $\xi \in \left(-\frac{\lambda}{p},\frac{\lambda}{q}\right)$, $f(x,y)$ is a non-negative function defined on $(0,\infty)\times(0,\infty)$,and $g(z) > 0$ on $(0,\infty)$; the constant $C = B\left(\frac{\lambda}{p}+\xi,\frac{\lambda}{q}-\xi\right)$ is the best possible.

In (2022), Al-Oushoush Nizar gave the discrete form of this study, see [10], also, he gave a new version of Hilbert integral inequality of three variables with a hyperbolic sine function [11].

2. Preliminries and Lemmas

Here, we use the improper integrals that represent the necessary functions that we need in this work:

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt,\ s > 0, \tag{2.1}$$

$$B(\eta,\tau) = \int_0^{\infty}\frac{t^{\eta-1}}{(t+1)^{\eta+\tau}}dt,\ \eta,\tau > 0. \tag{2.2}$$

Also, for the above famous special functions, we will use the other useful representations for them as follows:

$$\frac{1}{x^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-xt}dt. \tag{2.3}$$

$$B(\eta,\tau) = \frac{\Gamma(\eta)\Gamma(\tau)}{\Gamma(\eta+\tau)} \tag{2.4}$$

$$B(\eta,\tau) = B(\tau,\eta). \tag{2.5}$$

Next, the following lemmas are the main tools in proving the main result:

Lemma 2.1: Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $b_r>0$, then for $\omega>0$ and $0\le\vartheta<\lambda+\frac{1}{p}$, we have

$$\int_0^\infty z^\lambda e^{-z\omega} g(z)dz \le \omega^{\vartheta-\lambda-\frac{1}{p}}\Gamma^{\frac{1}{p}}(\lambda p-p\vartheta+1)\left(\int_0^\infty z^{q\vartheta}e^{-z\omega}g^q(z)dz\right)^{\frac{1}{q}}. \tag{2.6}$$

Proof: Using Hölder inequality, we get

$$\begin{aligned}\int_0^\infty z^\lambda e^{-z\omega} g(z)dz &= \int_0^\infty\left(z^{\lambda-\vartheta}e^{-\frac{z}{p}\omega}\right)\left(z^{\vartheta}e^{\frac{-z}{q}\omega}g(z)\right)dz\\ &\le\left(\int_0^\infty z^{\lambda p-\vartheta p}e^{-z\omega}dz\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\vartheta}e^{-z\omega}g^q(z)dz\right)^{\frac{1}{q}}\\ &=\omega^{\vartheta-\lambda-\frac{1}{p}}\Gamma^{\frac{1}{p}}(\lambda p-p\vartheta+1)\left(\int_0^\infty z^{q\vartheta}e^{-z\omega}g^q(z)dz\right)^{\frac{1}{q}}.\end{aligned}$$

Lemma 2.2: Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1, \omega>0$, $\lambda>0$, $0\le\mu<\lambda+\frac{1}{q}$ and $f(x,y)>0$ is a non-negative function defined and integrable on $(0,\infty)\times(0,\infty)$, we have

$$\begin{aligned}\left(\int_0^\infty\int_0^\infty f(x,y)\left(\frac{xy}{x+y}\right)^\lambda e^{-\frac{xy}{x+y}\omega}dxdy\right)^p &\le \omega^{p\mu-\lambda p}\Gamma^{\frac{p}{q}}(q\lambda-q\mu)\\ &\times\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\mu p}\frac{e^{-\frac{xy}{x+y}\omega}}{(x+y)^{-\frac{2p}{q}}}f^p(x,y)dxdy\end{aligned} \tag{2.7}$$

Proof: Using Hölder inequality, and using the substitutions $y=xu$, $x=\frac{1+u}{u}v$ (to evaluate the first double integral on the righthand of the inequality), we have

$$\begin{aligned}&\left(\int_0^\infty\int_0^\infty f(x,y)\left(\frac{xy}{x+y}\right)^\lambda e^{-\frac{xy}{x+y}\omega}dxdy\right)^p\\ &=\left(\int_0^\infty\int_0^\infty\left\{\left(\frac{xy}{x+y}\right)^{\lambda-\mu}\frac{e^{-\frac{xy}{x+y}\frac{\omega}{q}}}{(x+y)^{\frac{2}{q}}}\right\}\left\{\left(\frac{xy}{x+y}\right)^{\mu}\frac{e^{-\frac{xy}{x+y}\frac{\omega}{p}}}{(x+y)^{-\frac{2}{q}}}f(x,y)\right\}dxdy\right)^p\\ &\le\left(\left\{\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\lambda q-\mu q}\frac{e^{-\frac{xy}{x+y}\omega}}{(x+y)^2}dxdy\right\}^{\frac{1}{q}}\left\{\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\mu p}\frac{e^{-\frac{xy}{x+y}\omega}}{(x+y)^{-\frac{2p}{q}}}f^p(x,y)dxdy\right\}^{\frac{1}{p}}\right)^p\end{aligned}$$

$$=\left(\int_0^\infty\int_0^\infty\left(\frac{x(xu)}{x(1+u)}\right)^{\lambda q-\mu q}\frac{e^{-\frac{x(xu)}{x(1+u)}\omega}}{x^2(1+u)^2}dx(xdu)\right)^{\frac{p}{q}}\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\mu p}\frac{e^{-\frac{xy}{x+y}\omega}}{(x+y)^{-\frac{2p}{q}}}f^p(x,y)dxdy$$

$$=\left(\int_0^\infty\int_0^\infty\left(\frac{\left(\left(\frac{1+u}{u}v\right)u\right)}{(1+u)}\right)^{\lambda q-\mu q}\frac{e^{-\frac{\left(\frac{1+u}{u}v\right)u}{(1+u)}\omega}}{\left(\frac{u+1}{u}v\right)(1+u)^2}\left(\frac{u+1}{u}dv\right)du\right)^{\frac{p}{q}}$$

$$\times\left(\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\mu p}\frac{e^{-\frac{xy}{x+y}\omega}}{(x+y)^{-\frac{2p}{q}}}f^p(x,y)dxdy\right)$$

$$=\left(\int_0^\infty\int_0^\infty\frac{v^{\lambda q-\mu q-1}e^{-v\omega}}{(1+u)^2}dudv\right)^{\frac{p}{q}}\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\mu p}\frac{e^{-\frac{xy}{x+y}\omega}}{(x+y)^{-\frac{2p}{q}}}f^p(x,y)dxdy$$

$$=\omega^{p\mu-\lambda p}\Gamma^{\frac{p}{q}}(q\lambda-q\mu)\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\mu p}\frac{e^{-\frac{xy}{x+y}\omega}}{(x+y)^{-\frac{2p}{q}}}f^p(x,y)dxdy.$$

Remark: Note that if $0<p<1$ and $\frac{1}{p}+\frac{1}{q}=1$, then by the reverse form of Hölder inequality, we can prove the reverse form of (2.6) and (2.7), which we need in the theorem 3.2.

3. Main Result

Theorem 3.1: Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$, $-\frac{\lambda}{p}<\xi<\frac{\lambda}{q}$, $f(x,y)$ is a non-negative function defined on $(0,\infty)\times(0,\infty)$, and $g(z)$ is a positive function on $(0,\infty)$. If $\int_0^\infty\int_0^\infty\frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}}f^p(x,y)dxdy<\infty$, and $\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)<\infty$, then

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\lambda}}dxdydz\le C\left(\int_0^\infty\int_0^\infty\frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}}f^p(x,y)dxdy\right)^{\frac{1}{p}}\times\left(\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)\right)^{\frac{1}{q}} \tag{3.1}$$

where $C=B\left(\frac{\lambda}{p}+\xi,\frac{\lambda}{q}-\xi\right)$ is the best possible. In particular,

a) If we take $\lambda = \frac{3}{2}$, $\xi = \frac{1}{4}$, and $p = q = 2$, then (3.1) will be

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\frac{3}{2}}} dxdydz \le 2\left(\int_0^\infty\int_0^\infty (x+y)(xy)f^2(x,y)dxdy\right)^{\frac{1}{2}}\left(\int_0^\infty z^3 g^2(z)\right)^{\frac{1}{2}}$$

b) If we take $\lambda = 2$, $\xi = 0$, $p = \frac{4}{3}$, $q = 4$, then (3.1) will be

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\frac{1}{2}}} dxdydz \le \frac{\pi}{2}\left(\int_0^\infty\int_0^\infty (x+y)^{\frac{2}{3}} f^{\frac{4}{3}}(x,y)dxdy\right)^{\frac{3}{4}}\left(\int_0^\infty z^9 g^4(z)\right)^{\frac{1}{4}}$$

Proof: Let

$$\begin{aligned} I &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\lambda}} dxdydz \\ &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)(xyz)^{\lambda}}{(xy+(x+y)z)^{\lambda}} dxdydz \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty\int_0^\infty f(x,y)g(z)(xyz)^{\lambda}\left(\int_0^\infty t^{\lambda-1}e^{-[xy+(x+y)z]t}dt\right)dxdydz. \end{aligned} \tag{3.2}$$

Using the substitution $t = \frac{\omega}{x+y}$ in the (3.2), and applying Hölder inequality, we obtain:

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty\int_0^\infty f(x,y)g(z)(xyz)^{\lambda}\left(\int_0^\infty \left(\frac{\omega}{x+y}\right)^{\lambda-1} e^{-[xy+(x+y)z]\frac{\omega}{x+y}}\frac{d\omega}{x+y}\right)dxdydz \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)(xyz)^{\lambda}}{(x+y)^{\lambda}}\left(\int_0^\infty (\omega)^{\lambda-1} e^{-\left[\frac{xy}{x+y}+z)\right]\omega} d\omega\right)dxdydz \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)(xyz)^{\lambda}}{(x+y)^{\lambda}}\left(\int_0^\infty \omega^{\lambda-1} e^{-\left[\frac{xy}{x+y}+z\right]\omega} d\omega\right)dxdydz \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\left(\omega^{\frac{\lambda-1}{p}+\xi}\int_0^\infty\int_0^\infty f(x,y)\left(\frac{xy}{x+y}\right)^{\lambda} e^{-\frac{xy}{x+y}\omega}dxdy\right)\left(\omega^{\frac{\lambda-1}{q}-\xi}\int_0^\infty z^{\lambda}e^{-z\omega}g(z)dz\right)d\omega \end{aligned} \tag{3.3}$$

$$\le \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty \omega^{\lambda-1+\xi p}\left(\int_0^\infty\int_0^\infty f(x,y)\left(\frac{xy}{x+y}\right)^{\lambda} e^{-\frac{xy}{x+y}\omega}dxdy\right)^p d\omega\right)^{\frac{1}{p}}$$

$$\left(\int_0^\infty \omega^{\lambda-1-\xi q}\left(\int_0^\infty z^{\lambda}e^{-z\omega}g(z)dz\right)^q d\omega\right)^{\frac{1}{q}}.$$

Substitute (2.6) and (2.7) in (3.3), we get

$$I \le \frac{\Gamma^{\frac{1}{q}}(q\lambda - q\mu)\Gamma^{\frac{1}{p}}(\lambda p - p\vartheta + 1)}{\Gamma(\lambda)}\left(\int_0^\infty\int_0^\infty\left(\frac{xy}{x+y}\right)^{\mu p}\frac{f^p(x,y)}{(x+y)^{\frac{-2p}{q}}}dxdy\int_0^\infty \omega^{\lambda-1+p\xi-\lambda p+p\mu}e^{-\frac{xy}{x+y}\omega}d\omega\right)^{\frac{1}{p}}$$

$$\left(\int_0^\infty z^{q\vartheta}g^q(z)\left(\int_0^\infty \omega^{\lambda-1-\xi q+q\vartheta-q\lambda-\frac{q}{p}}e^{-z\omega}d\omega\right)dz\right)^{\frac{1}{q}}$$

$$= \frac{\Gamma^{\frac{1}{q}}(q\lambda - \mu q)\Gamma^{\frac{1}{p}}(\lambda p - p\vartheta + 1)\Gamma^{\frac{1}{q}}(\lambda - \xi q + q\vartheta - q\lambda - \frac{q}{p})\Gamma^{\frac{1}{p}}(\lambda + \xi p - \lambda p + \mu p)}{\Gamma(\lambda)}$$

$$\times\left[\left(\int_0^\infty\int_0^\infty \frac{\left(\frac{xy}{x+y}\right)^{\lambda p-\xi p-\lambda}}{(x+y)^{-\frac{2p}{q}}}f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+\xi q-\lambda+\frac{q}{p}}g^q(z)\right)^{\frac{1}{q}}\right]$$

$$= C_{\mu,\vartheta}\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-\xi p-\lambda}}{(x+y)^{\lambda p-\xi p-\lambda-\frac{2p}{q}}}f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+\xi q-\lambda+\frac{q}{p}}g^q(z)dz\right)^{\frac{1}{q}}$$

where $\mu = \dfrac{pq\lambda - \lambda - \xi p}{pq}$, and $\vartheta = \dfrac{pq\lambda - \lambda + \xi q + q}{pq}$, then $C_{\mu,\vartheta} = C = B\left(\dfrac{\lambda}{p} + \xi, \dfrac{\lambda}{q} - \xi\right)$.

The next step in this paper is to show that the constant C given in (3.1) is the best possible. Let ε be so very small real number; let us define two functions as follows: $f_\varepsilon(x,y) = 0$ on $(0,1)\times(0,1)$ and $f_\varepsilon(x,y) = \dfrac{q^{\frac{1}{q}}(xy)^{\xi-\frac{\lambda}{q}}}{(x+y)^{2+\frac{\varepsilon}{p}+\xi-\frac{\lambda}{q}}}$ on $[1,\infty)\times 1,\infty)$ and $g_\varepsilon(z) = 0$ on $(0,1)$ and $g_\varepsilon(z) = z^{\frac{\lambda-q\varepsilon}{q}-\xi-\lambda-1}$ on $[1,\infty)$.

Suppose that C is not the best possible, then there exists constant G where $0 < G < C$, such that:

$$I \le G\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-\xi p-\lambda}}{(x+y)^{\lambda p-\xi p-\lambda-\frac{2p}{q}}}f_\varepsilon^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+\xi q-\lambda+\frac{q}{p}}g_\varepsilon^q(z)dz\right)^{\frac{1}{q}}$$

$$= G\left(\int_1^\infty\int_1^\infty \frac{(xy)^{\lambda p-\xi p-\lambda}}{(x+y)^{\lambda p-\xi p-\lambda-\frac{2p}{q}}}\left(\frac{q^{\frac{1}{q}}(xy)^{\xi-\frac{\lambda}{q}}}{(x+y)^{2+\frac{\varepsilon}{p}+\xi-\frac{\lambda}{q}}}\right)^p dxdy\right)^{\frac{1}{p}}\left(\int_1^\infty z^{q\lambda+\xi q-\lambda+\frac{q}{p}}\left(z^{\frac{\lambda-q\varepsilon}{q}-\xi-\lambda-1}\right)^q dz\right)^{\frac{1}{q}}$$

$$< q^{\frac{1}{q}}G\left(\int_1^\infty\int_1^\infty \frac{(xy)^{\lambda p-\xi p-\lambda}}{(x+y)^{\lambda p-\xi p-\lambda-\frac{2p}{q}}}\frac{(xy)^{\xi p-\lambda p+\lambda}}{(x+y)^{\lambda p+\lambda+\varepsilon+\xi p+2p}}dxdy\right)^{\frac{1}{p}}\left(\int_1^\infty z^{q\lambda+\xi q-\lambda+\frac{q}{p}}\left(z^{\frac{\lambda-q\varepsilon}{q}-\xi-\lambda-1}\right)^q dz\right)^{\frac{1}{q}}$$

$$< q^{\frac{1}{q}} G \left(\int_1^\infty \int_1^\infty \frac{1}{(x+y)^{\varepsilon+2}} dxdy \right)^{\frac{1}{p}} \left(\int_1^\infty z^{-q\varepsilon-1} dz \right)^{\frac{1}{q}}$$

$$= \frac{q^{\frac{1}{q}} G}{2^{\frac{\varepsilon}{p}} (\varepsilon)^{\frac{1}{p}} (1+\varepsilon)^{\frac{1}{p}}} \left(\frac{1}{q\varepsilon} \right)^{\frac{1}{q}} = \frac{G}{2^{\frac{\varepsilon}{p}} \varepsilon (1+\varepsilon)^{\frac{1}{p}}} \tag{3.4}$$

On the other hand, to estimate the left-hand side of (3.1), let $u = z\left(\frac{x+y}{xy}\right)$, (note that $q^{\frac{1}{q}}(xy)^{-\varepsilon} = \frac{q^{\frac{1}{q}}}{(xy)^{\varepsilon}} > 1, \forall x, y \geq 1$), we find:

$$I = \int_0^\infty \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x,y) g_\varepsilon(z)}{\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^\lambda} dxdydz$$

$$= \int_1^\infty \int_1^\infty \int_1^\infty \frac{\left(\frac{q^{\frac{1}{q}} (xy)^{\xi - \frac{\lambda}{q}}}{(x+y)^{2+\frac{\varepsilon}{p}+\xi-\frac{\lambda}{q}}} \right) \left(z^{\frac{\lambda - q\varepsilon}{q} - \xi - \lambda - 1} \right) (xyz)^\lambda}{(xy + z(x+y))^\lambda} dxdydz$$

$$= \int_1^\infty \int_1^\infty \int_1^\infty \frac{q^{\frac{1}{q}} (xy)^{\xi - \frac{\lambda}{q}} (xy)^\lambda}{(x+y)^{2+\frac{\varepsilon}{p}+\xi-\frac{\lambda}{q}}} \frac{z^{\frac{\lambda}{q} - \varepsilon - \xi - 1}}{(xy)^\lambda \left(1 + z\frac{x+y}{xy}\right)^\lambda} dxdydz$$

$$= \int_1^\infty \int_1^\infty \int_1^\infty \frac{q^{\frac{1}{q}} (xy)^{\xi - \frac{\lambda}{q}}}{(x+y)^{2+\frac{\varepsilon}{p}+\xi-\frac{\lambda}{q}}} \frac{z^{\frac{\lambda}{q} - \varepsilon - \xi - 1}}{\left(1 + z\frac{x+y}{xy}\right)^\lambda} dxdydz$$

$$= \int_1^\infty \int_1^\infty \frac{q^{\frac{1}{q}} (xy)^{\xi - \frac{\lambda}{q}}}{(x+y)^{2+\frac{\varepsilon}{p}+\xi-\frac{\lambda}{q}}} \left(\int_1^\infty \frac{z^{\frac{\lambda}{q} - \varepsilon - \xi - 1}}{\left(1 + z\frac{x+y}{xy}\right)^\lambda} dz \right) dxdy$$

$$= \int_1^\infty \int_1^\infty \frac{q^{\frac{1}{q}} (xy)^{\xi - \frac{\lambda}{q}}}{(x+y)^{2+\frac{\varepsilon}{p}+\xi-\frac{\lambda}{q}}} \left(\frac{xy}{x+y} \right)^{\frac{\lambda}{q} - \varepsilon - \xi} \left(\int_{\frac{x+y}{xy}}^\infty \frac{u^{\frac{\lambda}{q} - \varepsilon - \xi - 1}}{(1+u)^\lambda} du \right) dxdy$$

$$= \int_1^\infty \int_1^\infty \frac{q^{\frac{1}{q}} (xy)^{-\varepsilon}}{(x+y)^{2+\frac{\varepsilon}{p}-\frac{\varepsilon}{q}}} \left(\int_{\frac{x+y}{xy}}^\infty \frac{u^{\frac{\lambda}{q} - \varepsilon - \xi - 1}}{(1+u)^\lambda} du \right) dxdy$$

$$I > \int_1^\infty\int_1^\infty \frac{1}{(x+y)^{2+\varepsilon}}\left(\int_0^\infty \frac{u^{\lambda-\varepsilon-\xi-1}}{(1+u)^\lambda}du - \int_0^{\frac{x+y}{xy}} u^{\frac{\lambda}{q}-\varepsilon-\xi-1}du\right)dxdy$$

$$= \int_1^\infty\int_1^\infty \frac{1}{(xy)^{2+\varepsilon}}\left(B\left(\frac{\lambda}{q}-\varepsilon-\xi,\frac{\lambda}{p}+\varepsilon+\xi\right) - \int_0^{\frac{x+y}{xy}} \frac{u^{\frac{\lambda}{q}-\varepsilon-\xi-1}}{(1+u)^\lambda}du\right)dxdy \quad (3.5)$$

$$= \frac{B\left(\frac{\lambda}{q}-\varepsilon-\xi,\frac{\lambda}{p}+\varepsilon+\xi\right)}{\varepsilon 2^\varepsilon(1+\varepsilon)} - O(1)$$

It is clear that when $\varepsilon \to 0^+$ from (3.4) and (3.5) leads to a contradiction. Therefore, we completely prove our theorem.

Next, we introduce the reverse form of the inequality given in Theorem 3.1

Theorem 3.2: Let $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$, $-\frac{\lambda}{p}<\xi<\frac{\lambda}{q}$, $f(x,y)$ is a non-negative function defined on $(0,\infty)\times(0,\infty)$ and $g(z)$ is a positive function on $(0,\infty)$. If $\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-\xi p-\lambda}}{(x+y)^{\lambda p-\xi p-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy<\infty$ and $\int_0^\infty z^{q\lambda+\xi q-\lambda+\frac{q}{p}} g^q(z)<\infty$, then

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdydz \ge C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-\xi p-\lambda}}{(x+y)^{\lambda p-\xi p-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy\right)^{\frac{1}{p}} \times\left(\int_0^\infty z^{q\lambda+\xi q-\lambda+\frac{q}{p}} g^q(z)\right)^{\frac{1}{q}} \quad (3.6)$$

where $C=B\left(\frac{\lambda}{p}+\xi,\frac{\lambda}{q}-\xi\right)$ is the best possible.

Proof: By using the reverse Hölder inequality, and following the same procedure as in the proof of Theorem 3.1, we can get the proof of Theorem 3.2.

4. Equivalent Forms

In this part, we give two equivalent forms of each of our main theorems, all with the best constant.

Theorem 4.1: Under the conditions of theorem 3.1, we give the following two inequalities:

$$\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^p dz \le C^p\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy. \quad (4.1)$$

and

$$\int_0^\infty\int_0^\infty \frac{(xy)^{-\lambda-p\xi-1}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^q dxdy \le C^q\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)dz \tag{4.2}$$

Inequalities (4.1) and (4.2) are equivalent to (3.1); also, here the constants C^p and C^q are best possible.

Proof: We start to prove (4.1), set

$$g(z)=z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^{p-1}$$

From (3.1), we get

$$\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^{p} dz$$

$$=\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^{p-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)dz$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdydz$$

$$\le C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}}f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)\right)^{\frac{1}{q}}$$

$$\le C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}}f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^{p} dz\right)^{\frac{1}{q}}$$

From that we get

$$\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^p dz \le C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^p dz\right)^{\frac{1}{q}} \tag{4.3}$$

dividing the two sides of (4.3) by $\left(\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^p dz\right)^{\frac{1}{q}}$ we get (4.1). Moreover, from Hölder inequality and (4.1), we get

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdydz$$

$$=\int_0^\infty\left(z^{-\frac{\lambda}{p}-\xi-\frac{1}{p}}\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)\left(z^{\frac{\lambda}{p}+\xi+\frac{1}{p}}g(z)\right)dz$$

$$\le\left(\int_0^\infty z^{-\lambda-p\xi-1}\left(\int_0^\infty\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdy\right)^p\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)dz\right)^{\frac{1}{q}}$$

$$\le C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)\right)^{\frac{1}{q}}$$

Now, we want to show the equivalence between (3.1) and (4.2), set

$$f(x,y)=\frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^{q-1}$$

Using the main inequality (3.1), we obtain

$$\int_0^\infty\int_0^\infty \frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^q dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^{q-1}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)dxdy$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dxdydz$$

$$\leq C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}}f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)\right)^{\frac{1}{q}}$$

$$=C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^q dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)\right)^{\frac{1}{q}}$$

Then

$$\int_0^\infty\int_0^\infty \frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^q dxdy$$
$$\leq C\left(\int_0^\infty\int_0^\infty \frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^q dxdy\right)^{\frac{1}{p}}\left(\int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}}g^q(z)\right)^{\frac{1}{q}} \tag{4.4}$$

Dividing the two sides of (4.4) by $\left(\int_0^\infty\int_0^\infty \frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}}\left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^\lambda}dz\right)^q dxdy\right)^{\frac{1}{p}}$ we get (4.2). Moreover, from Hölder inequality and (4.2), we get

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x,y)g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\lambda}}dxdydz = \int_0^\infty\int_0^\infty \left(\frac{(xy)^{-\frac{q\xi-\lambda}{q}}}{(x+y)^{-\frac{q\xi-\lambda+2}{q}}} f(x,y) \right)$$

$$\times \left(\frac{(xy)^{\frac{q\xi-\lambda}{q}}}{(x+y)^{\frac{q\xi-\lambda+2}{q}}} \int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\lambda}} dz \right) dxdy$$

$$\leq \left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy \right)^{\frac{1}{p}}$$

$$\left(\int_0^\infty\int_0^\infty \frac{(xy)^{q\xi-\lambda}}{(x+y)^{q\xi-\lambda+2}} \left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\lambda}} dz \right)^q dxdy \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy \right)^{\frac{1}{p}} \left(C^q \int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}} q(z)dz \right)^{\frac{1}{q}}$$

By this, we proved the equivalence between (4.2) and (3.1). Here the constants are the best possible as in (3.1). By this, we lead to the end of the proof of our theorem.

Theorem 4.2: Under the same conditions in theorem 3.2, we obtain:

$$\int_0^\infty z^{-\lambda-p\xi-1} \left(\int_0^\infty \frac{f(x,y)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\lambda}} dxdy \right)^p dz \geq C^p \int_0^\infty\int_0^\infty \frac{(xy)^{\lambda p-p\xi-\lambda}}{(x+y)^{\lambda p-p\xi-\lambda-\frac{2p}{q}}} f^p(x,y)dxdy. \tag{4.5}$$

and

$$\int_0^\infty\int_0^\infty \frac{(xy)^{-\lambda-p\xi-1}}{(x+y)^{q\xi-\lambda+2}} \left(\int_0^\infty \frac{g(z)}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{\lambda}} dz \right)^q dxdy \geq C^q \int_0^\infty z^{q\lambda+q\xi-\lambda+\frac{q}{p}} q(z)dz \tag{4.6}$$

where the constants here $C^p = B^p\left(\frac{\lambda}{p}+\xi, \frac{\lambda}{q}-\xi\right)$ and $C^q = B^q\left(\frac{\lambda}{p}+\xi, \frac{\lambda}{q}-\xi\right)$ are the best possible as in Theorem 3.2

Proof: Since the method of proof of the above inequalities is the same as the method in theorem 4.1, so, we leave it.

Availability of Data and Materials

The data which was used in this study, you can find it in the references listed at the end of the paper, some of them were identified in the introduction.

Competing Interests

The author declares that he has no competing interests.

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Authors' Contributions

We introduce a new form of Hilbert integral inequality for three variables with the new kernel, also, we prove that the constant which appear on the right hand of inequalitiy is the best constant. Also, we introduce the equivalent forms of our main theorems.

References

[1] Nizar, Al-O. Kh., Azar, L. E., and Ahmad H. A. Bataineh, *A Sharper Form of Half-Discrete Hilbert Inequality Related to Hardy Inequality*, Filomat, 32 (19), (2018), 6733–6740.

[2] Yang, B., *The norm of operator and Hilbert-Type Inequalities*, Scince press, Beijing, China, (2009).

[3] Hardy, G. H., Littlewood, J. E., and Polya, G., "*Inequalities*", Cambridge Univ. Press, London, (1952).

[4] Azar, L. E., *Tow new forms of Hilbert integral inequality*, Math. Inequal. Appl. 17, (2014).

[5] Krnić, M., *A refined discrete Hilbert inequality via the Hermite-Hadamard inequality*, Comput. Math. Appl. 63, (2012), 1587–1596.

[6] Krnic, M., Mingzhe, G., Pecaric, J., and Xuemei, G., *On the best constant in Hilbert's inequality*, Math. Inequal. Appl. 8 (2), 2005 317–329.

[7] Krnić, M. J., Pecarić, *Extension of Hilbert's inequality*, J. Math. Anal. Appl. 324, (2006), 150–160.

[8] Krnić, M. J. Pecarić, *General Hilbert's and Hardy's inequalities*, Math. Inequal. Appl. 8, (2005), 29–52.

[9] Batbold, T., and Azar, L. E., *A new form of Hilbert Integral inequaltity*, 12 (2), (2018), 379–390.

[10] Al-Oushoush, N. K., *New Discrete Form of Hilbert Inequality for Three Variables*, Journal of Mathematical Analysis, 13 (3), (2022), 56–67.

[11] Al-Oushoush, N. K., *A New Version of Hilbert Integral Inequality of Three Variables with Hyperbolic Sine Function*, Journal of Inequalities and Special Functions, 13 (4), 1–11.

[12] Batbold, T., and Azar, L. E., *Discrete analogues of Hilbert integral inequalities for three variables*, Journal of Inequalities and Special function, 9, (2018), 66–76.