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Superfluous ideals in module over nearrings

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Abstract

Nearrings are non-linear algebraic systems. Zero-divisor graphs based on algebraic structures like rings, module over rings are well-known. In this paper, we consider the module over a right nearring, (say, G). We define the superfluous ideal graph of G, denoted as $\mathcal{S}_G$. We obtain that if G has DCCI, then S_G has diameter at most 3. We characterize the set of ideals of G with degree 1 in S_G when G is completely reducible. Furthermore, we prove several properties of superfluous ideal graphs which involve connectivity, completeness, etc. with explicit examples of these notions.

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1. Introduction

The graph constructed from a ring structure was introduced in [1]. Later, the concept of zero-divisor graphs (see [2]) were studied. Based on ring structure, various types of graphs like annihilator essential graph (see [3]), essential graph (see [4]), co-maximal ideal graph (see [5]), and total graph (see [2]), zero-divisor graphs in nearrings (see [6]) etc. were studied.

In [7–10], the authors explored the properties of graphs with respect to ideals of a nearring, and ideals of a ring. In a commutative ring, the authors (see [11, 12]) studied the basic properties of the essential ideal graph (sum-essential graph, in case of modules over rings) and characterized rings (resp. modules over rings) based on the types of graphs. The authors considered the set of all non-trivial ideals as the vertex set in a commutative ring (resp. submodules in case of modules) and an edge is defined if the sum of two ideals (resp. submodules) is essential in a given ring (resp. module).

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Due to non-linearity in nearrings, the structure module over a nearring (also known as, N-group) has two substructures, namely, N-subgroups and ideals. Based on these substructures, two types of essential and superfluous ideals were defined and studied in [13, 14]. These concepts were further generalized in [15, 16]. Eventually, the graphs obtained from these will yield different structural properties.

In section 2, we provide necessary definitions of nearrings and notions of graphs from [17, 18]. In section 3, we introduce the superfluous ideal graph of G and consider a subgraph of the superfluous ideal graph, induced by the set of all non-superfluous ideals. We also prove an equivalent condition for G to be completely reducible in terms of S_G. We show that the superfluous ideal graph is always connected with a diameter not greater than 3. Also, we prove that if the superfluous ideal graph of G has a unique universal vertex, then there must exist a non-zero superfluous ideal in G.

2. Preliminaries

A right nearring N is a set having two binary operations under which N is an additive group and multiplicative semigroup with right distributive law holds. In general, for some $x \in N$, $x \cdot 0 \neq 0$, and so we call N is zero-symmetric if $x \cdot 0 = 0$ for every $x \in N$. We denote the set of all elements of N satisfying this property by N_0. A normal subgroup L of a nearring N is an ideal of N (denoted by $L \trianglelefteq N$) if $LN \subseteq L$, and $s(k+i) - sk \in L$ for all $s, k \in N$, $i \in L$. An additive group G with a mapping $N \times G \to G$, satisfying $(n + n_1)g = ng + n_1 g$ and $(n \cdot n_1)g = n(n_1 g)$ for all $n, n_1 \in N$ and $g \in G$ is called an N-group. Throughout, we use G for an N-group. A subgroup P of G is an N-subgroup (denoted as, $P \leq_N G$) of G if $NP \subseteq P$; and a normal subgroup S of G is an ideal (denoted as, $S \trianglelefteq_N G$) of G if $a(x+s) - ax \in I$, for all $a \in N$, $x \in G$ and $s \in S$. $I \trianglelefteq_N G$ is superfluous in G if $I + K = G$ and $K \trianglelefteq_N G$ implies $K = G$ (denoted as $I \ll G$) and an N-group G is hollow if every proper ideal of G is superfluous in G [13]. G is simple if it has no ideals other than (0) and G, and is completely reducible if G is the direct sum of simple ideals. If there exists $B = \{x_1, x_2, \cdots, x_n\} \subset G$ such that $\langle B \rangle = G$, then G is finitely generated. An N-group is said to satisfy decreasing chain condition on ideals (abbr. DCCI) if every chain of ideals has a minimal ideal [18]. Furthermore, we consider simple graphs. A vertex v is universal if the degree of $v = n-1$, where n is the vertex number of vertices of the given graph. A null graph is a graph whose vertex set is empty and an empty graph is a graph having at least one vertex and the edge set empty. The edge between u and v of a graph is denoted as $u \sim v$.

For basic definitions in nearrings and N-groups, we refer to [13, 14, 18, 19, 20], and for graph-theoretical notions, we refer to [17, 19].

3. Main Results

In this section, the superfluous ideal graph of G is defined and some of its properties are discussed.

Definition 3.1: *The superfluous ideal graph $\mathcal{S}_G$ of G, is a graph with vertex set* $V = \{I : (0) \neq I \subsetneq G$ is an ideal of $G\}$ *and the edge set* $E = \{I \sim J : I \neq J$ and $I \cap J \ll G\}$.

Example 3.2: *$\mathcal{S}_G$ is a null graph where G is a simple N-group. Moreover, N-groups of type 0, type 1 and type 2 are simple and monogenic see,* [18]. *Hence the corresponding superfluous ideal graph is a null graph.*

Example 3.3: *Consider the $\mathbb{Z}$-group $\mathbb{Z}_{p^n}$, where p is prime and the addition and multiplication are carried out modulo p^n. Then the proper ideals of G are $\langle p^i \rangle$ where $0 \leq i \lneq n$, which are superfluous in G since $\langle p^i \rangle \subseteq \langle p^j \rangle$ for all $0 \leq j \leq i \lneq n$ and therefore $\langle p^i \rangle + \langle p^j \rangle = \langle p^{max(i,j)} \rangle \neq G$ for all $0 \leq i, j \lneq n$. Therefore, the corresponding superfluous ideal graph is a complete graph.*

Example 3.4: *Let* $N = (\dfrac{\mathbb{Z}_2(t)}{\langle t^3 + t \rangle}, +, \cdot) = \{at^2 + bt + c : a, b, c \in \mathbb{Z}_2\}$, *and $H = N$. The non-trivial ideals of H are $\langle t \rangle, \langle t+1 \rangle, \langle t^2 + t \rangle$ and $\langle t^2 + 1 \rangle$. The ideal $\langle t^2 + t \rangle$ is superfluous. The corresponding superfluous ideal graph is given in the Figure 1.*

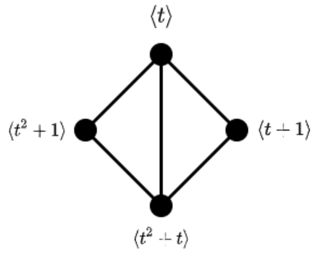

Figure 1:

Table 1:

*	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1	a_1	a_1	a_3	a_3
a_3	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_4	a_1	a_1	a_1	a_1	a_1	a_1	a_3	a_3
a_5	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_6	a_1	a_1	a_1	a_1	a_1	a_1	a_3	a_3
a_7	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_8	a_1	a_1	a_1	a_1	a_1	a_1	a_3	a_3

Example 3.5: *Consider $N = (\mathbb{Z}_{p^n}, +, \star)$ where p is prime and addition modulo p^n and $\star$ is any near-ring multiplication on Z_{p^n} and let $G = N$. Then the possible non-trivial ideals of G are of the form $\langle p^i \rangle$, where $0 < i \lneq n$. In this case, if G has any non-zero proper ideal, then $\mathcal{S}_G$ is a complete graph, otherwise, $\mathcal{S}_G$ is a null graph.*

Example 3.6: *Let $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with the multiplication table given in Table 1 (SONATA [21] GTW8 3,18). Let $N = G$. For convenience, the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are denoted as* $(0,0,0) = a_1$, $(0,0,1) = a_2$, $(0,1,0) = a_3$, $(0,1,1) = a_4$, $(1,0,0) = a_5$, $(1,0,1) = a_6$, $(1,1,0) = a_7$, $(1,1,1) = a_8$.

Ideals are $I_1 = \{a_1\}$, $I_2 = \{a_1, a_2, a_3, a_4\}$, $I_3 = \{a_1, a_3, a_5, a_7\}$, $I_4 = \{a_1, a_3, a_6, a_8\}$, $I_5 = \{a_1, a_3\}$, $I_6 = \{a_1, a_2\}$ and $I_7 = G$. In this case, $I_2 + I_3 = G$, $I_4 + I_3 = G$ and $I_6 + I_3 = G$. Therefore the ideals I_2, I_3, I_4 and I_6 are not superfluous. The only non-zero superfluous ideal is I_5 since $I_5 + I_6 = I_3$, $I_5 + I_2 = I_2$, $I_5 + I_3 = I_3$ and $I_5 + I_4 = I_4$. Superfluous ideal graph is given in Figure 2.

Example 3.7: *Let $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $N = G$. Consider the notations given in Example 3.6. The multiplication table is given in the Table 2 (SONATA [21] GTW8 3,810).*

Ideals are $S_1 = \{a_1\}$, $S_2 = \{a_1, a_2, a_5, a_6\}$, $S_3 = \{a_1, a_2, a_3, a_4\}$, $S_4 = \{a_1, a_3, a_5, a_7\}$, $S_5 = \{a_1, a_5\}$, $S_6 = \{a_1, a_3\}$, $S_7 = \{a_1, a_2\}$ and $S_8 = G$. In this case, we have $S_2 + S_6 = G$, $S_3 + S_5 = G$ and $S_4 + S_7 = G$. Therefore, all non-zero ideals are non-superfluous. The corresponding superfluous ideal graph is given in Figure 3.

Example 3.8: *Let $N = \mathbb{Z}$ and $G = \mathbb{Z}_{24}$ with addition and scalar multiplication defined modulo* 24.

Then the non-trivial ideals of G are $2\mathbb{Z}_{24}$, $3\mathbb{Z}_{24}$, $4\mathbb{Z}_{24}$, $6\mathbb{Z}_{24}$, $8\mathbb{Z}_{24}$ and $12\mathbb{Z}_{24}$. The ideals $\langle 6 \rangle$ and $\langle 12 \rangle$ are non-zero superfluous ideals. The superfluous ideal graph is given in Figure 4.

Lemma 3.9: *If $C \ll G$, then $C \cap D \ll G$ for any ideal D of G.*

Proof. Let $C \ll G$ and $D \trianglelefteq_N G$. To prove $C \cap D \ll G$, let $K \trianglelefteq_N G$ such that $(C \cap D) + K = G$. Now $G = (C \cap D) + K \subseteq C + K$ implies $C + K = G$. Since $C \ll G$, we get $K = G$. Therefore, $C \cap D \ll G$.

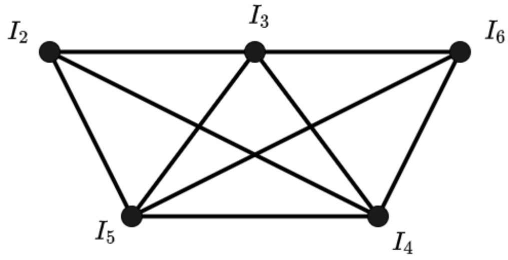

Figure 2:

Table 2:

*	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_3	a_1	a_2	a_1	a_2	a_1	a_2
a_3	a_1	a_1	a_3	a_3	a_1	a_1	a_3	a_3
a_4	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4
a_5	a_5	a_5	a_5	a_5	a_5	a_5	a_5	a_5
a_6	a_5	a_6	a_5	a_6	a_5	a_6	a_5	a_6
a_7	a_5	a_5	a_7	a_7	a_5	a_5	a_7	a_7
a_8	a_5	a_6	a_7	a_8	a_5	a_6	a_7	a_8

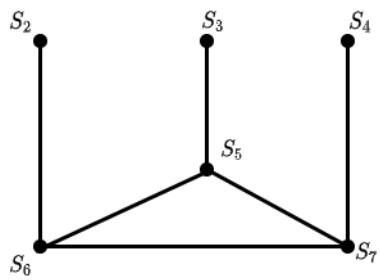

Figure 3:

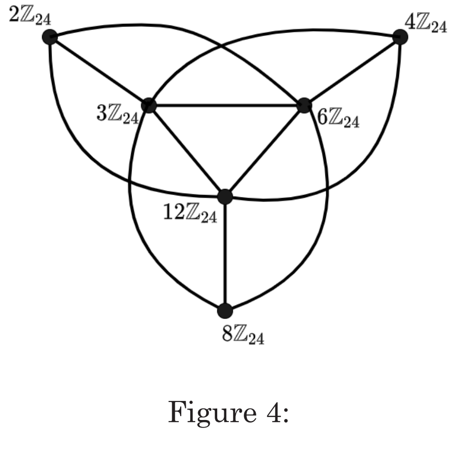

Figure 4:

Lemma 3.10: *Any non-zero superfluous ideal of G is a universal vertex in $\mathcal{S}_G$.*

Proof. Let $(0) \neq C \ll G$ and $D \in V(\mathcal{S}_G)$. We prove $C \sim D \in E(\mathcal{S}_G)$. Since, $C \ll G$, by Lemma 3.9, we have $C \cap D \ll G$. Therefore $C \sim D \in E(\mathcal{S}_G)$. Since D is arbitrary, $C \sim D \in E(\mathcal{S}_G)$ for all $D \in V(\mathcal{S}_G)$. Hence, C is a universal vertex in $\mathcal{S}_G$.

Remark 3.11: *In Example 3.8,* $6\mathbb{Z}_{24}$ *and* $12\mathbb{Z}_{24}$ *are superfluous and therefore* $6\mathbb{Z}_{24}$, $12\mathbb{Z}_{24}$ *are universal in* $\mathcal{S}_G$*. The vertex* $3\mathbb{Z}_{24}$ *is a universal vertex in* $\mathcal{S}_G$*, but* $3\mathbb{Z}_{24}$ *is not superfluous since* $3\mathbb{Z}_{24}+2\mathbb{Z}_{24}=\mathbb{Z}_{24}$*, which shows that the converse of Lemma 3.10 need not be true.*

Proposition 3.12: *[(i)] Let* G *be an* N*-group having DCCI. If* M *is the unique minimal ideal of* G*, then* M *is a universal vertex in* $\mathcal{S}_G$

[(ii)] Let $A \in V(\mathcal{S}_G)$ *be a universal vertex. If* A *is not superfluous in* G*, then* A *is a maximal ideal of* G.

Proof. [(i)] We prove that I is a superfluous ideal of G. Let $K \trianglelefteq_N G$ such that $M+K=G$. Since M is a proper ideal, K must be non-zero. Since G has DCCI, there must be a minimal ideal of G which is contained in K. Since M is the unique minimal ideal of G, M must be contained in K and therefore $G=K+M=K$. Hence M is a superfluous ideal of G. By Lemma 3.10, we can say that M is a universal vertex of S_G.

[(ii)] Let K be a proper ideal of G such that $A \subsetneq K$. Since A is a universal vertex of G, $A \sim K \in E(S_G)$ which implies that $A \cap K = A$ is superfluous in G, a contradiction to the hypothesis. Therefore A is a maximal ideal of G.

Lemma 3.13: *The graph* $\mathcal{S}_G[Min(G)]$ *induced by* $Min(G)$*, where* $Min(G)$ *is the set of all minimal ideals of* G*, is a clique.*

Proof. Case 1: G has only one minimal ideal.
Then $S_G[Min(G)]$ is K_1, a complete graph having one vertex.

Case 2: $|Min(G)| \geq 2$. Let $M_1, M_2 \in Min(G)$. Then $M_1 \cap M_2 \subsetneq M_1$ and $M_1 \cap M_2 \subsetneq M_2$. Since M_1 and M_2 are minimal, we get $M_1 \cap M_2 = (0) \ll G$ and therefore $M_1 \sim M_2 \in E(S_G)$. From this we conclude that there exists edge between every two minimal ideals. Hence $S_G[Min(G)]$ is a clique.

Definition 3.14: *Let* $I \trianglelefteq_N G$*. The dual annihilator of* I*, (denoted as* $ann_d(I)$*) is the intersection of all ideals* J *of* G *such that* $I+J=G$*. That is,*

$$ann_d(I) = \bigcap_{J \trianglelefteq_N G, I+J=G} J$$

Example 3.15: $N = D_8$ *with the multiplication given in the Table 3 given below. Let* $G = N$.

The ideals of G are $I_1 = G$, $I_2 = \{e, r^2, r^3, r\}$, $I_3 = \{e, sr^3, r^2, sr\}$, $I_4 = \{e, sr^2, s, r^2\}$, $I_5 = \{e, r^2\}$ and $I_6 = \{e\}$. We have $I_2 + I_3 = N$ and $I_2 + I_4 = N$. Therefore $ann_d(I_2) = \bigcap\{I_3, I_4\} = I_5$.

Example 3.16: *In the Example 3.7, the dual annihilator of* S_7 *is* S_4 *and that of* S_2 *is* S_6.

Table 3:

*	e	r	r^2	r^3	s	sr^3	sr^2	sr
e	e	e	e	e	e	e	e	e
r	e	e	e	e	e	r^2	e	e
r^2	e	e	e	e	e	e	e	e
r^3	e	e	e	e	e	r^2	e	e
s	e	e	e	e	e	e	e	e
sr^3	e	e	e	e	e	r^2	e	e
sr^2	e	e	e	e	e	e	e	e
sr	e	e	e	e	e	r^2	e	e

Proposition 3.17: *Let $I \trianglelefteq_N G$. Then $I \cap ann_d(I) \ll G$.*

Proof. Let $K \trianglelefteq G$ such that $(I \cap ann_d(I)) + K = G$. Since $(I \cap ann_d(I)) \subseteq I$, we have $I + K = G$, which implies $ann_d(I) \subseteq K$ and so $(I \cap ann_d(I)) \subseteq K$. Now $K = K + (I \cap ann_d(I)) = G$. Therefore, $(I \cap ann_d(I)) \ll G$.

Proposition 3.18: *Let G be an N-group having DCCI. Then $\mathcal{S}_G$ is an empty graph if and only if G has exactly one non-zero proper ideal.*

Proof. If G has exactly one non-zero proper ideal, then $\mathcal{S}_G$ is K_1, an empty graph. Conversely, suppose $\mathcal{S}_G$ is an empty graph. Then by Lemma 3.13, we get $|Min(G)| = 1$. Let I be the unique minimal ideal of G. Then by Proposition 3.12, I is a universal vertex. Let J be a non-zero proper ideal of G other than I. Then $I \sim J \in \mathcal{S}_G$, a contradiction since $\mathcal{S}_G$ is an empty graph. Therefore, G has exactly one non-zero proper ideal.

Definition 3.19: *Let $H \trianglelefteq_N G$. An ideal K of G is said to an i-supplement of H in G if $H + K = G$ and $H + K' \neq G$ for any $K' \trianglelefteq_N G$ such that $K' \subset K$.*

Lemma 3.20: *Let H be a non-superfluous ideal of G and K be its i-supplement. Then $H \cap K \ll G$.*

Proof. Let $L \trianglelefteq_N G$ such that $(H \cap K) + L = G$. Then $((H \cap K) + L) \cap K = K$. Using Modular law, we get $(H \cap K) + (L \cap K) = K$. Now $G = H + K = H + (H \cap K) + (L \cap K) = H + (L \cap K)$. Since K is a i-supplement of H, we have $K \cap L = K$ which implies $K \subseteq L$. Therefore $G = (H \cap K) + L = L$. Hence $H \cap K \ll G$.

Proposition 3.21: *Let G be an N-group with DCCI. Every proper non-superfluous ideal of G is adjacent to its i-supplement in $\mathcal{S}_G$.*

Proof. Let I be a non-superfluous ideal of G. Then there exists a proper ideal J of G, such that $I + J = G$. Consequently, I has a proper i-supplement, say I' and so $I \cap I' \ll G$. Therefore, $I \sim I' \in E(\mathcal{S}_G)$.

Proposition 3.22: *Let G be an N-group having DCCI. Then $\mathcal{S}_G$ is a connected graph of diameter less than 4.*

Proof. Suppose G has a non-zero superfluous ideal, say K. Then K is a universal vertex in $\mathcal{S}_G$ (by Lemma 3.10). Therefore, $\mathcal{S}_G$ is connected. Suppose G has no non-zero superfluous ideal. Let P and L be two non-zero proper ideals of G.

Case 1: $P \cap L \ll G$. Then $P \sim L \in E(S_G)$.

Case 2: $P \cap L$ is not superfluous in G. Since P and L are not superfluous, we have $P + L$ is not superfluous in G. Let C be the i-supplement of $P \cap L$. If $P + L \neq G$, then $P + L$ has a non-zero i-supplement, say D. Then $(P + L) \cap D \ll G$ which implies $P \cap D \ll G$ and $L \cap D \ll G$. Now we have a path $P \sim D \sim L$ from P to L. Suppose $P + L = G$. Then we claim that $L \cap C$ and $P \cap C$ are non-zero. Suppose on the contrary, $L \cap C = (0)$. Since C is a i-supplement of $P \cap L$ and G has no non-zero superfluous ideals, we get $(P \cap L) + C = G$ and $(P \cap L) \cap C = (0)$, which implies that the sum $(P \cap L) \cap C = G$ is direct. Now since $P \cap L \subseteq L$, we get $L + C = G$. $L \cap C = (0)$ implies that the $L \oplus C = G$ is direct. The direct sums $L \oplus C$ and $(P \cap L) \oplus C$ yield G and therefore we get $P \cap L = L$, which implies $L \subseteq P$. Now $P + L = G$ implies $P = G$, a contradiction. Therefore $L \cap C \neq (0)$. Similarly, we can prove that $P \cap C \neq (0)$. Since $P \cap L \cap C = (0)$, we have a path $P \sim L \cap C \sim P \cap C \sim L$ from P to L.

Proposition 3.23: *$\mathcal{S}_G$ is a complete graph if and only if every proper, non-superfluous ideal is a maximal ideal.*

Proof. Assume that every proper, non-superfluous ideal of G is a maximal ideal. To prove $\mathcal{S}_G$ is a complete graph, let I and J be two distinct vertices of $\mathcal{S}_G$. Then I and J are proper in G.

Case 1: Either I or J is superfluous in G. Then by Lemma 3.9, $I \sim J \in E(\mathcal{S}_G)$.
Case 2: Neither I nor J is superfluous in G.

Since I and J are proper, by the hypothesis I and J are maximal. Then $I \cap J \subseteq I \neq G$, implies $I \cap J$ is not maximal. Again from the hypothesis, $I \cap J$ is superfluous, and hence $I \sim J \in E(\mathcal{S}_G)$.

Conversely, assume that $\mathcal{S}_G$ is a complete graph. We prove every proper non-superfluous ideal is maximal. On the contrary, let I be a proper non-superfluous ideal, which is not maximal. Then there exists an ideal $J \neq G$ such that $I \subset J$. Now, since $\mathcal{S}_G$ is a complete graph, $I \sim J \in E(\mathcal{S}_G)$, which implies $I = I \cap J \ll G$, a contradiction.

Proposition 3.24: *If $\mathcal{S}_G$ has exactly one universal vertex, then G has a unique non-zero superfluous ideal.*

Proof. Suppose $\mathcal{S}_G$ has a unique universal vertex, say I. Then I is a non-zero proper ideal of G. We prove that I is minimal. On the contrary, suppose that I is not minimal. Then I contains some ideal K of G. Since I is a universal vertex, we get $I \sim K \in E(\mathcal{S}_G)$, which implies $I \cap K = K \ll G$, a contradiction since $\mathcal{S}_G$ has only one universal vertex. Therefore, I is minimal. If I is not maximal, then there exists an ideal $(0) \neq J$ of G such that $I \subsetneq J$. Since I is a universal vertex, $I \sim J \in E(\mathcal{S}_G)$, which implies $I \cap J = I \ll G$. If I is maximal, we prove that I is superfluous. On the contrary, suppose that I is not superfluous, then there exists a proper ideal J of G, $I + J = G$. Since I is maximal, we have $I \nsubseteq J$. Since, G is finitely generated, there exists a maximal ideal M' such that $J \subseteq M'$. Now, since I and M' are maximal, we get $I + M' = G$. Also, since I is minimal, we get $I \cap M' = (0)$. Then we can easily verify that G is isomorphic to $\frac{G}{I} \times \frac{G}{M'}$. Since I and M' are maximal, $\frac{G}{I}$ and $\frac{G}{M'}$ are simple. Hence G has only two proper ideals and hence $\mathcal{S}_G = K_2$, which has two universal vertices, a contradiction. Therefore, I is superfluous.

Remark 3.25: *The converse of Proposition 3.24 need not be true. This can be observed in Example 3.6. I_5 is the only non-zero superfluous ideal. But in the corresponding graph there are two universal vertices I_3 and I_5.*

Proposition 3.26: *Let I be a non-zero proper ideal of G and $J \trianglelefteq_N G$. If $I \sim J \in E(\mathcal{S}_G)$, then $I \sim K \in E(\mathcal{S}_G)$ for any proper ideal K of G which is contained in J.*

Proof. Let $(0) \neq K$ be a proper ideal of G such that $K \subseteq J$. Let $P \trianglelefteq_N G$ be such that $(I \cap K) + P = G$. Then $G = (I \cap K) + P \subseteq (I \cap J) + P$. Since $I \sim J \in E(\mathcal{S}_G)$, $I \cap J \ll G$, and we get $P = G$. Therefore, $I \cap K \ll G$, hence $I \sim K \in E(\mathcal{S}_G)$.

Definition 3.27: *The proper superfluous ideal graph $\mathcal{P}_G$ of G is a subgraph of $\mathcal{S}_G$, induced by the vertices which are non-superfluous ideals of G.*

Proposition 3.28

1. $\mathcal{P}_G$ is a null graph if and only if G is hollow.
2. $\mathcal{P}_G$ cannot be an empty graph.

Proof.

1. It is clear by the definition of a hollow N-group.
2. Let $I \in V(\mathcal{P}_G)$. This means I is non-superfluous and I has a proper i-supplement, say C, and so $I \cap C \ll G$, hence $I \sim C \in E(\mathcal{P}_G)$.

Theorem 3.29: *(Theorem 2.48 of* [18]*). Let $N = N_0$. Then, G is completely reducible if and only if each ideal of G is a direct summand.*

Proposition 3.30: *Suppose $N = N_0$ and G is not simple. Then (1)–(3) are equivalent.*

1. G is completely reducible.
2. $\mathcal{S}_G = \mathcal{P}_G$.
3. There exists a vertex X of $\mathcal{P}_G$ with $deg_{\mathcal{S}_G}(X) = deg_{\mathcal{P}_G}(X)$.

Proof. $(1) \Rightarrow (2)$: Suppose that G is completely reducible. Then if I is any non-trivial ideal of G, by Theorem 3.29, there exists a non-trivial ideal J of G such that $I + J = G$. Therefore, I is not superfluous, which implies $V(\mathcal{S}_G) = V(\mathcal{P}_G)$. Since $\mathcal{P}_G$ is an induced subgraph of $\mathcal{S}_G$, we get $\mathcal{S}_G = \mathcal{P}_G$.

$(2) \Rightarrow (3)$ follows directly.

$(3) \Rightarrow (1)$: Suppose that there exists a vertex X of $\mathcal{P}_G$ such that $deg_{\mathcal{S}_G}(X) = deg_{\mathcal{P}_G}(X)$. To prove G is completely reducible, we show every ideal of G is a direct summand. Suppose G has a superfluous ideal I. Then $deg_{\mathcal{S}_G}(X) = deg_{\mathcal{P}_G}(X) + 1$, as I is a universal vertex in $\mathcal{S}_G$ and $X \sim I \in E(\mathcal{S}_G)$, which is a contradiction to the assumption. Hence, every proper ideal of G is not superfluous. Now, let J be a proper ideal of G. Since J is not superfluous, J has a proper i-supplement, say K. Then $J \cap K \ll G$ and since G has no non-zero superfluous ideal, we get $J \cap K = (0)$. Therefore J is a direct summand. Since J is arbitrary, we conclude that every ideal of G is a direct summand. Hence G is completely reducible.

Remark 3.31 *Let $N = N_0$ and G be completely reducible. Using Theorem 3.29, one can easily deduce that G has no non-zero superfluous ideals.*

Proposition 3.32 *Let $B \in \mathcal{S}_G$. If $deg_{\mathcal{S}_G}(B) = 1$, then either B is a maximal ideal or $B \subset A$ for some maximal ideal A of G and $V(\mathcal{S}_G) = \{A, B\}$.*

Proof. If B is maximal, then we are done. Suppose B is not maximal. Then there exists a proper ideal A of G such that $B \subseteq A$. If $B \ll G$, then B is a universal vertex. Since $deg_{\mathcal{S}_G}(B) = 1$, we get $V(\mathcal{S}_G) = \{A, B\}$. Suppose B is not superfluous in G, then B has a i-supplement, say C. Then, since $B \cap C \ll G$, we have $B \sim C \in E(\mathcal{S}_G)$. Now $B \cap (A \cap C) \ll G$ and so $B \sim (A \cap C) \in E(\mathcal{S}_G)$. If $A \cap C \neq (0)$, then since $deg_{\mathcal{S}_G}(B) = 1$, we have $A \cap C = C$ which implies $C \subseteq A$. Now $G = B + C \subseteq A + C = A$, a contradiction. Thus $A \cap C = (0)$ and since $B \subseteq A$, we get $B \cap C = (0)$. Since $B + C = G$, we get $A + C = G$. Therefore $B = A$, a contradiction. Hence B is maximal.

Remark 3.33: *Let B be a direct summand of G. If $deg_{\mathcal{S}_G}(B) = 1$, then B is maximal.*

Proposition 3.34: *Let $N = N_0$ and G be completely reducible and $B \trianglelefteq_N G$. Then $deg_{\mathcal{S}_G}(B) = 1$ if and only if B is maximal and has a unique proper i-supplement.*

Proof. Suppose that $deg_{\mathcal{S}_G}(B) = 1$. Since G is completely reducible, by the Theorem 3.29, we get that B is a direct summand. Hence, by Proposition 3.33, we have B is maximal. Since G is completely reducible, by Remark 3.31, B is not superfluous. Since G has DCCI, we get B has a proper i-supplement. Suppose C and D are two proper i-supplements of B. Then, by Remark 3.21, $B \sim C$, $B \sim D \in E(\mathcal{S}_G)$, and since $deg_{\mathcal{S}_G}(B)=1$, we have $C=D$. Therefore the i-supplement of B in G is unique. To prove the other part, suppose B is maximal, and has a unique i-supplement, say S_B in G. Then $B \ll S_B \ll G$, which implies $B \sim S_B \in E(\mathcal{S}_G)$. We now show that B is not adjacent to any other ideal of G. Let K be a non-zero proper ideal of G other than S_B. Since G is completely reducible, by Remark 3.31, K is not superfluous. Since B is maximal, we get $B+K=B$ or $B+K=G$. If $B+K=B$, then $K \subseteq B$. Since K is not superfluous, we get B is not superfluous. Therefore $B \sim K \notin E(\mathcal{S}_G)$. If $B+K=G$, then $S_B \subsetneq K$. We claim that $B \oplus K$ is not superfluous in G. On the contrary, suppose $B+K \ll G$. Then since G has no non-zero superfluous ideal, we get $B \oplus K = G$. Now, $B \oplus K = G$ and $B \oplus S_B = G$, and B has a unique i-supplement, implying that $K = S_B$, a contradiction. Therefore B is adjacent to only S_B. Hence, $deg_{\mathcal{S}_G}(B)=1$.

4. Conclusion

We have introduced the superfluous ideal graph of N-group G and proved that if G has DCCI, then S_G has diameter at most 3. Further, we have characterized the set of ideals of G with degree 1 in S_G when G is completely reducible. Several properties are investigated with explicit examples. As future scope, one can explore the study of the properties of module analogue aspects of lattices, which have been motivated by the authors in [22, 23].

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References

[1] Beck, I., *Colouring of commutative rings*, Journal of Algebra 116, (1988), 208–216.
[2] Anderson D. F., Asir T., Badavi A., Tamizh Chelvam T., *Graphs from rings*, Springer Nature (2021).
[3] Babaei, S., Payrovi, S., Sevim, E. S., *On the annihilator submodules and the annihilator essential graph*, Acta Mathematica Vietnamica 44, (2019), 905–914.
[4] Nikmehr, M. J., Nikandish, R., Bakhtyiari, M., *On the essential graph of a commutative ring*, Journal of Algebra and Its Applications 16(5), (2017).
[5] Sharma, P. K., Bhatwadekar, S. M., *A note on graphical representation of rings*, Journal of Algebra 176, (1995), 124–127.
[6] Cannon, G. A., Neuerburg, K. M., Redmond, S. P., *Zero-divisor graphs of nearrings and semigroups*, In: Kiechle H., Kreuzer A., Thomsen M. J. (eds) Nearrings and Nearfields Springer, Dordrecht, (2005).
[7] Bhavanari, S., Kuncham, S. P., Kedukodi, B. S., *Graph of a nearring with respect to an ideal*, Communications in Algebra 38(5), (2010), 1957–1967.
[8] Kedukodi., B. S., Jagadeesha, B., Kuncham, S. P., *Different prime graphs of a nearring with respect to an ideal*, In: Nearrings, Nearfields and Related Topics. World Scientific, Singapore, (2017), 185–203.
[9] Bhavanari, S., Kuncham, S. P., Nagaraju, D., *Prime graph of a ring, Journal of Combinatorics*, Information and System Sciences 35, (2010). 27–42. ISSN 0250-9628.
[10] Rajani, S., Kedukodi, B. S., Harikrishnan, P. K., Kuncham, S. P., *Essential ideal of a matrix nearring and ideal related properties of graphs*. Bol. Soc. Paran. Mat. (accepted for publication).
[11] Amjadi, J., *The essential ideal graph of a commutative ring*, Asian-European Journal of Mathematics, 11(4), (2018).
[12] Matczuk, J., Majidinya, A., *Sum-essential graphs of modules*, Journal of Algebra and Its Applications 20(11), (2021).
[13] Reddy, Y. V., Bhavanari, S., *Finite spanning dimension in N-groups*, The Mathematics Student 56, (1988), 75–80.
[14] Reddy, Y. V., Bhavanari, S., *A note on N-groups*, Indian Journal Pure and Applied Mathematics 19, (1988), 842–845.
[15] Tapatee, S., Davvaz, B., Panackal, H., Kedukodi, B. S., Kuncham, S. P., *Relative essential ideals in N-groups*, Tamkang Journal of Mathematics, 54, (2021), https://doi.org/10.5556/j.tkjm.54.2023.4136
[16] Rajani, S., Tapatee, S., Kedukodi, B. S., Harikrishnan, P., Kuncham, S.P., *Superfluous ideals of N-groups*. Rend. Circ. Mat. Palermo, II. Ser (2023). https://doi.org/10.1007/s12215-023-00888-2.
[17] Bhavanari, S., Kuncham, S. P., *Discrete mathematics and graph Theory*, PHI learning (2009).
[18] Pilz, G., *Nearrings: the theory and its applications*, North Holland Publishing Company 23, (1983).

[19] Bhavanari, S., Kuncham, S. P., *Nearrings, fuzzy ideals, and graph theory*, Chapman and Hall (2013) Taylor and Francis Group (London, New York) ISBN 13: 9781439873106.
[20] Bhavanari, S., *Goldie dimension and spanning dimension in modules and N-groups*, In: Near-rings, Nearfields and Related topics (Review Volume) World Scientific Singapore (2017), 26–41.
[21] Archinger, J., Binder, F., Ecker, J., Mayr, P., Nobauer, C., *System of nearrings and their applications*, Version 2.9.1 (GAP package. 2.6, 2012).
[22] Nimbhorkar, S., Deshmukh, V., The essential element graph of a lattice, Asian European Journal of Mathematics 13(1) (2020).
[23] Tapatee, S., Harikrishnan, P. K., Kedukodi, B. S., Kuncham, S. P., *Graphs with respect to superfluous elements in a lattice*, Miskolc Mathematical Notes 23(2), (2022), 929–945, doi:10.1142/S1793557120500230.