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Caputo-type of two parameters for fuzzy fractional differential equations using OHAM technique

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Abstract

Fractional derivative gives important tools to fit the real data with mathematical models because of the fractional parameters. This article introduces an algorithm for the approximation of solutions to linear and nonlinear fuzzy fractional initial value problems, specifically those involving the Caputo-Katugampola (CK) derivative, a generalized fractional derivative. The CK fractional derivative, characterized by two parameters, extends the capabilities of Caputo and Caputo-Hadamard fractional derivatives. The Optimal Homotopy Asymptotic Method (OHAM) is employed as an approximate analytic technique, offering multiple convergent control parameters to fine-tune solution convergence and accuracy. The article also addresses the representation of environmental uncertainty within the solution using Zadeh's fuzzy theory extension principle. This algorithm not only introduces the fuzzy fractional differential with the CK derivative but also provides a convergent analytic solution with minimal residual error. This contribution aims to support researchers in refining mathematical models to better align with real-world data. Three examples are considered to demonstrate the efficiency of the algorithm with several figures and tables.

Key words and phrases: Caputo-Katugampola Derivative, Fractional Derivative, Optimal Homotopy Asymptotic Method, Fuzzy Fractional Differential Equations.

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1. Introduction

The context for real-world mathematical modeling procedures is provided by fuzzy fractional differential equations (FFDEs). To develop a more realistic and flexible model, FFDEs account for the fact that knowledge about the behavior of a dynamical system may be ambiguous or contain ambiguous parameters [1]. FFDEs are used in various applications such as civil engineering, medicine, population models, and particle systems. Agarwal [2] used fuzzy fractional differential equations for the first time in 2010, Agarwal's shareholding changed into drawing the manner for scholars to deal with fuzzy fractional differential equations via investigating more fantastic applications can be discovered in fluid dynamics [3], physics [4]. One of the greatest benefits of using fractional derivatives for modeling real-life phenomena is the long-term memory effect in the system [5]. For example, Wawrzkiewicz et al [6] used the fractional derivative for simple random-walk models of ion-channel gate dynamics reflecting long-term memory. Because of the fast improvement applications of physical for FFDEs, numerous analytical strategies have been applied to solve FFDEs.

In most Fuzzy fractional differential equations especially nonlinear ones, there are no exact analytical solutions known in closed form, so approximate analytical and numerical methods must be applied. Newly, many studies have been focused on the numerical and analytical solution of Fuzzy fractional differential equations. Some numerical and analytical methods have been developed, such as the Power series expansion method [7], Laplace transform method [8, 9], Sumudu transform [10, 11], and semi-analytical methods such as Variation Iteration Method (VIM) [12, 13], Adomian Decomposition Method (ADM) [14] and Modified Homotopy Perturbation Method (MHPM) [15], Gegenbauer Wavelet Polynomials [16], and successive iterations [17].

The concept of the OHAM for handling nonlinear problems without relying only on small/large parameters was presented by Marinca [18, 19, 20]. Recently, the OHAM has been hired on several engineering and physical [21, 22]. The interesting benefit of OHAM is that it no longer requires determining the starting estimate or the curve of the convergence control parameter as HAM does. Instead, OHAM has a built-in convergence criterion, comparable to HAM but with a larger degree of flexibility [23]. This technique does not necessitate the linearization or discretization of the variables, nor does it suffer from computation round-off errors, nor does it necessitate a timer. OHAM is also parameter unfastened and provides higher accuracy than the approximate analytical strategies on the identical order of approximation.

Caputo-Katugampola (CK) fractional integral and derivative is a novel notion of fractional integral and derivative [24], which generalizes Caputo and Caputo-Hadamard fractional derivatives. The utilized derivative is considerably impacted by the parameter values, introducing a useful tool for developing fractional calculus models. CK derivative has two fractional parameters which give more generalization than other fractional definitions in the modeling and simulations. For instance, the Caputo and Caputo-Hadamard fractional derivatives are special cases of CK derivative. This sort of generalized derivative appears to be more similar to ordinary derivatives than others and proposes an adaptive predictor-corrector approach for numerically solving generalized Caputo-type initial value problems [25]. The value has a significant impact on the fractional derivative's features of the second parameter ρ, as a result, it gives a new route for control applications in [26] the authors study the existence and uniqueness findings of an initial value issue solution for (CK) fuzzy fractional differential equations.

This work constructs a new effective algorithm for solving FFDEs that contain CK derivative with two parameters using OHAM. This algorithm is based on choosing proper parameters that guarantee the convergence of solutions that appear in the discussed examples. Combining the several parameters fractional derivative form as CK definition with a strong algorithm that controls the convergence parameters (OHAM) will guarantee an accurate solution that obeys the real-life data of the models. To the best of our knowledge, this is the first contribution that suggests an approximate analytic solution for FFDEs including the CK fractional derivative.

2. Preliminaries

In this section, some Preliminaries will be briefly presented which will be used throughout the paper. Here we should mention fundamental definitions and substantial properties for generalized CK derivative and integral, and basic definitions for the fuzzy set concept.

2.1. Caputo-Katugampola fractional derivative

The CK derivative is a novel fractional operator that generalizes the idea of Caputo and Caputo-Hadamard fractional derivatives.

Definition 2.1. [27] *Both left and right generalized fractional integrals of the function f , called the Katugampola fractional integrals are respectively given by:*

$$I_{a+}^{\alpha,\rho}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x s^{\rho-1}(x^\rho-s^\rho)^{\alpha-1}f(s)ds,$$

and

$$I_{b-}^{\alpha,\rho}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b s^{\rho-1}(x^\rho-s^\rho)^{\alpha-1}f(s)ds,$$

where $0<a<b<\infty$, $f:[a,b]\to \mathrm{R}$ *is an integrable function, and* $\alpha>0$ *and* $\rho>0$ *two fixed real numbers.*

The following properties of the fractional integral operator $I^{\alpha,\rho}$ where $\alpha\geq 0$, $\rho>0$, and for constant $c\in \mathrm{R}$, holds:

(1) $I^{\alpha,\rho}I^{\beta,\rho}f(x)=I^{\alpha+\beta,\rho}f(x)=I^{\beta+\alpha,\rho}f(x)=I^{\beta,\rho}I^{\alpha,\rho}f(x)$,
(2) $I^{\alpha,\rho}(cf(x))=cI^{\alpha,\rho}(f(x))$,
(3) $I^{\alpha,\rho}(x^n)=\dfrac{\rho^{-\alpha}\Gamma\left(\frac{n}{\rho}+1\right)}{\Gamma\left(\alpha+\frac{n}{\rho}+1\right)}x^{n+\alpha\rho}$

The first two properties are found in [28].

Proof property (3)

Proof. By Definition (2.1) and set $a=0$ we have

$$I^{\alpha,\rho}(x^n)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x s^{\rho-1}(x^\rho-s^\rho)^{\alpha-1}(s^n)ds.$$

Using the Beta function B(., .), and change variables $u=\frac{s^\rho}{x^\rho}$, we obtain

$$I^{\alpha,\rho}(x^n)=\frac{\rho^{-\alpha}}{\Gamma(\alpha)}x^{\alpha\rho+n}\int_0^1(1-u)^{\alpha-1}u^{\frac{n}{\rho}}du.$$

$$=\frac{\rho^{-\alpha}}{\Gamma(\alpha)}x^{\alpha\rho+n}\mathcal{B}(\alpha,\frac{n}{\rho}+1)$$

$$=\frac{\rho^{-\alpha}}{\Gamma(\alpha)}x^{\alpha\rho+n}\frac{\Gamma(\alpha)\Gamma(\frac{n}{\rho}+1)}{\Gamma(\alpha+\frac{n}{\rho}+1)}$$

$$=\frac{\rho^{-\alpha}\Gamma(\frac{n}{\rho}+1)}{\Gamma(\alpha+\frac{n}{\rho}+1)}x^{n+\alpha\rho}.$$

Definition 2.2. [27] *Let* $0 < a < b < \infty$, $f : [a, b] \to \mathrm{R}$ *is an integrable function, and for* $m - 1 < \alpha \le m$, $m \in \mathrm{N}$ *and* $\rho > 0$. *Both left and right CK fractional derivatives of order* α, ρ *are respectively defined by:*

$$ {}^{C}D_{a+}^{\alpha,\rho} f(x) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^x \frac{s^{(\rho-1)(1-m)}}{(x^\rho - s^\rho)^{\alpha-m+1}} f^{(m)}(s)ds, $$

and

$$ {}^{C}D_{b-}^{\alpha,\rho} f(x) = \frac{(-1)^m \rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_x^b \frac{s^{(\rho-1)(1-m)}}{(x^\rho - s^\rho)^{\alpha-m+1}} f^{(m)}(s)ds, $$

respectively.

The following properties of CK fractional derivative can be held for constant $c \in \mathrm{R}$, $a < x \le b$ where $a \ge 0$, $\rho > 0$, $m - 1 < \alpha \le m$ and $f \in C^j [a, b]$, first three properties can refer to [25, 27]:

(1) ${}^{C}D^{\alpha,\rho}(c) = 0$,

(2) ${}^{C}D_{a+}^{\alpha,\rho} I_{a+}^{\alpha,\rho} f(x) = f(x)$,

(3) $I_{a+}^{\alpha,\rho}\, {}^{C}D_{a+}^{\alpha,\rho} f(x) = f(x) - \sum_{j=0}^{m-1} \frac{1}{\rho^j (j)!} (x^\rho - a^\rho)^j \left[\left(x^{1-\rho} \frac{d}{dx} \right)^j f(x) \right] |_{x=a}$,

(4) ${}^{C}D^{\alpha,\rho}(x^n) = \dfrac{\Gamma(n+1)\Gamma(\frac{n}{\rho} - m + 1)}{\Gamma(n-m+1)\Gamma(\frac{n}{\rho} - \alpha + 1)} \rho^{\alpha-m} x^{n-\alpha\rho}$.

We going to prove property (4).

Proof. By Definition (2.2) and set $a = 0$ we have

$$ {}^{C}D^{\alpha,\rho}(x^n) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_0^x \frac{s^{(\rho-1)(1-m)}}{(x^\rho - s^\rho)^{\alpha-m+1}} \left(\frac{d^m}{dx^m} x^n \right) ds. $$

Using the Beta function B(., .), and change variables $u = \dfrac{s^\rho}{x^\rho}$, we obtain

$$ \begin{aligned} {}^{C}D^{\alpha,\rho}(x^n) &= \frac{\Gamma(n+1)\rho^{\alpha-m}}{\Gamma(n-m+1)\Gamma(m-\alpha)} x^{n-\alpha\rho} \int_0^1 (1-u)^{m-\alpha-1} u^{\frac{n}{\rho}-m} du \\ &= \frac{\Gamma(n+1)\rho^{\alpha-m}}{\Gamma(n-m+1)\Gamma(m-\alpha)} x^{n-\alpha\rho} \mathcal{B}(m-\alpha, \frac{n}{\rho} - m + 1) \\ &= \frac{\Gamma(n+1)\rho^{\alpha-m}}{\Gamma(n-m+1)\Gamma(m-\alpha)} x^{n-\alpha\rho} \frac{\Gamma(m-\alpha)\Gamma(\frac{n}{\rho} - m + 1)}{\Gamma(\frac{n}{\rho} - \alpha + 1)} \\ &= \frac{\Gamma(n+1)\Gamma(\frac{n}{\rho} - m + 1)}{\Gamma(n-m+1)\Gamma(\frac{n}{\rho} - \alpha + 1)} \rho^{\alpha-m} x^{n-\alpha\rho} \end{aligned} $$

Remark 2.3. *When* $\rho = 1$, *the CK fractional derivative is simplified to the usual Caputo.*

2.2. Fuzzy set

A fuzzy set idea that was studied by Zadeh in 1965 [29] and is regarded as a generalization of crisp (classical) set theory [30]. The membership of items on the topic of a set is classified in binary terms in the crisp sets concept; an element either belongs to or does not belong to the set. Fuzzy set theory, on the other hand, provides for the progressive assessment of the membership of elements on the topic of a set, which is defined with the valuable resource of a membership function valued with inside interval [0, 1]. Fuzzy sets are an extension of the classical set principle because, given a positive universe, a membership characteristic can also work as an indicator function, translating all components to either 0 or 1. A crisp set is often defined as a collection of items or objects *xinX* that may or may not be countable.

Definition 2.4. [31] *Let $\tilde{A}: X \to [0,1]$ be a fuzzy set. The ζ-level (ζ-cut) representation of a fuzzy set $\tilde{A}$ is defined as:*

$$[\tilde{A}]_{\zeta} = \{s \in X \mid \delta\tilde{A}(s) > \zeta\}, \quad \zeta \in [0,1]$$

Definition 2.5. [32] *A triangular fuzzy number is a fuzzy number defined as triple numbers $\alpha_1 < \alpha_2 < \alpha_3$ with the base on the interval $[\alpha_1, \alpha_3]$ and at $x = \alpha_2$ as a peak point and membership function is as the following:*

$$\delta(s;\alpha_1,\alpha_2,\alpha_3) = \begin{cases} 0, & x < \alpha_1; \\ \dfrac{s-\alpha_1}{\alpha_2-\alpha_1}, & \alpha_1 \le x \le \alpha_2; \\ \dfrac{\alpha_3 - s}{\alpha_3-\alpha_2}, & \alpha_2 \le x \le \alpha_3; \\ 0, & x > \alpha_3. \end{cases}$$

such that the ζ-level as follows:

$$[\delta(s)]_{\zeta} = [\alpha_1 + \zeta(\alpha_2 - \alpha_1), \alpha_3 - \zeta(\alpha_3 - \alpha_2)], \qquad \zeta \in [0, 1]$$

Definition 2.6. [33] *Let $\tilde{S}$ be the set of all normal upper semi-continuous convex fuzzy numbers with ζ-level bounded intervals that satisfy the following condition:*

$$[\delta(s)]_{\zeta} = \{s \in \mathbb{R} : \delta \ge \zeta\}, \qquad \zeta \in [0, 1]$$

An arbitrary fuzzy number is represented by an ordered pair of membership functions $[\delta(s)]_{\zeta} = [\underline{\delta}(x), \bar{\delta}(x)]_{\zeta}$ for all which is satisfying

(1) *$\delta(s)$ is normal: there exists $s_0 \in \mathbb{R}$ such that $\delta(s_0) = 1$.*
(2) *$\bar{\delta}(s)$ is convex: $\forall s, t \in \mathbb{R}$ and $\lambda \in [0, 1]$, it holds that:*

$$\delta(\lambda s + (1-\lambda)t) \ge \min\{(\delta(s), \delta(t)\}$$

(3) *δ is upper semi continues: for any $s_0 \in \mathbb{R}$, it satisfied that $\delta(s_0) \ge \lim_{s \to s_0^{\pm}} \delta(s)$.*
(4) *$\{s \in \mathbb{R} : \delta \ge \zeta\}$ is compact subset of $\mathbb{R}$.*

In the parametric form, which is represented by an ordered pair of functions $[\delta]\zeta = [\underline{\delta}(s), \bar{\delta}(s)]_{\zeta} = [\delta(s;\zeta), \delta(s;\zeta)]$, $\zeta \in [0, 1]$, that hold the below conditions:

(1) $\underline{\delta}(s;\zeta)$ is a bounded left continuous non-decreasing in [0, 1].
(2) $\bar{\delta}(s;\zeta)$ is a bounded left continuous non-increasing in [0, 1].
(3) $\underline{\delta}(s;\zeta) \le \bar{\delta}(s;\zeta)$.

Definition 2.7. [34] *Let $\tilde{h}$: $M \to \tilde{S}$ be a map, so, for interval $M \subseteq \tilde{S}$ denote a fuzzy function with crisp variable, and we define ζ-level set as*

$$[\tilde{h}(s)]_\zeta = [\underline{h}(s;\zeta), \bar{h}(s;\zeta)] \qquad s \in M, \zeta \in [0, 1].$$

where $\tilde{S}$ sets all upper semi-continuous normal convex fuzzy numbers. That is, the fuzzifying function is a mapping from a domain to a set of fuzzy ranges. In a mathematical sense, the fuzzifying function and the fuzzy relation coincide.

Definition 2.8. [35] *Given a function $g : S \to T$, where $S = S_1 \times S_2 \times \ldots \times S_n$ and let $\tilde{A} = \tilde{A}_1 \times \tilde{A}_2 \times \ldots \times$ An, where $\tilde{A}_i$, $i = 1, 2, \ldots, n$, be n-fuzzy subset in S and $t = g(s_1, s_2, \ldots, s_n)$ in T. Then, the extension principle allows defining a fuzzy subset $\tilde{B} = g(\tilde{A})$ in T by:*

$$\tilde{B} = \{(t, \delta_{\tilde{B}}(t)) : t = g(s_1, s_2, \ldots, s_n), s_1, s_2, \ldots, s_n \in S\}.$$

such that,

$$\delta_{\tilde{B}}(t) = \begin{cases} \sup\limits_{s_1, s_2, \ldots, s_n \in g^{-1}(t)} \min\{\delta_{\tilde{A}_1}(s_1), \delta_{\tilde{A}_2}(s_2), \ldots, \delta_{\tilde{A}_n}(s_n)\}, & if \quad g^{-1}(t) \neq \phi, \\ 0, & Otherwise. \end{cases}$$

the extension principle can be written for $n = 1$ as

$$\tilde{B} = \{(t, \delta_{\tilde{B}}(t)) : t = g(s), s \in S\}.$$

such that,

$$\delta_{\tilde{B}}(t) = \begin{cases} \sup\limits_{s \in g^{-1}(t)} \{\delta_{\tilde{A}}(s)\} & if, \quad g^{-1}(t) \neq \phi, \\ 0, & Otherwise. \end{cases}$$

For $s, t \in \tilde{S}$, and $\lambda \in$ R, the sum $s + t$ is $[s + t]_\zeta = [s]_\zeta + [t]_\zeta$ and the product $\lambda.s$ is $[\lambda.s]_\zeta = \lambda[s]_\zeta$, and the diameter of the ζ-level set of s as $diam[s]_\zeta = [\underline{s}(\zeta) - \bar{s}(\zeta)]$.

Definition 2.9. [36] *Let $s, t \in \tilde{S}$. If there is $r \in \tilde{S}$: $s = t + r$, then r is said to be the Hukuhara difference of s and t and it is denoted by $s \ominus t$.*

Definition 2.10. [36] *Let s, t be two fuzzy numbers then the distance $D[s, t]$ (Hausdorff distance) is defined as*

$$D[s,t] = \sup_{0 \le \zeta \le 1} \max\{|\underline{s}(\zeta) - \underline{t}(\zeta)|, |\bar{s}(\zeta) - \bar{t}(\zeta)|\}.$$

Definition 2.11. [37] *Let $g : I \to \tilde{E}$ and $s_0 \in I$. Then the fuzzy function g is said to be Hukuhara differentiable (H-differentiable) at s_0, if there is $g'(s_0) \in \tilde{E}$, and for $h > 0$, there are $g(s_0 + h) \ominus g(s_0)$ and $g(s_0) \ominus g(s_0 - h)$ such that*

$$g'(s_0) = lim_{h \to 0} \frac{g(s_0 + h) \ominus g(s_0)}{h} = lim_{h \to 0} \frac{g(s_0) \ominus g(s_0 - h)}{h}.$$

Definition 2.12. [38] *Let $g : I \to \tilde{E}$ for $s \in I \subseteq$ R. The nth order Hukuhara differentiable functions at t*

$$[\tilde{g}(t)]_\zeta = [\underline{g}(t;\zeta), \bar{g}(t;\zeta)], \forall \zeta \in [0, 1].$$

The functions $\underline{g}(t;\zeta)$, $\bar{g}(t;\zeta)$ are both n^{th} order Hukuhara differentiable functions and

$$[\tilde{g}^{(n)}]_\zeta = [\underline{g}^{(n)}(t;\zeta), \bar{g}^{(n)}(t;\zeta)], \forall \zeta \in [0, 1].$$

3. Methodology

3.1. Defuzzification of Fuzzy Fractional IVP with CK derivative

Consider the following general nonlinear FFIVP with the ordinary differential equation:

$$\begin{cases} D^{\alpha,\rho}\tilde{\upsilon}(s) = \tilde{A}_f\big(s,\tilde{\upsilon}(s)\big) + \tilde{w}(s), \quad s \in [s_0, S], \\ \tilde{\upsilon}(s_0) = \tilde{\upsilon}_0 \end{cases} \tag{1}$$

such that $\tilde{\upsilon}(s)$ is the fuzzy function of the crisp variable s, $D^{\alpha,\rho}\tilde{\upsilon}(s)$ fuzzy fractional H-derivative of order α and ρ, $\tilde{A}_f$ be as a fuzzy fractional function of fuzzy variable $w(s)$ is nonhomogeneous segment of FFDE in (1), $D^{\alpha,\rho}\tilde{\upsilon}(s)$ fuzzy fractional H-derivative of $\tilde{\upsilon}(s)$ and is an initial condition by the fuzzy form which constitutes a triangular fuzzy number[39].

Using the defuzzification properties and ζ-cut set of $\tilde{\upsilon}(s)$ to get:

$$\begin{cases} D^{\alpha,\rho}\tilde{\upsilon}(s;\zeta) = \tilde{A}_f\big(s,\tilde{\upsilon}(s;\zeta);\zeta\big) + \tilde{w}(s;\zeta), s \in [s_0, S], \\ \tilde{\upsilon}(s_0;\zeta) = \tilde{\upsilon}_0 \end{cases} \tag{2}$$

such that,

$$\begin{cases} \tilde{\upsilon}(s) = \tilde{\upsilon}(s;\varsigma) = [\underline{\upsilon}(s;\varsigma), \overline{\upsilon}(s;\varsigma)] \\ \tilde{A}_f(s,\tilde{\upsilon}(s)) = \tilde{A}_f(s,\tilde{\upsilon}(s;r);\varsigma) = [\underline{A}_f(s,\tilde{\upsilon}(s;\varsigma);\varsigma), \overline{A}_f(s,\tilde{\upsilon}(s;\varsigma);\varsigma)] \\ \tilde{\upsilon}(s_0) = \tilde{\upsilon}(s_0;\varsigma) = [\underline{\upsilon_0}, \overline{\upsilon_0}] \\ \tilde{w}(s) = \tilde{w}(s;\varsigma) = [\underline{w}(s;\varsigma), \overline{w}(s;\varsigma)] \end{cases} \tag{3}$$

the fuzzy fractional H-derivative $D^{\alpha,\rho}\tilde{\upsilon}(s;\zeta)$, $\forall\zeta \in [0, 1]$:

$$[D^{\alpha,\rho}\tilde{\upsilon}(s;\zeta)] = [D^{\alpha,\rho}\underline{\upsilon}(s;\zeta), D^{\alpha,\rho}\overline{\upsilon}(s;\zeta)], \tag{4}$$

and the fuzzy fractional function $\tilde{A}_f$ can be written as follows:

$$\begin{aligned} \underline{\mathcal{A}}_f\big(s,\tilde{\upsilon}(s;\zeta);\zeta\big) &= \mathcal{F}[s,\underline{\upsilon},\overline{\upsilon}]_\zeta, \\ \overline{\mathcal{A}}_f\big(s,\tilde{\upsilon}(s;\zeta);\zeta\big) &= \mathcal{G}[s,\underline{\upsilon},\overline{\upsilon}]_\zeta, \end{aligned} \tag{5}$$

By (1), $D^{\alpha,\rho}\tilde{\upsilon}(s;\zeta) = \tilde{A}_f((s,\tilde{\upsilon}(s;\zeta);\zeta))$, and by utilized the extension fuzzy principle in [40],

$$\begin{aligned} \mathcal{F}[s,\underline{\upsilon},\overline{\upsilon}]_\zeta &= \min\big\{\tilde{\mathcal{A}}_f\big(s,\tilde{\delta}(\zeta)\big) : \tilde{\delta}(\zeta) \in \tilde{\upsilon}(s;\zeta)\big\}, \\ \mathcal{G}[s,\underline{\upsilon},\overline{\upsilon}]_\zeta &= \max\big\{\tilde{\mathcal{A}}_f\big(s,\tilde{\delta}(\zeta)\big) : \tilde{\delta}(\zeta) \in \tilde{\upsilon}(s;\zeta)\big\} \end{aligned} \tag{6}$$

$$\begin{aligned} \underline{\mathcal{A}}_f\big(s,\tilde{\upsilon}(s;\zeta);\zeta\big) &= \mathcal{F}\big(s,\underline{\upsilon}(s;\zeta),\overline{\upsilon}(s;\zeta)\big) = \mathcal{F}\big(s,\tilde{\upsilon}(s;\zeta)\big), \\ \overline{\mathcal{A}}_f\big(s,\tilde{\upsilon}(s;\zeta);\zeta\big) &= \mathcal{G}\big(s,\underline{\upsilon}(s;\zeta),\overline{\upsilon}(s;\zeta)\big) = \mathcal{G}\big(s,\tilde{\upsilon}(s;\zeta)\big), \end{aligned} \tag{7}$$

The last procedure is the defuzzification technique to approximate the solution of FFDEs

3.2. General fuzzy fractional IVP and convergent analysis OHAM

OHAM is an extension of the HAM that is dependent on minimizing residual error. Consider the following FFIVP for the fuzzy fractional process of OHAM:

$$\begin{cases} D^{\alpha,\rho}\tilde{\upsilon}(s;\zeta) = \tilde{A}_f\big(s,\tilde{\upsilon}(s;\zeta);\zeta\big) + \tilde{w}(s;\zeta), \quad s \in [s_0, S], \\ \mathcal{B}\left(\tilde{\upsilon}(s;\zeta), \dfrac{\partial\tilde{\upsilon}}{\partial s}\right) = 0 \end{cases} \tag{8}$$

where, $D^{\alpha,\rho}\tilde{\upsilon}$ is the linear operator for Eq. (8), $\tilde{\upsilon}(s;\zeta)$ is the anonymous fuzzy function with crisp variable s, $\tilde{w}(s;\zeta)$ represent anonymous fuzzy function, $\mathcal{B}$ is the boundary condition, and $\tilde{A}_f$ is the fuzzy fractional function of crisp variable s and fuzzy variable $\tilde{\upsilon}$.

Refer to OHAM idea in [41] the formulation of fuzzy fractional OHAM $\phi(s;q;\zeta):[s0,S]\times[0,1]\to\mathbb{R}$ which satisfies the following:

$$\begin{cases}\mathcal{H}(s;q;\zeta)=(1-q)\left[\tilde{\mathcal{L}}\left(\tilde{\phi}(s;q;\zeta)\right)-\tilde{w}(s;\zeta)\right]-\tilde{\mathcal{H}}(q;\zeta)\\ \left[\ \tilde{\mathcal{L}}\left(\tilde{\phi}(s;q;\zeta)\right)-\tilde{\mathcal{A}}_f\left(s,\tilde{\upsilon}(s;\zeta)\right)-\tilde{w}(s;\zeta)\right],\\ \mathcal{B}\left(\tilde{\upsilon}(s;\zeta),\dfrac{\partial\tilde{\upsilon}}{\partial s}\right)=0,\end{cases}\tag{9}$$

where, $q\in[0,1]$ is an embedding parameter, $\tilde{\mathcal{H}}(q;\zeta)=[\underline{\mathcal{H}}(q;\zeta),\overline{\mathcal{H}}(q;\zeta)]$ is the auxiliary function (when $q\neq 0$) and $\tilde{\phi}(s;q;\zeta)$ is the unknown fuzzy function,

$$\tilde{\mathcal{L}}=[\underline{\mathcal{L}},\overline{\mathcal{L}}]=[D^{\alpha,\rho}(\underline{\phi}(s;q;\zeta),D^{\alpha,\rho}(\bar{\phi}(s;q;\zeta))],$$

for $q=0$ we have $\tilde{\phi}(s;0;\zeta)=\tilde{\upsilon}_0(s)$ which is the initial guess, and for $q=1$, we have $\tilde{\phi}(s;1;\zeta)=\tilde{\upsilon}(s)$ which is the exact solution.

The dynamic of OHAM for solving first-order FFDEs introduced by reformulation Eq. (8) and refer to Section (3.1) in the lower and upper bound respectively:

$$\begin{cases}\underline{\mathcal{L}}\left(\underline{\upsilon}(s;\zeta)\right)-\mathcal{F}\left(s,\tilde{\upsilon}(s;\zeta)\right)-\underline{w}(s;\zeta)=0,\\ \mathcal{B}\left(\underline{\upsilon}(s),\dfrac{\partial\underline{\upsilon}}{\partial s}\right)=0,\end{cases}\tag{10}$$

$$\begin{cases}\overline{\mathcal{L}}(\overline{\upsilon}(s;\zeta))-\mathcal{G}(s,\tilde{\upsilon}(s;\zeta))-\overline{w}(s;\zeta)=0,\\ \mathcal{B}\left(\overline{\upsilon}(s;\zeta),\dfrac{\partial\overline{\upsilon}}{\partial s}\right)=0.\end{cases}\tag{11}$$

According to the defuzzification of Eq. (1). The formulation for the lower and upper bound of Eq. (9) fuzzy fractional OHAM

$$\begin{cases}(1-q)\underline{\mathcal{L}}(\underline{\phi}(s;q;\zeta))-\underline{w}(s;\zeta)]=\underline{\mathcal{H}}(q;\zeta)[\underline{\mathcal{L}}(\underline{\phi}(s;q;\zeta))\\ -\mathcal{F}s,\tilde{\upsilon}(s;\zeta)-\underline{w}(s;\zeta)],\\ \mathcal{B}\left(\underline{\phi}(s;q;\zeta),\dfrac{\partial\underline{\phi}(s;q;\zeta)}{\partial s}\right)=0,\end{cases}\tag{12}$$

$$\begin{cases}(1-q)[\overline{\mathcal{L}}(\bar{\phi}(s;q;\zeta))-\overline{w}(s;\zeta)]=\overline{\mathcal{H}}(q;\zeta)[\overline{\mathcal{L}}(\bar{\phi}(s;q;\zeta))\\ -\mathcal{G}(s,\tilde{\upsilon}(s;\zeta))-\overline{w}(s;\zeta)],\\ \mathcal{B}\left(\bar{\phi}(s;q;\zeta),\dfrac{\partial\bar{\phi}(s;q;\zeta)}{\partial s}\right)=0.\end{cases}\tag{13}$$

where the lower and upper fuzzy fractional linear operators are the lower and upper auxiliary fuzzy function and $[\underline{\phi}(s;q;\zeta),\bar{\phi}(s;q;\zeta)]$ the lower and upper unknown fuzzy function respectively. Obviously, when $q=0$ and $q=1$ respectively we have:

$$\bar{\phi}(s;0;\zeta)=\bar{\upsilon}_0(s;\zeta),\qquad \underline{\phi}(s;1;\zeta)=\underline{\upsilon}(s;\zeta).\tag{14}$$

$$\bar{\phi}(s;0;\zeta)=\bar{\upsilon}_0(s;\zeta),\qquad \bar{\phi}(s;1;\zeta)=\bar{\upsilon}(s;\zeta).\tag{15}$$

Therefor, when q increase from 0 to 1, the solution $\tilde{\phi}(s;q;\zeta)$ vary from $\tilde{v}_0(s;\zeta)$ to the exact solution. Now when $q = 0$, the lower and upper bounds of zeroth-order:

$$\begin{cases} \underline{\mathcal{L}}\left(\underline{\phi}(s;0;\zeta)\right) = \underline{w}(s;\zeta), & \mathcal{B}\left(\underline{v}(s;\zeta), \dfrac{\partial \underline{v}(s;\zeta)}{\partial s}\right) = 0, \\ \overline{\mathcal{L}}\left(\overline{\phi}(s;0;\zeta)\right) = \overline{w}(s;\zeta), & \mathcal{B}\left(\overline{v}(s;\zeta), \dfrac{\partial \overline{v}(s;\zeta)}{\partial s}\right) = 0, \end{cases} \tag{16}$$

Now, the auxiliary function $\tilde{\mathcal{H}}(q;\zeta)$ for Eq. (12) and (13) as:

$$\begin{cases} \underline{\mathcal{H}}(q;\zeta) = \sum_{j=1}^{\infty} \underline{K}_j(\zeta) q^j = \underline{K}_1(\zeta) q^1 + \underline{K}_2(\zeta) q^2 + \dots \\ \overline{\mathcal{H}}(q;r) = \sum_{j=1}^{\infty} \overline{K}_j(\zeta) q^j = \overline{K}_1(\zeta) q^1 + \overline{K}_2(\zeta) q^2 + \dots \end{cases} \tag{17}$$

where, $\tilde{K}_1(\zeta) = [\underline{K}_1(\zeta), \overline{K}_1(\zeta)], \tilde{K}_2(\zeta) = [\underline{K}_2(\zeta), \overline{K}_2(\zeta)], \dots,$ is the auxiliary convergence constants. Expanding the solution $\tilde{\phi}(s;q;\zeta)$ about q by Taylor's series get the series approximate solution via fuzzy fractional OHAM

$$\begin{cases} \underline{\phi}\left(s;q;\underline{K}_j(\zeta);\zeta\right) = \underline{v}_0(s;\zeta) + \sum_{j=1}^{\infty} \underline{v}_j\left(s;\underline{K}_j(\zeta);\zeta\right) q^j. \\ \overline{\phi}\left(s;q;\overline{K}_j(\zeta);\zeta\right) = \overline{v}_0(s;\zeta) + \sum_{j=1}^{\infty} \overline{v}_j\left(s;\overline{K}_j(\zeta);\zeta\right) q^j. \end{cases} \tag{18}$$

Substitute Eq. (17) and (18) in Eq. (12) and (13) and then collect the coefficient of like powers of q to find the lower and upper bound. This procedure gives us a system of linear equations, for the zeroth-order given in (16) and for the first order

$$\begin{cases} \underline{\mathcal{L}}(\underline{v}_1(s;\zeta)) - \underline{\mathcal{L}}(\underline{v}_0(s;\zeta)) + \underline{w}(s) = \underline{K}_1(\zeta)(\underline{L}\,(\underline{v}_0(s;\zeta) - \mathcal{F}_0(\underline{v}_0(s;\zeta)) - \underline{w}(s;\zeta)), \\ \overline{\mathcal{L}}(\underline{v}_1(s;\zeta)) - \overline{\mathcal{L}}(\underline{v}_0(s;\zeta)) + \overline{w}(s) = \overline{K}_1(\zeta)(\overline{L}\,(\overline{v}_0(s;\zeta) - \mathcal{G}_0(\overline{v}_0(s;\zeta)) - \overline{w}(s;\zeta)), \\ \mathcal{B}\left(\tilde{v}_1(s;\zeta), \dfrac{\vartheta v_1(s;\zeta)}{\vartheta s}\right) = 0 \end{cases} \tag{19}$$

The problem of second-order

$$\begin{cases} \underline{\mathcal{L}}(\underline{v}_2(s;\zeta)) - \underline{\mathcal{L}}(\underline{v}_1(s;\zeta)) = \underline{C}_2(\zeta)\mathcal{F}_0(\underline{v}_0(s;\zeta)) + \underline{K}_1(r)[\underline{\mathcal{L}}(\underline{v}_1(s;\zeta)) - \mathcal{F}_1(\underline{v}_0(s;\zeta), \underline{v}_1(s;\zeta)) - \underline{w}(s;\zeta)], \\ \overline{\mathcal{L}}(\overline{v}_2(s;\zeta)) - \overline{\mathcal{L}}(\overline{v}_1(s;\zeta)) = \overline{C}_2(\zeta)\mathcal{F}_0(\overline{v}_0(s;\zeta)) + \overline{K}_1(\zeta)[\overline{\mathcal{L}}(\overline{v}_1(s;\zeta)) - \mathcal{G}_1(\overline{v}_0(s;\zeta), \underline{v}_1(s;\zeta)) - \overline{w}(s;\zeta)], \\ \mathcal{B}\left(\tilde{v}_2(s;\zeta), \dfrac{\vartheta \tilde{v}_2(s;\zeta)}{\vartheta s}\right) \end{cases} \tag{20}$$

the general form of the governing problem via OHAM of k^{th} order

$$\begin{cases} \underline{\mathcal{L}}\left(\underline{v}_k(s;\zeta)\right) - \underline{\mathcal{L}}\left(\underline{v}_{k-1}(s;\zeta)\right) = \underline{K}_k(\zeta)\mathcal{F}_0\left(\underline{v}_0(s;\zeta)\right) - \sum_{i=1}^{k-1} \underline{K}_i(\zeta)\Big[\underline{\mathcal{L}}\left(\underline{v}_{k-i}(s;\zeta)\right) \\ -\mathcal{F}_{k-i}\left(\underline{v}_0(s;\zeta), \underline{v}_1(s;\zeta), \dots, \underline{v}_i(s;\zeta)\right) - \underline{w}(s;\zeta)\Big], \\ \overline{\mathcal{L}}\left(\overline{v}_k(s;\zeta)\right) - \overline{\mathcal{L}}\left(\overline{v}_{k-1}(s;\zeta)\right) = \overline{K}_k(\zeta)\mathcal{G}_0\left(\overline{v}_0(s;\zeta)\right) - \sum_{i=1}^{k-1} \overline{K}_i(\zeta)\Big[\overline{L}\left(\underline{v}_{k-i}(s;r)\right) \\ -\mathcal{G}_{k-i}\left(\overline{v}_0(s;\zeta), \overline{v}_1(s;\zeta), \dots, \overline{v}_i(s;\zeta)\right) - \overline{w}(s;\zeta)\Big], \\ \mathcal{B}\left(\tilde{v}_k(s;\zeta), \dfrac{\partial \tilde{v}_k(s;\zeta)}{\partial s}\right) = 0, \end{cases} \tag{21}$$

where $\mathcal{F}_{k-i}(\tilde{v}_0(s;\zeta), \tilde{v}_1(s;\zeta), \ldots, \tilde{v}_i(s;\zeta))$ are the lower bound coefficient q^k and $\mathcal{G}_{k-i}(\tilde{v}_0(s;\zeta), \tilde{v}_1(s;\zeta), \ldots, \tilde{v}_i(s;\zeta))$ are the upper bound coefficient q^k. Dependent on parameter $K_1(\zeta), K_2(\zeta), \ldots, K_k(\zeta)$, and at $q = 1$ we have:

$$\begin{cases} \underline{v}\left(s,\underline{K}_1(\zeta)\underline{K}_2(\zeta),\ldots;\zeta\right) = \underline{v}_0(s;\zeta) + \sum_{i=1}^{\infty} \underline{v}_i\left(s,\underline{K}_1(\zeta),\underline{K}_2(\zeta),\ldots;\zeta\right) \\ \bar{v}\left(s,\bar{K}_1(\zeta),\bar{K}_2(\zeta),\ldots;\zeta\right) = \bar{v}_0(s;\zeta) + \sum_{i=1}^{\infty} \bar{v}_i\left(s,\bar{K}_1(\zeta),\bar{K}_2(\zeta),\ldots;\zeta\right) \end{cases} \tag{22}$$

Approximate the series solution for k term as:

$$\begin{cases} \underline{v}_*\left(s,\underline{K}_1(\zeta)\underline{K}_2(\zeta),\ldots;\underline{K}_k(\zeta);\zeta\right) = \underline{v}_0(s;\zeta) + \sum_{i=1}^{k} \underline{v}_i\left(s,\underline{K}_1(\zeta),\underline{K}_2(\zeta),\ldots;\underline{K}_i(\zeta);\zeta\right) \\ \bar{v}_*\left(s,\bar{K}_1(\zeta),\bar{K}_2(\zeta),\ldots\bar{\zeta}_k(\zeta);\zeta\right) = \bar{v}_0(s;\zeta) + \sum_{i=1}^{k} \bar{v}_i\left(s,\bar{K}_1(\zeta),\bar{K}_2(\zeta),\ldots;\bar{K}_i(\zeta);\zeta\right) \end{cases} \tag{23}$$

We can obtain of the residual error $\tilde{R} = [\underline{R}, \bar{R}]$ by substituting Eq. (23) in Eq. (10) and (11)

$$\begin{aligned} \underline{R}\left(s,\underline{K}_1(\zeta),\ldots,\underline{K}_i(\zeta);\zeta\right) = {} & \underline{\mathcal{L}}\left(\underline{v}_*\left(s,\underline{K}_1(\zeta),\ldots,\underline{K}_k(\zeta);\zeta\right)\right) \\ & -\mathcal{F}\left(\underline{v}_*\left(s,\underline{K}_1(\zeta),\ldots,\underline{K}_k(\zeta);\zeta\right)\right) \\ & -\underline{w}(s;\zeta) \end{aligned} \tag{24}$$

$$\begin{aligned} \bar{R}\left(s,\bar{K}_1(\zeta),\ldots,\bar{K}_i(\zeta);\zeta\right) = {} & \bar{\mathcal{L}}\left(\bar{v}_*\left(s,\bar{K}_1(\zeta),\ldots,\bar{K}_k(\zeta);\zeta\right)\right) \\ & -\mathcal{G}\left(\bar{v}_*\left(s,\bar{K}_1(\zeta),\ldots,\bar{K}_k(\zeta);\zeta\right)\right) \\ & -\bar{w}(s;\zeta) \end{aligned} \tag{25}$$

In case $\tilde{R} = 0$ where $\tilde{R} = [\underline{R}, \bar{R}]$, then $\tilde{v}_*$ yields the exact solution. To determinations auxiliary constants $\tilde{K}_1(\zeta), \tilde{K}_2(\zeta), \ldots, \tilde{K}_k(\zeta)$, we apply the least squares method on the interval $s \in [s_0, S]$:

$$\begin{cases} \underline{M}\left(s,\underline{K}_1(\zeta),\ldots,\underline{K}_k(\zeta);\zeta\right) = \int_{s_0}^{s} \underline{R}^2\left(s,\underline{K}_1(\zeta),\ldots,\underline{K}_k(\zeta);\zeta\right)ds \\ \bar{M}\left(s,\bar{K}_1(\zeta),\ldots,\bar{K}_k(\zeta);\zeta\right) = \int_{s_0}^{s} \bar{R}^2\left(s,\bar{K}_1(\zeta),\ldots,\bar{K}_k(\zeta);\zeta\right)ds \end{cases} \tag{26}$$

where, a and b are set based on the given problem, $\tilde{M} = [\underline{M}, \bar{M}]$ and the optimal values for $\tilde{K}_1(\zeta)$, $\tilde{K}_2(\zeta), \ldots, \tilde{K}_K(\zeta)$, can be determent by following:

$$\frac{\partial \tilde{M}}{\partial \tilde{K}_1} = \frac{\partial \tilde{M}}{\partial \tilde{K}_2} = \ldots = \frac{\partial \tilde{M}}{\partial \tilde{K}_k} = 0. \tag{27}$$

4. Applications

In this part, we provide three examples to illustrate the effectiveness of the OHAM for solving linear and nonlinear FFIVPs using generalized Caputo-type fractional sense (CK) derivative.

Example 4.1. *Consider the following FFIVP:*

$$\begin{cases} D^{\alpha,\rho}\tilde{v}(s) = s^2, \quad 0 \le s \le 1, \\ \tilde{v}(0) = \tilde{v}_0(0,\zeta) = [\zeta, 2-\zeta], \quad \zeta \in [0,1]. \end{cases} \tag{28}$$

The general exact solution for Eq. (28) by refer to section (2.1), for arbitrary α, $\rho > 0$ and $\zeta \in [0, 1]$:

$$\tilde{v}(s;\zeta) = \tilde{v}_0(0,\zeta) + \frac{\rho^{-\alpha}\Gamma(\frac{2}{\rho}+1)}{\Gamma(\alpha+\frac{2}{\rho}+1)} s^{2+\alpha\rho}.$$

We can construct the following homotopy series by referring to section (3.2) as follows:

$$(1-q)\left[D^{\alpha,\rho}\tilde{v}(s;q;\zeta) - s^2\right] = \tilde{\mathcal{H}}(q;\zeta)\left[D^{\alpha,\rho}\tilde{v}(s;q;\zeta) - s^2\right]. \tag{29}$$

Now, employing (17) and (18) into (29), and equating the coefficient of the same powers of q, we have the following governing equations for example (4.1):

$$q^0 : D^{\alpha,\rho}\tilde{v}_0(s,\zeta) - s^2 = 0, \quad \tilde{v}_0(0,\zeta). \tag{30}$$

$$\begin{aligned} q^1 : D^{\alpha,\rho}\tilde{v}_1(s,\tilde{K}_1(\zeta);\zeta) &= (1+\tilde{K}_1(\zeta))D^{\alpha,\rho}\tilde{v}_0(s,\zeta) \\ &-(1+\tilde{K}_1(\zeta))s^2, \\ &\tilde{v}_1(0,\zeta) = 0. \end{aligned} \tag{31}$$

$$\begin{aligned} q^2 : D^{\alpha,\rho}\tilde{v}_2(s,\tilde{K}_1(\zeta),\tilde{K}_2(\zeta);\zeta) &= (1+\tilde{K}_1(\zeta))D^{\alpha,\rho}\tilde{v}_1(s,\zeta) \\ &+\tilde{K}_2(\zeta)D^{\alpha,\rho}\tilde{v}_0(s;\zeta) \\ &\tilde{v}_2(0,\zeta) = 0. \\ &\vdots \end{aligned} \tag{32}$$

$$\begin{aligned} q^k : D^{\alpha,\rho}\tilde{v}_k(s,\tilde{K}_1,\ldots,\tilde{K}_k(\zeta);\zeta) &= (1+\tilde{K}_1(\zeta))D^{\alpha,\rho}\tilde{v}_{k-1}(s,\zeta) \\ &+\sum_{i=1}^{k}\tilde{K}_i(\zeta)D^{\alpha,\rho}\tilde{v}_{k-i}(s,\zeta) \\ &-\tilde{K}_k(\zeta)s^2, \quad \tilde{v}_k(0,\zeta) = 0. \\ &\vdots \end{aligned} \tag{33}$$

Now by applying the operator $I^{\alpha,\rho}$ on (30)–(34), using initial condition given in (28) we conclude that:

$$\tilde{v}_0(s,\zeta) = \tilde{v}_0(0,\zeta) + I^{\alpha,\rho}(s^2). \tag{34}$$

$$\begin{aligned} &\tilde{v}_1(s,\tilde{K}_1(\zeta);\zeta) = 0, \\ &\vdots \end{aligned} \tag{35}$$

$$\begin{aligned} &\tilde{v}_k(s,\tilde{K}_1(\zeta),\ldots,\tilde{K}_k(\zeta);\zeta) = 0, \\ &\vdots \end{aligned} \tag{36}$$

In our example, we can see that the zeroth order problem of the homotopy series gives the exact solution for Eq. (28), which is an indication that the OHAM method is effective, accurate and superior under (CK) sense.

Example 4.2. *Consider the following linear FFIVP:*

$$\begin{cases} D^{\alpha,\rho}\tilde{v}(s,\zeta) = \tilde{v}(s,\zeta), \quad 0 \le s \le 1, \\ \tilde{v}(0) = \tilde{v}_0(s,\zeta) = [0.2\zeta + 0.8, 1.2 - 0.2\zeta], \quad \zeta \in [0,1]. \end{cases} \tag{37}$$

By referring to section (3.2) we can get of series homotopy of eighth order by

$$(1-q)D^{\alpha,\rho}\tilde{v}(s;q;\zeta) = \tilde{\mathcal{H}}(q;\zeta)[D^{\alpha,\rho}\tilde{v}(s;q;\zeta) - \tilde{v}(s;q;\zeta)], \tag{38}$$

where

$$\tilde{v}(s;q;\zeta) = \tilde{v}_0(s;\zeta) + \sum_{i=1}^{8} \tilde{v}_i(s;\tilde{K}_i(\zeta);\zeta)q^i, \tag{39}$$

$$\tilde{\mathcal{H}}(q;\zeta) = \sum_{i=1}^{8} \tilde{K}_i(\zeta)q^i. \tag{40}$$

Now, substituting (39) and (40) into (38), and equating the coefficient of the same powers of q, we have the following:

$$q^0 : D^{\alpha,\rho}\tilde{v}_0(s,\zeta) = 0, \quad \tilde{v}_0(0,\zeta). \tag{41}$$

$$\begin{aligned} q^1 : D^{\alpha,\rho}\tilde{v}_1(s,\tilde{K}_1(\zeta);\zeta) &= (1+\tilde{K}_1(\zeta))D^{\alpha,\rho}\tilde{v}_0(s,\zeta) \\ &\quad -\tilde{K}_1(\zeta)\tilde{v}_0(s;\zeta), \tilde{v}_1(0,\zeta) = 0. \end{aligned} \tag{42}$$

$$\begin{aligned} q^2 : D^{\alpha,\rho}\tilde{v}_2(s,\tilde{K}_1(\zeta),\tilde{K}_2(\zeta);\zeta) &= (1+\tilde{K}_1(\zeta))D^{\alpha,\rho}\tilde{v}_1(s,\zeta) + \tilde{K}_2(\zeta)D^{\alpha,\rho}\tilde{v}_0(s;\zeta) \\ &\quad -\tilde{K}_1(\zeta)\tilde{v}_1(s;\zeta) + \tilde{K}_2(\zeta)\tilde{v}_0(s;\zeta), \tilde{v}_2(0,\zeta) = 0. \\ &\;\;\vdots \end{aligned} \tag{43}$$

$$\begin{aligned} q^k : D^{\alpha,\rho}\tilde{v}_k(s,\tilde{K}_1(\zeta),\ldots,\tilde{K}_k(\zeta);\zeta) &= D^{\alpha,\rho}\tilde{v}_{k-1}(s,\zeta) + \sum_{i=1}^{k} \tilde{K}_i(\zeta)D^{\alpha,\rho}\tilde{v}_{k-i}(s;\zeta) \\ &\quad -\sum_{i=1}^{k} \tilde{K}_i(\zeta)\tilde{v}_{k-i}(s;\zeta), \ \tilde{v}_k(0,\zeta) = 0. \end{aligned} \tag{44}$$

Now by applying the operator $I^{\alpha,\rho}$ on equations in (41)–(44), and using initial condition given in (37), we have the fuzzy linear equations as follows:

$$\tilde{v}_0(s;\zeta) = \tilde{v}_0(0;\zeta). \tag{45}$$

$$\tilde{v}_1(s,\tilde{K}_1(\zeta),\zeta) = (1+\tilde{K}_1(\zeta))\tilde{v}_0(s;\zeta) - I^{\alpha,\rho}[\tilde{K}_1(\zeta)\tilde{v}_0(s;\zeta)]. \tag{46}$$

$$\begin{aligned} \tilde{v}_2(s,\tilde{K}_1(\zeta),\tilde{K}_2(\zeta),\zeta) &= (1+\tilde{K}_1(\zeta))\tilde{v}_1(s;\zeta) + \tilde{K}_2(\zeta)\tilde{v}_0(s;\zeta) \\ &\quad -I^{\alpha,\rho}\left[\tilde{K}_2(\zeta)\tilde{v}_0(s;\zeta) + \tilde{K}_1(\zeta)\tilde{v}_1(s;\zeta)\right]. \\ &\;\;\vdots \end{aligned} \tag{47}$$

$$\begin{aligned} \tilde{v}_k\left(s,\tilde{K}_1(\zeta),\ldots,\tilde{K}_k(\zeta);\zeta\right) &= \tilde{v}_{k-1} + \sum_{i=1}^{k} \tilde{K}_i(\zeta)\tilde{v}_{k-i}(s;\zeta) \\ &\quad -I^{\alpha,\rho}\left[\sum_{i=1}^{k} \tilde{K}_i(\zeta)\tilde{v}_{k-i}(s;\zeta)\right]. \end{aligned} \tag{48}$$

We note that formula (48) satisfying for $k \ge 2$.

The approximate solution of Eq. (37) by 8 order fuzzy fractional OHAM as follows:

$$\tilde{y}(s;\zeta) = \tilde{v}_0(s;\zeta) + \sum_{i=1}^{8} \tilde{v}_i(s, \tilde{K}_i(\zeta);\zeta). \tag{49}$$

In this problem, we solved example (4.2) for $\alpha = 0.8$, $\rho = 0.95$ and $s = 0.2$ for different values of $\zeta \in [0, 1]$, using Mathematica software. Here we use the 8-order OHAM series solution to obtain fuzzy solutions and accuracy of the lower and upper bounds of (45)–(48) as shown in Table (2), and choosing the optimal convergence parameters $\tilde{K}_i$ at $\zeta = 1$ given below in Table (1). Figure (1) explain the three dimensional for the lower and upper fuzzy solution for Eq. (37). Figures (2) and (3) shows the residual error for lower and upper fuzzy fractional OHAM respectively in three-dimensional form. We can obtain the residual error by:

$$\tilde{E}(s;\, \zeta) = D^{\alpha,\rho} v_*(s;\, \zeta) - v_*(s;\, \zeta). \tag{50}$$

Example 4.3. *Fuzzy Riccati equation* [42]:

$$\begin{cases} D^{\alpha,\rho}\tilde{v}(s) = \overline{v}^2(s) + s^2, \quad s \geq 0, \\ \tilde{v}(0) = \tilde{v}_0(s,\zeta) = [0.1\zeta - 0.1, 0.1 - 0.1\zeta], \quad \zeta \in [0,1]. \end{cases} \tag{51}$$

solution of Eq. (37), at $\alpha = 0.8$, $\rho = 0.95$, $s \in [0, 0.2]$ and $\zeta \in [0, 1]$

Table 1. OHAM 8-order optimal values of $\tilde{K}_i(\zeta)$ for $\zeta = 1$ and $s = 0.2$ of Eq. (37).

i	1	2	3	4	5	6	7	8
$\underline{\zeta_i}(\zeta)$	–0.991	–0.00186	0.00063	–0.000066	3.5×10^{-6}	9.5×10^{-7}	7.8×10^{-8}	8.8×10^{-9}
$\bar{K}_i(\zeta)$	–0.954	–0.00382	0.00016	–0.000086	0.000018	-2.9×10^{-6}	6.4×10^{-7}	8.8×10^{-9}

Table 2. OHAM lower and upper solution and residual error for equation (37), at $\alpha = 0.8$, $\rho = 0.95$ and $s = 0.2$.

ζ	$\underline{v}(s;\zeta)$	$\underline{E}(s;\zeta)$	$\bar{v}(s;\zeta)$	$\bar{E}(s;\zeta)$
0	1.12473	3.25723×10^{-9}	1.68709	1.09421×10^{-7}
0.2	1.18096	3.42009×10^{-9}	1.63085	1.05774×10^{-7}
0.4	1.23720	3.58295×10^{-9}	1.57462	1.02127×10^{-7}
0.6	1.29344	3.74582×10^{-9}	1.51838	9.84796×10^{-8}
0.8	1.34967	3.90868×10^{-9}	1.46214	9.48321×10^{-8}
1	1.40591	4.07154×10^{-9}	1.40591	9.11847×10^{-8}

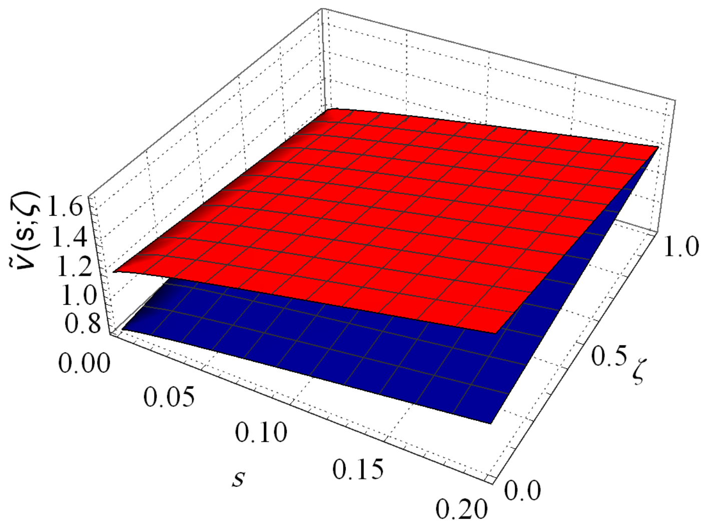

Figure 1. Upper and lower fuzzy fractional OHAM.

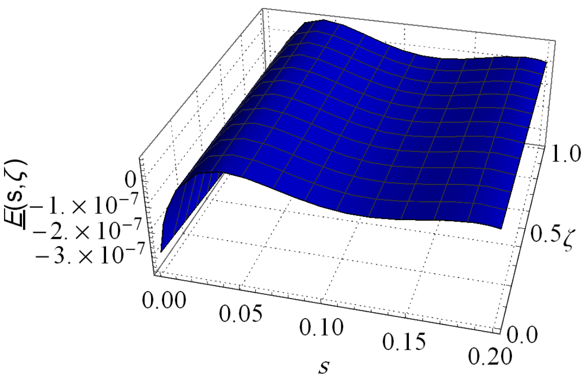

Figure 2. Residual error for lower fuzzy fractional OHAM of Eq. (37) for $\alpha = 0.8$, $\rho = 0.95$ and $\zeta \in [0, 1]$, $s \in [0, 0.2]$.

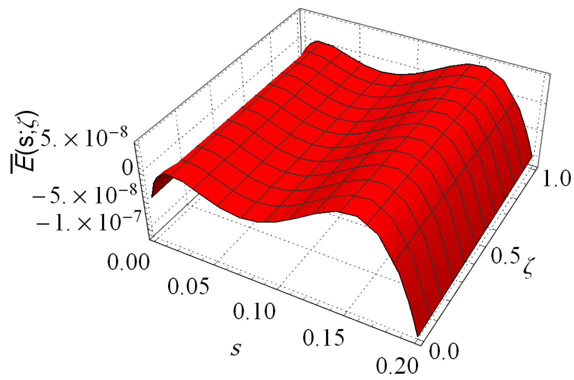

Figure 3. Residual error for upper fuzzy fractional OHAM of Eq. (37) for $\alpha = 0.8$, $\rho = 0.95$ and $\zeta \in [0, 1]$, $s \in [0, 0.2]$.

By refer to section (3.2) we can get of series homotopy

$$(1-q)\left[D^{\alpha,\rho}\tilde{v}(s;q;\zeta)-s^2\right]=\tilde{\mathcal{H}}(q;\zeta)\left[D^{\alpha,\rho}\tilde{v}(s;q;\zeta)-\tilde{v}^2(s;q;\zeta)-s^2\right], \tag{52}$$

where

$$\tilde{v}(s;q;\zeta)=\tilde{v}_0(s;\zeta)+\sum_{i=1}^{4}v_i(s;\tilde{K}_i(\zeta);\zeta)q^i, \tag{53}$$

$$\tilde{\mathcal{H}}(q;\zeta)=\sum_{i=1}^{4}\tilde{K}_i(\zeta)q^i. \tag{54}$$

By, substituting (53) and (54) into (52), and equating the coefficient of the same powers of q, we have the following equations:

Zeroth order problem

$$D^{\alpha,\rho}\tilde{v}_0(s,\zeta)=s^2, \qquad \tilde{v}_0(0,\zeta), \tag{55}$$

by applying the $I^{\alpha,\rho}$ on the equation (51), we have

$$\tilde{v}_0(s;\zeta) = I^{\alpha,\rho}(s_2). \tag{56}$$

First order problem

$$\begin{cases}\tilde{v}_1(s;\zeta)=\left(1+K_1(\zeta)\right)\tilde{v}_0(s;\zeta)-I^{\alpha,\rho}\left[K_1(\zeta)\left(\tilde{v}_0^2(s;\zeta)+s^2\right)+s^2\right],\\ \tilde{v}_1(0;\zeta)=0.\end{cases} \tag{57}$$

Second order problem

$$\begin{cases}\tilde{v}_2(s;\zeta)=\tilde{v}_1(s;\zeta)+\sum_{i=1}^{2}K_i(\zeta)\tilde{v}_{2-i}(s;\zeta)-I^{\alpha,\rho}[2\tilde{K}_1(\zeta)\\ (\tilde{v}_0(s;\zeta)\tilde{v}_1(s;\zeta))]-I^{\alpha,\rho}\left[\tilde{K}_2(\zeta)\left(\tilde{v}_0^2(s;\zeta)+s^2\right)\right]\ \tilde{v}_2(0;\zeta)=0,\end{cases} \tag{58}$$

Third order problem

$$\begin{cases} \tilde{v}_3(s;\zeta) = \tilde{v}_2(s;\zeta) + \sum_{i=1}^{3} K_i(\zeta)\tilde{v}_{3-i}(s;\zeta) - I^{\alpha,\rho}\Big[2\tilde{K}_1(\zeta) \\ \left(\tilde{v}_0(s;\zeta)\tilde{v}_2(s;\zeta) + \tilde{v}_1^2(s;\zeta)\right)\Big] - I^{\alpha,\rho}\Big[2\tilde{K}_2(\zeta) \\ \left(\tilde{v}_0(s;\zeta)\tilde{v}_1(s;\zeta)\right) + \tilde{K}_3(\zeta)\left(\tilde{v}_0^2(s;\zeta) + s^2\right)\Big]\tilde{v}_3(0;\zeta) = 0. \end{cases} \tag{59}$$

Fourth order problem

$$\begin{cases} \tilde{v}_4(s;\zeta) = \tilde{v}_3(s;\zeta) + \sum_{i=1}^{4}\left(K_i(\zeta)\tilde{v}_{4-i}(s;\zeta)\right) - I^{\alpha,\rho}\Big[2\tilde{K}_1(\zeta) \\ \left(\tilde{v}_0(s;\zeta)\tilde{v}_3(s;\zeta) + \tilde{v}_1(s;\zeta)\tilde{v}_2(s;\zeta)\right) + \tilde{K}_2(\zeta)\left(2\tilde{v}_0(s;\zeta)\tilde{v}_2(s;\zeta) + \tilde{v}_1^2(s;\zeta)\right) \\ +2\tilde{K}_3(\zeta)\left(\tilde{v}_0(s;\zeta)\tilde{v}_1(s;\zeta)\right) + \tilde{K}_4(\zeta)\left(\tilde{v}_0^2(s;\zeta) + s^2\right)\Big]\tilde{v}_4(0;\zeta) = 0, \end{cases} \tag{60}$$

The approximate solution of equation (51) using Fourth order become:

$$\tilde{v}_*\left(s, \tilde{K}_1(\zeta), \ldots, \tilde{K}_4(\zeta); \zeta\right) = \tilde{v}_0(s;\zeta) + \sum_{i=1}^{4}\tilde{v}_i\left(s, \tilde{K}_1(\zeta), \ldots, \tilde{K}_i(\zeta); \zeta\right). \tag{61}$$

we can obtain the residual error by:

$$\tilde{E}(s;\zeta) = D^{\alpha,\rho}v_*(s;\zeta) - y_*^2(s;\zeta) - s^2. \tag{62}$$

For example (51), we construct a fourth-order of homotopy series and use Mathematica software. The fuzzy fractional OHAM for lower and upper solution and accuracy given in Table (3) at $\alpha = 0.8$, $\rho = 0.95$ and $s = 0.3$ for $\zeta \in [0, 1]$, the lower and upper auxiliary convergent parameters shown in Tables (4) and (5) respectively. Figure (4) gives us the lower and upper solution at $\alpha = 0.8$, $\rho = 0.95$, $s \in [0, 0.3]$ and $\zeta \in [0, 1]$, the residual error for upper and lower bounds explain in figures (5) and (6)

Table 3. OHAM lower and upper solution and residual error for example (51), at $\alpha = 0.8$, $\rho = 0.95$ and $s = 0.3$.

ζ	$\underline{v}(s;\zeta)$	$\underline{E}(s;\zeta)$	$\bar{v}(s;\zeta)$	$\bar{E}(s;\zeta)$
0	–0.0812111	-2.89021×10^{-9}	0.120079	-5.52908×10^{-9}
0.2	–0.0626852	6.92484×10^{-10}	0.0982036	1.45209×10^{-9}
0.4	–0.0438324	-2.60771×10^{-10}	0.076751	-7.67054×10^{-10}
0.6	–0.0246426	-8.13365×10^{-11}	0.0557067	-2.27483×10^{-10}
0.8	–0.00510512	5.73839×10^{-12}	0.0350577	1.5787×10^{-10}
1	0.0147911	-1.81204×10^{-10}	0.0147911	-1.24098×10^{-10}

Table 4. Lower auxiliary convergence parameters $\tilde{K}_i$ of Eq. (51) for forth order OHAM at $\alpha = 0.85$, $\rho = 0.9$, for $s \in [0, 0.3]$.

ζ	$\underline{K}_1$	$\underline{K}_2$	$\underline{K}_3$	$\underline{K}_4$
0	–0.9849	–0.00185	–0.000089	-4.63×10^{-6}
0.2	–0.9620	–0.00025	-9.49×10^{-7}	1.84×10^{-9}
0.4	–0.9633	–0.00023	-8.54×10^{-7}	1.64×10^{-9}
0.6	–0.9646	–0.0002	-7.62×10^{-7}	1.48×10^{-9}
0.8	–0.9659	–0.0002	-6.8×10^{-7}	1.297×10^{-9}
1	–0.9672	–0.00019	-5.99×10^{-7}	1.167×10^{-9}

Table 5. Upper auxiliary convergence parameters $\tilde{K}_i$ of Eq. (51) for forth order OHAM at $\alpha = 0.85$, $\rho = 0.9$, for $s \in [0, 0.3]$.

ζ	$\bar{K}_1$	$\bar{K}_3$	$\bar{K}_3$	$\bar{K}_4$
0	–1.0373	–0.00025	8.71×10^{-7}	2.80×10^{-9}
0.2	–1.0296	–0.00016	4.30×10^{-7}	1.23×10^{-9}
0.4	–1.0234	0.001240	–0.000019	-1.54×10^{-6}
0.6	–1.0183	9.2×10^{-6}	2.06×10^{-7}	-4.75×10^{-9}
0.8	–1.0041	0.00015	1.63×10^{-6}	-8.05×10^{-8}
1	–0.9998	–0.00014	1.11×10^{-6}	-2.12×10^{-8}

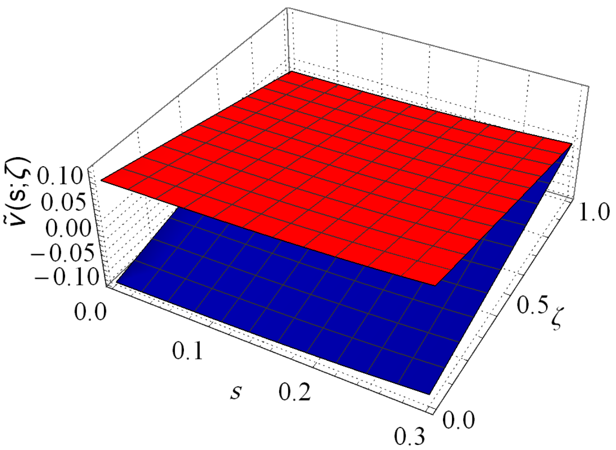

Figure 4. Upper and lower fuzzy fractional OHAM solution of Eq. (51), at $\alpha = 0.85$, $\rho = 0.9$, $s \in [0, 0.3]$ and $\zeta \in [0, 1]$.

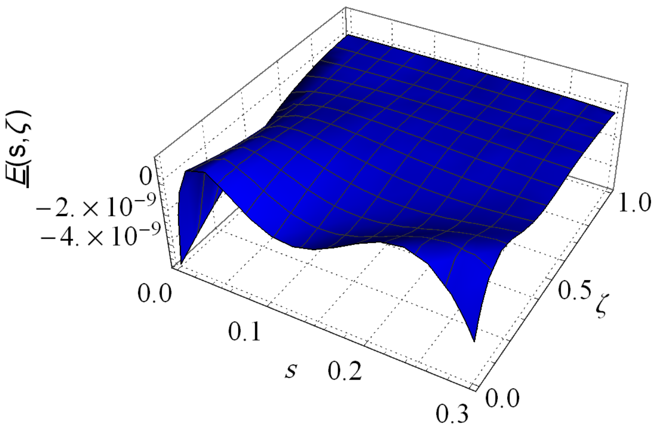

Figure 5. Residual error for lower fuzzy fractional OHAM of Eq. (51) for $\alpha = 0.85$, $\rho = 0.9$, and $\zeta \in [0, 1]$, $s \in [0, 0.3]$.

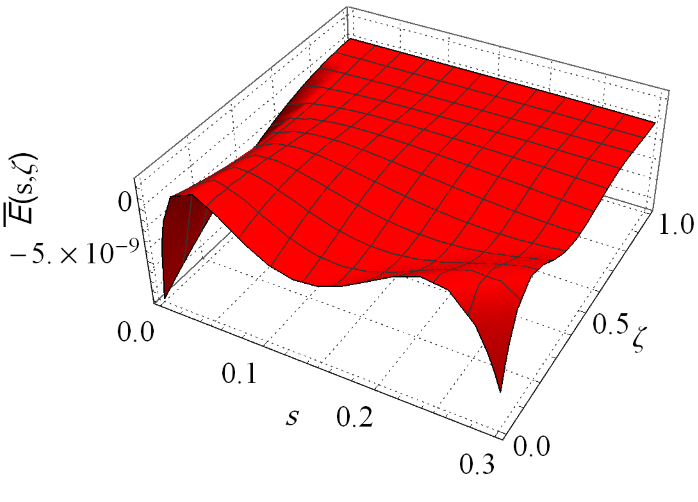

Figure 6. Residual error for upper fuzzy fractional OHAM of Eq. (51) for $\alpha = 0.85$, $\rho = 0.9$, and $\zeta \in [0, 1]$, $s \in [0, 0.3]$.

respectively. One can see that from Tables (6) and (7), third-order OHAM lower and upper solution and accuracy with various values for parameters α and ρ give us more flexibility and many tools to get fit data as is evident the lower solution in Figures 7,8 and the upper solution in Figures 9,10.

Conclusion

The OHAM successfully created a novel approach for solving fractional order fuzzy differential equations using the CK fractional derivative. The effect of the two parameters of the fractional derivatives

Table 6. Three order OHAM lower and upper solution and accuracy of Eq. (51), at $\zeta = 1$, $\rho = 1$ and $s = 0.5$.

α	$\underline{v}(s;\zeta)$	$\underline{E}(s;\zeta)$	$\bar{v}(s;\zeta)$	$\bar{E}(s;\zeta)$
0.50	0.109895	-2.15803×10^{-5}	0.1099004	-2.54972×10^{-6}
0.75	0.0678964	-1.06149×10^{-6}	0.0678966	8.59075×10^{-8}
1.00	0.0417911	-6.22067×10^{-8}	0.0417911	2.59737×10^{-8}

Table 7. Three order OHAM lower and upper solution and accuracy of Eq. (51), at $\zeta = 1$, $\alpha = 1$ and $s = 0.5$.

ρ	$\underline{v}(s;\zeta)$	$\underline{E}(s;\zeta)$	$\bar{v}(s;\zeta)$	$\bar{E}(s;\zeta)$
0.50	0.0713605	-6.02267×10^{-6}	0.0713622	6.86773×10^{-6}
0.75	0.0543347	-5.23302×10^{-7}	0.0543347	3.06924×10^{-7}
1.00	0.0417911	-6.22067×10^{-8}	0.0417911	2.59737×10^{-8}

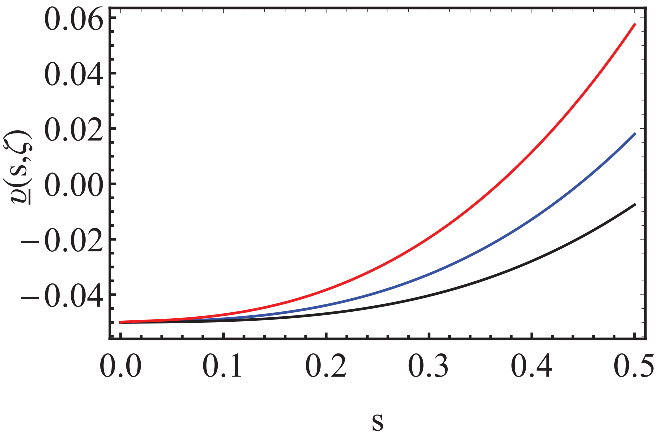

Figure 7. Lower third-order OHAM solution for Eq. (51) for $\zeta = 0.5$ and $s \in [0, 0.5]$, with $\rho = 1$ and $\alpha = 0.5$:(Red), $\alpha = 0.75$:(Blue), $\alpha = 1$:(Black).

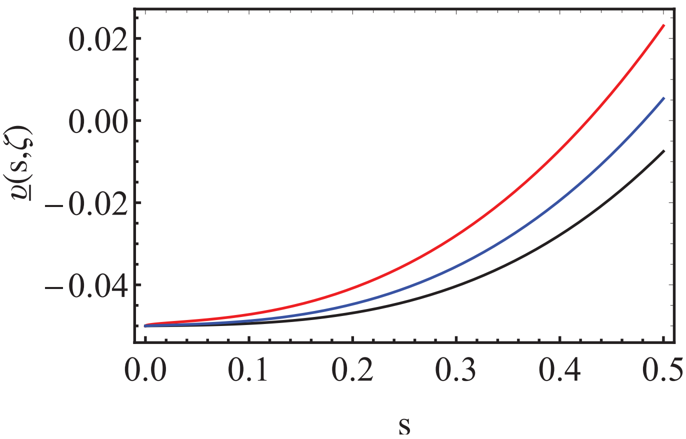

Figure 8. Lower third-order OHAM solution for Eq. (51) for $\zeta = 0.5$ and $s \in [0, 0.5]$, with $\alpha = 1$, $\rho = 0.5$:(Red), $\rho = 0.75$:(Blue), $\rho = 1$:(Black).

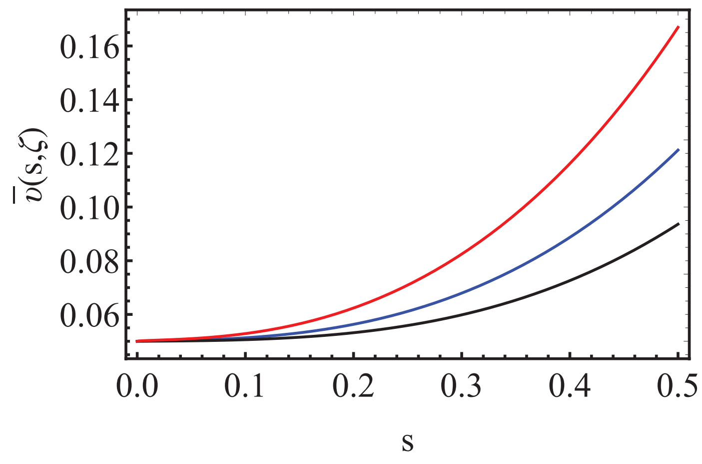

Figure 9. Upper third-order OHAM solution for Eq. (51) for $\zeta = 0.5$ and $s \in [0, 0.5]$, with $\rho = 1$ and $\alpha = 0.5$:(Red), $\alpha = 0.75$:(Blue), $\alpha = 1$:(Black).

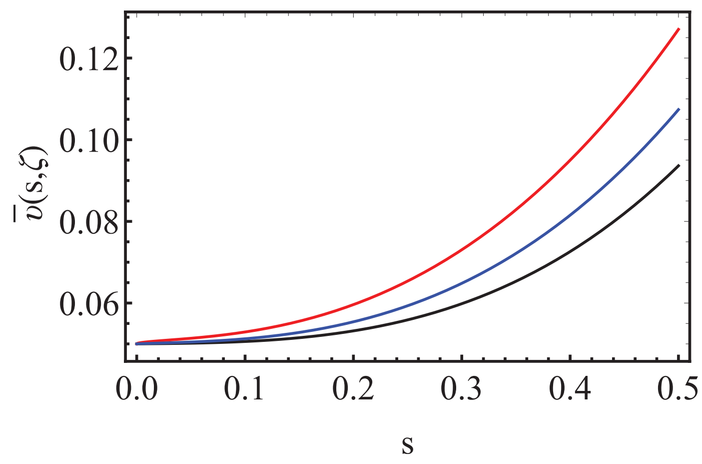

Figure 10. Upper third-order OHAM solution for Eq. (51) for $\zeta = 0.5$ and $s \in [0, 0.5]$, with $\alpha = 1$, $\rho = 0.5$:(Red), $\rho = 0.75$:(Blue), $\rho = 1$:(Black).

on the solution behavior is discussed. The parameters have a substantial impact on the used derivative, introducing an excellent tool for creating fuzzy fractional models. The results are compared with the exact solution in the standard case, and the residual error is calculated for the fractional one. The results prove that the residual errors are close to zero for different fractional parameters. The algorithm's numerical findings satisfy the fuzzy number characteristics by taking the convex fuzzy number form. Furthermore, the approach offers benefits over other existing analytical approximation methods.

References

[1] James J Buckley and Thomas Feuring. Fuzzy initial value problem for nth-order linear differential equations. *Fuzzy Sets and Systems*, 121(2):247–255, 2001.

[2] Ravi P Agarwal, V Lakshmikantham, and Juan J Nieto. On the concept of solution for fractional differential quations with uncertainty. *Nonlinear Analysis: Theory, Methods & Applications*, 72(6):2859–2862, 2010.

[3] Soheil Salahshour, Ali Ahmadian, Norazak Senu, Dumitru Baleanu, and Praveen Agarwal. On analytical solutions of the fractional differential equation with uncertainty: application to the basset problem. *Entropy*, 17(2):885–902, 2015.

[4] Soheil Salahshour and Tofigh Allahviranloo. Applications of fuzzy laplace transforms. *Soft computing*, 17(1):145–158, 2013.

[5] Manal Alqhtani, Kolade M Owolabi, Khaled M Saad, and Edson Pindza. Spatiotemporal chaos in spatially extended fractional dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*, 119:107118, 2023.

[6] Agata Wawrzkiewicz, Krzysztof Pawelek, Przemyslaw Borys, Beata Dworakowska, and Zbigniew J Grzywna. On the simple random-walk models of ion-channel gate dynamics reflecting long-term memory. *European Biophysics Journal*, 41:505–526, 2012.

[7] M Shahidi and A Khastan. Solving fuzzy fractional differential equations by power series expansion method. In *2018 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS)*, pages 37–39. IEEE, 2018.

[8] Asim Kumar Das and Tapan Kumar Roy. Exact solution of some linear fuzzy fractional differential equation using laplace transform method. *Glob J Pure Appl Math*, 13(9):5427–5435, 2017.

[9] Rahman Abdollahi, Mohammad Bagher Farshbaf Moghimi, Alireza Khastan, and Mohammad Reza Hooshmandasl. Linear fractional fuzzy differential equations with caputo derivative. *Computational Methods for Differential Equations*, 7(2):252–265, 2019.

[10] Norazrizal Aswad Abdul Rahman and Muhammad Zaini Ahmad. Solving fuzzy fractional differential equations using fuzzy sumudu transform. *J. Nonlinear Sci. Appl*, 10(5):2620–2632, 2017.

[11] Norazrizal Aswad Abdul Rahman and Muhammad Zaini Ahmad. Applications of the fuzzy sumudu transform for the solution of first order fuzzy differential equations. *Entropy*, 17(7):4582–4601, 2015.

[12] Ekhtiar Khodadadi, Mesut Karabacak, and Ercan Çelik. Solving fuzzy fractional riccati differential equations by the variational iteration method. *International Journal of Engineering and Applied Sciences*, 2(11):35–40, 2015.

[13] A Panahi. Approximate solution of fuzzy fractional differential equations. *International Journal of Industrial Mathematics*, 9(2):111–118, 2017.

[14] Ebrahim Esmailzadeh, Davood Younesian, and Hassan Askari. Integral based methods. In *Analytical Methods in Nonlinear Oscillations*, pages 197–247. Springer, 2019.

[15] Najeeb Alam Khan, Fatima Riaz, and Oyoon Abdul Razzaq. A comparison between numerical methods for solving fuzzy fractional differential equations. *Nonlinear Engineering*, 3(3):155–162, 2014.

[16] Manal Alqhtani, Mohamed M Khader, and Khaled Mohammed Saad. Numerical simulation for a high-dimensional chaotic lorenz system based on gegenbauer wavelet polynomials. *Mathematics*, 11(2):472, 2023.

[17] Sara Al Fahel, Dumitru Baleanu, Qasem M Al-Mdallal, and Khaled M Saad. Quadratic and cubic logistic models involving caputo–fabrizio operator. *The European Physical Journal Special Topics*, pages 1–5, 2023.

[18] Nicolae Herişanu, Vasile Marinca, Toma Dordea, and Gheorghe Madescu. A new analytical approach to nonlinear vibration of an electrical machine. In *Proc. Rom. Acad. Ser. A*, volume 9, pages 229–236, 2008.

[19] Vasile Marinca, Nicolae Herişanu, and Iacob Nemeş. Optimal homotopy asymptotic method with application to thin film flow. *Open Physics*, 6(3):648–653, 2008.

[20] Vasile Marinca, Nicolae Herişanu, Constantin Bota, and Bogdan Marinca. An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate. *Applied Mathematics Letters*, 22(2):245–251, 2009.

[21] Mohammad Hamarsheh, Ahmad Ismail, and Zaid Odibat. Optimal homotopy asymptotic method for solving fractional relaxation-oscillation equation. *Journal of Interpolation and Approximation in Scientific Computing*, 2:98–111, 2015.

[22] N Ratib Anakira, AK Alomari, and Ishak Hashim. Numerical scheme for solving singular two-point boundary value problems. *Journal of Applied Mathematics*, 2013, 2013.

[23] Hampiran Persamaan Riccati, F Mabood, Ahmad Izani, Md Ismai, and Ishak Hashim. Application of optimal homotopy asymptotic method for the approximate solution of riccati equation. *Sains Malaysiana*, 42(6):863–867, 2013.

[24] Udita N Katugampola. New approach to a generalized fractional integral. *Applied Mathematics and Computation*, 218(3):860–865, 2011.

[25] Zaid Odibat and Dumitru Baleanu. Numerical simulation of initial value problems with generalized caputo-type fractional derivatives. *Applied Numerical Mathematics*, 156:94–105, 2020.

[26] Ngo Van Hoa, Ho Vu, and Tran Minh Duc. Fuzzy fractional differential equations under caputo–katugampola fractional derivative approach. *Fuzzy Sets and Systems*, 375:70–99, 2019.

[27] Ricardo Almeida, Agnieszka B Malinowska, and Tatiana Odzijewicz. Fractional differential equations with dependence on the caputo–katugampola derivative. *Journal of Computational and Nonlinear Dynamics*, 11(6), 2016.

[28] Minh Duc Tran, Vu Ho, and Hoa Ngo Van. On the stability of fractional differential equations involving generalized caputo fractional derivative. *Mathematical Problems in Engineering*, 2020, 2020.

[29] Lotfi A Zadeh. Fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 394–432. World Scientific, 1996.

[30] Lotfi Asker Zadeh. The concept of a linguistic variable and its application to approximate reasoning—i. *Information sciences*, 8(3):199–249, 1975.

[31] Luciana T Gomes and Laécio C Barros. A note on the generalized difference and the generalized differentiability. *Fuzzy Sets and Systems*, 280:142–145, 2015.

[32] Didier Dubois and Henri Prade. Towards fuzzy differential calculus part 3: Differentiation. *Fuzzy sets and systems*, 8(3):225–233, 1982.

[33] S Siah Mansouri and N Ahmady. A numerical method for solving nth-order fuzzy differential equation by using characterization theorem. *Communication in Numerical Analysis*, 2012:12, 2012.

[34] Xiaobin Guo and Dequan Shang. Approximate solution of th-order fuzzy linear differential equations. *Mathematical Problems in Engineering*, 2013, 2013.

[35] Giangiacomo Gerla and Luisa Scarpati. Extension principles for fuzzy set theory. *Information Sciences*, 106(1-2):49–69, 1998.

[36] Vangipuram Lakshmikantham and Ram N Mohapatra. *Theory of fuzzy differential equations and inclusions*. CRC press, 2004.

[37] Y Chalco-Cano, H Roman-Flores, and MA Rojas-Medar. Fuzzy differential equations with generalized derivative. In *NAFIPS 2008-2008 Annual Meeting of the North American Fuzzy Information Processing Society*, pages 1–5. IEEE, 2008.

[38] Jameel A. F. *Numerical And Approximate – Analytical Solutions of Fuzzy Initial Value Problems*. PhD thesis, Universiti Utara Malaysia (UUM), 2015.

[39] C Duraisamy and B Usha. Another approach to solution of fuzzy differential equations. *Applied Mathematical Sciences*, 4(16):777–790, 2010.

[40] Marina Tuyako Mizukoshi, LC d Barros, Yurilev Chalco-Cano, Heriberto Román-Flores, and Rodney Carlos Bassanezi. Fuzzy differential equations and the extension principle. *Information Sciences*, 177(17):3627–3635, 2007.

[41] AF Jameel, Azizan Saaban, SA Altaie, NR Anakira, AK Alomari, and N Ahmad. Solving first order nonlinear fuzzy differential equations using optimal homotopy asymptotic method. *International Journal of Pure and Applied Mathematics*, 118(1):49–64, 2018.

[42] AF Jameel, Azizan Saaban, SA Altaie, NR Anakira, AK Alomari, and N Ahmad. Solving first order nonlinear fuzzy differential equations using optimal homotopy asymptotic method. *International Journal of Pure and Applied Mathematics*, 118(1):49–64, 2018.