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Uniquely solvable problems for the Laplace-Beltrami operator on a sphere punctured by a curve

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Abstract

A fixed smooth arc from a two-dimensional sphere of three-dimensional space is removed. The Laplace-Beltrami operator on the resulting surface is studied. In the paper the boundary conditions on a remote arc are found, which guarantee the existence of a unique solution to the inhomogeneous Laplace-Beltrami equation.

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1. Introduction

It is known [1] that the two-dimensional sphere S in the three-dimensional space $\mathbb{R}^3$ represents a Riemannian manifold. The Laplace-Beltrami operator Δ_S on the indicated sphere in a standard way is introduced. Since the sphere S is a smooth manifold without boundary, the problem is well defined.

$$u(x) - \Delta_S u(x) = f(x), \quad x \in S.$$

The sphere S of the closed curve C is split into two non-intersecting parts S_2 and S_1 [2, 4].

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The following Dirichlet problem studied in the work [2]

$$-\Delta_S u(x) = 0, \qquad x \in S_2,$$
$$u(x) = g(x), \quad x \in C.$$

Using the potentials of a simple and double layer, the existence of a solution to the indicated Dirichlet problem is proved.

Another problem is studied in the work [4]

$$-\Delta_S u(x) = \omega(x), \quad x \in S_2, \tag{1}$$

$$-\frac{1}{2}u(x) + \int_C u(y)\underline{curl}_S \varepsilon(x,y)\cdot \vec{t}(y)ds_y - \int_C \varepsilon(x,y)\underline{curl}_S u(y)\cdot \vec{t}(y)ds_y = 0, \quad x \in C. \tag{2}$$

Here $\varepsilon(x,y) = -\frac{1}{4\pi}\ln|1-\langle x,y\rangle|$ represents the fundamental solution of the Laplace-Beltrami operator. It is shown in the work [4], that the solution to problem (1),(2) is written as

$$u(x) = \int_{S_2} \varepsilon(x,y)\omega(y)d\sigma_y, \quad x \in S_2.$$

We also note the paper [5], where a boundary value problem for the Laplace-Beltrami operator on a punctured sphere was studied. A punctured sphere is a sphere from which one point has been removed. In this case, the problem arises: What additional conditions must the solution at the remote point satisfy in order to guarantee the uniqueness of the solution?

In the present paper, a fixed arc from the sphere is removed and the same question is studied: what additional conditions must the solution on the removed arc satisfy in order to guarantee the uniqueness of such a solution?

2. Preliminaries

Consider a closed curve C on the two-dimensional sphere S that divides S into two parts S_1 and S_2. In particular, the curve $C = \partial S_1$ is the boundary of S_1.

Let us introduce some vector identities on the sphere. Let $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$ be ordinary unit vectors in spherical coordinates. According to [2], we define the surface gradient of the scalar f on S and the surface divergence for the vector function $\vec{V} = V_r\vec{e}_r + V_\varphi\vec{e}_\varphi + V_\theta\vec{e}_\theta$ on a sphere using the following formulas:

$$\nabla_S f(x) = \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\vec{e}_\varphi + \frac{\partial f}{\partial\theta}\vec{e}_\theta,$$
$$div_S\vec{V}(x) = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\varphi}V_\varphi(\varphi,\theta) + \frac{\partial}{\partial\theta}(\sin\theta V_\theta(\varphi,\theta))\right).$$

This implies the identity:

$$\Delta_S u(x) = div_S \nabla_S u(x).$$

We introduce [2] a vector surface rotation for a scalar field f on a sphere:

$$\underline{curl}_S f(x) = -\frac{\partial f}{\partial\theta}\vec{e}_\varphi + \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\vec{e}_\theta$$

and (scalar) rotation of the surface of the vector field $\vec{V}$ as

$$curl_S\vec{V}(x) = \frac{1}{\sin\theta}\left(-\frac{\partial}{\partial\varphi}V_\theta(\varphi,\theta) + \frac{\partial}{\partial\theta}(\sin\theta V_\varphi(\varphi,\theta))\right).$$

Another vector identity for the Laplace-Beltrami operator is given in the work [2]:

$$\Delta_S u(x) = -curl_S \underline{curl}_S u(x) \text{for} x \in S.$$

We introduce the scalar product in the function space $L_2(S)$

$$\langle u,v\rangle_{L_2(S)} = \int_S u(x)v(x)d\sigma_x = \int_0^{2\pi}\int_0^{\pi} u(x(\varphi,\theta))v(x(\varphi,\theta))\sin\theta d\theta d\varphi.$$

Now we derive Green's identity. Integrating by parts the scalar product $\langle -\Delta_S u,v\rangle_{L_2(S)}$, we obtain the following representation:

$$\langle -\Delta_S u,v\rangle_{L_2(S)} = a_S(u,v) = a_S(v,u) = \langle u,-\Delta_S v\rangle_{L_2(S)},$$

where the symmetrical bilinear form is introduced

$$\begin{aligned} a_S(u,v) \quad &:= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}u(\varphi,\theta)\frac{\partial}{\partial\varphi}v(\varphi,\theta) + \sin\theta\frac{\partial}{\partial\theta}v(\varphi,\theta)\frac{\partial}{\partial\theta}u(\varphi,\theta)\right]d\theta d\varphi \\ &= \int_S \nabla_S u(x)\cdot\nabla_S v(x)d\sigma_x. \end{aligned}$$

The Stokes theorem [3] for a positively oriented curve C and a domain S_2 can be written as

$$\int_{S_2} curl_S \vec{V}(x)d\sigma_x = \int_C \vec{V}(x)\cdot\vec{t}(x)ds_x.$$

Here $\vec{t}$ is the unit tangent vector to C. Note that a similar identity holds for S_1, but the direction of the tangent is changed. Now setting $\vec{V} = v(x)\vec{W}(x)$ and applying the product rule, we get $\int_{S_2}\underline{curl}_S v(x)\cdot\vec{W}(x)d\sigma_x = -\int_C v(x)[\vec{W}(x)\cdot\vec{t}(x)]ds_x + \int_{S_2} v(x)curl_S\vec{W}(x)d\sigma_x$. Using $\vec{W}(x) = \underline{curl}_S u(x)$, we finally obtain Green's first formula for Laplace-Beltrami operator,

$$\int_{S_2}\underline{curl}_S v(x)\cdot\underline{curl}_S u(x)d\sigma_x = \int_{S_2} v(x)(-\Delta_S u(x))d\sigma_x - \int_C v(x)\left[\underline{curl}_S u(x)\cdot\underline{t}(x)\right]ds_x. \tag{3}$$

In this formula, the role of the bilinear form $a_{S^2}(u,v)$ is played by the expression $\int_{S_2}\underline{curl}_S v(x)\cdot\underline{curl}_S u(x)d\sigma_x$. Let's rewrite the formula (3) by swapping the functions $u(x)$ and $v(x)$. As a result, we get

$$\int_{S_2}\underline{curl}_S u(x)\cdot\underline{curl}_S v(x)d\sigma_x = \int_{S_2} u(x)(-\Delta_S v(x))d\sigma_x - \int_C u(x)\left[\underline{curl}_S v(x)\cdot\underline{t}(x)\right]ds_x. \tag{4}$$

Subtracting the formula (4) from (3), we obtain the second Green formula for the Laplace-Beltrami operator in the form

$$\int_{S_2}[u(x)\Delta_S v(x) - v(x)\Delta_S u(x)]d\sigma_x == \int_C [v(x)\underline{curl}_S u(x) - u(x)\underline{curl}_S v(x)]\cdot\vec{t}(x)ds_x. \tag{5}$$

To conclude this section, we present the definitions of the simple and double layer potentials introduced in [2].

The potential of a simple layer with a sufficiently smooth density function σ is determined by the formula:

$$(\widetilde{V}\sigma)(x) := \int_C \varepsilon(x,y)\sigma(y)ds_y \quad \text{for } x \notin C.$$

The potential of a double layer with a sufficiently smooth density function μ is given by the formula:

$$(\widetilde{W}\mu)(x) := \int_C \mu(y)\left[\underline{curl}_S\varepsilon(x,y)\cdot\vec{t}(y)\right]ds_y \quad \text{for } x \notin C.$$

The introduced potentials have the standard properties of single and double layer potentials.

For $x \notin C$, the simple layer potential satisfies to the condition:

$$\Delta_S(\tilde{V}\sigma)(x) = \frac{1}{4\pi}\int_C \sigma(y)ds_y.$$

Similarly, the double layer potential satisfies the Laplace-Beltrami equation for $x \notin C$

$$\Delta_S(\widetilde{W}\mu)(x) = 0,$$

without additional restrictions on the density μ.

3. Main results

Correct formulation of a boundary value problem for the Laplace-Beltrami operator on a sphere with a cut

Let $\delta > 0$. In this section, as a closed curve C, we choose the curve C_δ on the sphere S, which is shown in Figure 1. The curve C_δ divides the sphere S into two parts $S_{1\delta}$ and $S_{2\delta}$. In Figure 1, the surface $S_{2\delta}$ is shaded. The unshaded part of the sphere S in $\mathbb{R}^3$ is denoted by $S_{1\delta}$.

We set $S_{2\delta} = S \setminus \overline{S}_{1,\delta}$ and also denote by C_δ its boundary $S_{2\delta}$.

According to [2], the fundamental solution of the Laplace-Beltrami operator Δ_S is given by

$$\varepsilon(x,y) = -\frac{1}{4\pi}\log|1-(x,y)| = -\frac{1}{4\pi}\log[1-\cos(\varphi_x - \varphi_y)\sin\theta \sin_x\theta_y - \cos\theta_x \cos\theta_y].$$

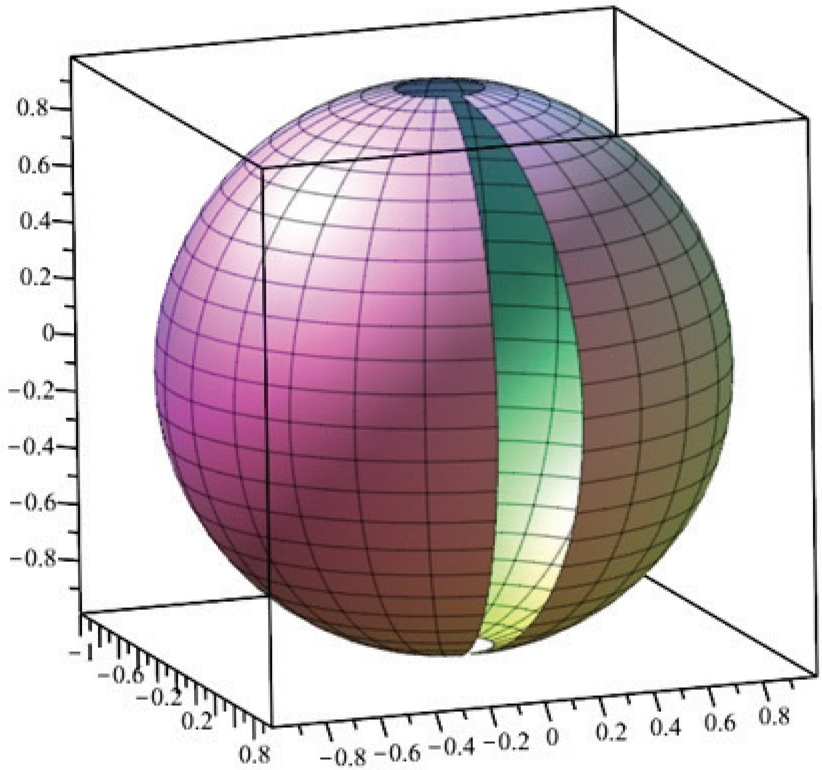

Figure 1: A two-dimensional sphere with a closed curve that breaks it into two parts.

In particular [2, 4],

$$\Delta_S \varepsilon(x,y) = \frac{1}{4\pi} + \delta(x-y).$$

for $x = x(\varphi,\theta), y = y(\varphi_0,\theta_0) \in S$. Here $\delta(x-y)$ is the Dirac delta function.

Consider the Laplace-Beltrami equation on the surface $S_{2\delta}$ from S to $\mathbb{R}^3$. Further $\lim_{\delta\to 0} S_{2\delta}$ will be denoted by S_{20}. The limiting surface S_{20} is a sphere with a cut along the arc. The boundary S_{20} is denoted by C_0. We introduce a class of the function $W^2_{2,loc}(S_{20}) = \bigcup_{\delta>0} W_2^2(S_{2\delta})$. Assuming that h belongs to $W^2_{2,loc}(S_{20})$, by direct calculation for any $x \in S_{2\delta}$ we find

$$\begin{aligned} T(x) &\overset{def}{=} \int_{S_{2\delta}} \varepsilon(x,y)(-\Delta_S h(y))d\sigma_y \\ &= \int_{S_{2\delta}} h(y)(-\Delta_S \varepsilon(x,y))d\sigma_y + \int_{C_\delta} h(y)\underline{curl}_S \varepsilon(x,y)t(y)ds_y - \int_{C_\delta} \varepsilon(x,y)\underline{curl}_S h(y)t(y)ds_y \\ &= h(x) - \frac{1}{4\pi}\int_{S_{2\delta}} h(y)d\sigma_y + \int_{C_\delta} h(y)\underline{curl}_S \varepsilon(x,y)t(y)ds_y - \int_{C_\delta} \varepsilon(x,y)\underline{curl}_S h(y)t(y)ds_y, \end{aligned}$$

where $t(y)$ is the unit tangent vector to C_δ at the point y.

From the last relation we get

$$T(x) - h(x) = -\frac{1}{4\pi}\int_{S_{2\delta}} h(y)d\sigma_y + \int_{C_\delta} h(y)\underline{curl}_S \varepsilon(x,y)t(y)ds_y - \int_{C_\delta} \varepsilon(x,y)\underline{curl}_S h(y)t(y)ds_y. \quad (6)$$

Subsequently, we assume that

$$\int_{S_{2\delta}} h(y)d\sigma_y = 0.$$

Thus, from (6) we find

$$T(x) - h(x) = \int_{C_\delta} h(y)\underline{curl}_S \varepsilon(x,y)\vec{t}(y)ds_y - \int_{C_\delta} \varepsilon(x,y)\underline{curl}_S h(y)\vec{t}(y)ds_y, \quad x \in S_{2\delta}. \quad (7)$$

The relation (7) plays an important role in our reasoning. For further purposes, we need to calculate the limit of the right side of the equality (7) at $\delta \to 0$. Therefore, it is convenient to introduce the following linear operators using the formulas

$$\begin{aligned} (L_s h)(x) &:= \lim_{\delta\to 0}\int_{C_\delta} \varepsilon(x,y)\underline{curl}_S h(y)\vec{t}(y)ds_y \quad \text{for} x \notin C_0, \\ (L_d h)(x) &:= \lim_{\delta\to 0}\int_{C_\delta} h(y)\underline{curl}_S \varepsilon(x,y)\vec{t}(y)ds_y \quad \text{for} x \notin C_0. \end{aligned}$$

Note that the operator L_s corresponds to the single layer potentials, while the operator L_d corresponds to the double layer potential.

Let us present one useful property of the operators L_d and L_s.

Let $x_0 \in S_{20}$, i.e. $x_0 \bar{\in} C$. Denote by $\psi_1(y) = \varepsilon(x_0,y)$, $\vec{\Phi}(y) = \underline{curl}_S \psi(y)$. Let us calculate the values of $(L_s\psi_1)(x)$ and $(L_d\psi_1)(x)$ for $x \in S_{20}$. In order to calculate these values, we introduce the function

$$R(x) = \lim_{\delta\to 0}\int_{S_{2\delta}} \varepsilon(x,y)(-\Delta_S \psi_1(y))d\sigma_y.$$

Since

$$-\Delta_S\psi_1(y) = -\Delta_S\varepsilon(x_0, y) = -\delta(x_0 - y) - \frac{1}{4\pi},$$

that

$$R(x) = -\varepsilon(x, x_0) - \frac{1}{4\pi}\lim_{\delta\to 0}\int_{S_{2\delta}}\varepsilon(x, y)d\sigma_y.$$

This implies for any $x, x_0 \in S_{20}$

$$(L_d\psi_1)(x) - (L_s\psi_1)(x) = -2\varepsilon(x, x_0),$$

where $\psi_1(y) = \varepsilon(x_0, y), \quad y \in S_{20}$.

$$-\varepsilon(x, x_0) - \frac{1}{4\pi}\lim_{\delta\to 0}\int_{S_{2\delta}}\varepsilon(x, y)d\sigma_y = \varepsilon(x_0, x) - \frac{1}{4\pi}\lim_{\delta\to 0}\int_{S_{2\delta}}\varepsilon(x, y)d\sigma_y + \lim_{\delta\to 0}\int_{C_\delta}\psi_1(y)\underline{curl}_S\varepsilon(x, y)\vec{t}(y)ds_y$$
$$-\lim_{\delta\to 0}\int_{C_\delta}\varepsilon(x, y)\underline{curl}_S\psi_1(y)\vec{t}(y)ds_y.$$

Thus, the difference $(L_d\psi_1 - L_s\psi_1)$ has a singularity on C of the same character as the fundamental solution.

We remind, the functions $g(x)$ that satisfy to the equation $\Delta_S g(x) = 0, \quad x \in S_{20}$ are called harmonic. Denote by $H(S_{20})$ the class of harmonic functions on S_{20} that can have a singularity of the same nature as the fundamental solution $\varepsilon(x, y)$.

As well as necessary to introduce a class of the function

$$W^2_{2L}(S_{20}) = \{h \in W^2_{2,loc}(S_{20}) : \lim_{\delta\to 0}\int_{S_{20}} h(y)d\sigma_y = 0,$$
$$(L_d h)(x) - (L_s h)(x) \in H(S_{20}), \Delta_S h \in L_2(S)\}.$$

Take an arbitrary function $h(x)$ from the class $W^2_{2L}(S_{20})$. Since $\Delta_S h \in L_2(S)$, that $T(x) \in W^2_2(S)$ and $\Delta_S T = \Delta_S h$. Thus, $T(x)$ represents a regularization of the function $h(x)$. In fact, the function $h(x)$ could have singularities on the curve C. At the same time, its regularization $T(x)$ no longer has singularities on the full sphere S, that is, it belongs to the space $W^2_2(S)$.

In particular, the assertion is true.

Lemma 3.1: *An arbitrary element h of the class $W^2_{2L}(S_{20})$ for $x \in S_{20}$ can be represented as $h(x) = T(x) + (L_s h)(x) - (L_d h)(x)$, where $T(x)$ is a function from $W^2_2(S)$.*

Now we can formulate the statement.

Theorem 3.2: *For any function $f \in L_2(S)$ and any function g from $H(S_{20})$, the problem*

$$-\Delta_S u(x) = f(x), \quad x \in S_{20},$$
$$(L_d u)(x) - (L_s u)(x) = g(x), \quad x \in S_{20}$$

has a unique solution in the class $W^2_{2L}(S_{20})$.

Statements similar to Lemma 3.1 and Theorem 3.2 were proved in [5].

Theorem 3.2 implies the main assertion of the present article.

Theorem 3.3: *Let K be a continuous linear operator from the space $L_2(S)$ to the space $H(S_{20})$. Then for any f from $L_2(S)$ the following problem*

$$-\Delta_S u(x) = f(x), \quad x \in S_{20}, \tag{8}$$

$$(L_d u)(x) - (L_s u)(x) = K(-\Delta_S u(x)), \quad x \in S_{20} \tag{9}$$

has a unique solution from the class $W^2_{2,L}(S_{20})$.

The proof of Theorem 3.3 repeats the arguments given in the proof of a similar theorem [5]. Also problem for perturbed harmonic oscillator were considered in [6] and for Laplace operator in [7].

Finally, note that conditions (9) for $x \in S_{20}$ can be rewritten for $x_0 \in C$ as boundary conditions

$$\lim_{\substack{x\to x_0 \\ x\in S_{20}}} ((L_d u)(x) - (L_s u)(x)) = \lim_{\substack{x\to x_0 \\ x\in S_{20}}} K(-\Delta_S u(x)).$$

In conclusion, we give an example. We take the following operator as an operator K

$$Kf(x) = \int_S f(\tau) g(\tau) d\sigma_\tau \cdot \int_C \varepsilon(x,\tau) p(\tau) d\sigma_\tau,$$

where the fixed function $g \in L_2(S)$, $p(\tau)$ is a polynomial on the arc C. In this case, problem (8), (9) is equivalent to a boundary value problem on S_{20}

$$-\Delta_S u(x) = f(x), \quad x \in S_{20},$$
$$\lim_{x\to x_0 x\in S_{20}} ((L_d u)(x) - (L_s u)(x)) = \int_C \varepsilon(x_0,\tau) p(\tau) d\sigma_\tau \cdot \int_S g(\tau)(-\Delta_S u(\tau)) d\sigma_\tau. \tag{10}$$

If $g(\tau)$ represents a harmonic function on S, then the boundary condition (10) can be written in a more convenient form. To do this, the formula is used:

$$\int_S g(\tau)(-\Delta_S u(\tau)) d\sigma_\tau = \int_C g(y) \underline{curl}_S u(y) \cdot \vec{t}(y) ds_y - \int_C u(y) \underline{curl}_S g(y) \vec{t}(y) ds_y.$$

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