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η- approximation, η-co-approximation and δ-orthogonal in bounded space

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Abstract

This article aims to establish some relationships have been established of the concepts η -approximation, η -co-approximation and η -orthogonal with the images under isometry mappings, whereas these results are generalization and expansion of η -approximation, η -co-approximation and δ -orthogonal.

Key words: η -approximation, η -co-approximation, η -orthogonal, bounded function and bounded space.
Mathematics Subject Classication (2010): 41-XX, 65D15

1. Introduction

The study of best approximation for bounded functions in metric space, normed space and weighed spaces has been the center of sober research activity. In actuality, the approximation theory field has practical importance in mathematics.

In 1988 Roshed [1], studied M-metric space in terms of proximinality properties of certain sets, Cybenk in 1989 [2] explain that limited number of linear combinations of mix two concepts of stable unvaried and set affine mappings can estimated every connected function of n real mutable with backing in the unite hypercabe, Rezapour in 2007 [3] given some results about characterization of best

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approximation in cone-metric space, Konovolov in 2008 [4] obtain on the degree of best approximation of monotone radial functions in measurable space by polynomials, Gumus in 2012 [5] presented the set of all t-better approximation on uncertain n-areas of normed, Naser in 2015 [6] proposed the problem of t-better approximation compact sets and t-aproximal sets in fuzzy metric space, in 2015 Narange [7] discussed some results about the existence and uniqueness of best approximation and best co-approximation in metric linear space, also Auad and Al. Jumaili in 2017 [8] established the set of polynomials to approximate unbounded functions in measured space via modulus of smoothness, Auad and Abduljabbar in 2018 [9] they investigated that degree of best algebraic approximation equivalent degree of best trigonometric approximation in the same space.

Auad and Abdulsttar in 2019 [10] they provide some ideas of best simultaneous approximation of unbounded functions with multi variables in weighted space by using two different definitions and found the relationship between best simultaneous approximation with best approximation, Auad and Fayyadh in 2021 [11] they introduce the outline of direct and invers theorems of unbounded functions in weighted spaces through the modulus of smoothens functions.

In this work, we introduced some concepts that related approximation of functions as η-approximation, η-co-approximation and δ-orthogonal through which the bounded by using isometry operators within bounded functions space. Also.

Let $S_{b,\eta}$ be the space of all bounded functions and $\Upsilon_{b,\eta}$ subset of it. For any $\eta > 0$, the function $\mu_0 \in \Upsilon_{b,\eta}$ is said to be an η-approximation (η-co-approximation) respectively to function $\varphi_b \in S_{b,\eta}$, if

$$\| \mu_0 - \varphi_b \|_{b,\aleph} = \inf \left\{ \| \mu - \varphi_b \|_{b,} + \delta, \ \forall \ \mu \in \Upsilon_{b,} \right\} \tag{1}$$

$$\| \mu_0 - \mu \|_{b,\eta} = inf \left\{ \| \mu - \varphi_b \|_{b,\eta} + \delta, \ \forall \ \mu \in \Upsilon_{b,\eta} \right\}. \tag{2}$$

The set of all η-approximation (η-co-approximation) from subset $\Upsilon_{b,\eta}$ to function φ_b in $S_{b,\eta}$ are called η-proximal (η-co-proximal) respectively and denoted of them via $F_{b,\eta}(\varphi_b)$ & $G_{b,\eta}(\varphi_b)$ respectively.

If $F_{b,\eta}(\varphi_b)$ & $G_{b,\eta}(\varphi_b)$ are both non-empty sets and for φ_b belong to finite dimensional space $S_{b,\delta}$, then its easy to see that be function of best δ-approximation of φ_b always exists but the function of η-co-approximation may or may not found.

For $\varphi_{b,1}, \varphi_{b,2} \in S_{b,\delta}$, if $\varphi_{b,1}$ orthogonal to $\varphi_{b,2}$ and denoted by $\varphi_{b,1} \perp_\eta \varphi_{b,2}$, then

$$\| \varphi_{b,1} \|_{b,\eta} \leq \| \varphi_{b,1} - \alpha \varphi_{b,2} \|_{b,\eta}, \text{for all scalar } \alpha. \tag{3}$$

In general, if $U_{b,\eta}$ & $V_{b,\eta}$ are orthogonal non-empty subset of $S_{b,\delta}$ and denoted by $U_{b,\eta} \perp_\eta V_{b,\eta}$, then $u_b \perp_\delta v_b$ for $u_b \in U_{b,\eta}$ and $v_b \in V_{b,\eta}$.

We defined the following sets:

$$\Upsilon_{b,\eta}{}^{\perp_\delta} = \left\{ \varphi_b \in S_{b,\eta}, \varphi_b \perp_\eta \Upsilon_{b,\eta} \right\}.$$

$$\Upsilon_{\perp_\eta}^{b,\eta} = \left\{ \varphi_b \in S_{b,\eta}, \ \Upsilon_{b,\eta} \perp_\eta \varphi_b \right\}$$

We can define the isometry mapping by the following:

Let $\left(S_{b,\eta}, \| . \|_{b,\eta} \right)$ be a normed space. Then the mapping $\Psi : \left(S_{b,\eta} \| . \|_{b,\eta} \right) \to \left(S_{b,\eta}, \| . \|_{b,\eta} \right)$ is named isometry mapping (preserving mapping) if, for $p_{b,1}, p_{b,2} \in S_{b,\eta}$, $\| p_{b,1} - p_{b,2} \|_{b,\eta} = \| \Psi\left(p_{b,1}\right) - \Psi\left(p_{b,2}\right) \|_{b,\eta}$ and $\Psi\left(\| p_{b,1} \|_{b,\eta} \right) = \| p_{b,1} \|_{b,\eta}$.

2. Auxiliary Lemmas

In this section, some lemmas that we need it in our main results have been proven.

Lemma 2.1: Let $\Upsilon_{b,\eta_{\perp}}$ be a subset of $S_{b\eta}$ and $\vartheta_0 \in \Upsilon_{b,\eta}$. Then m_0 is η-approximation to $p_b \in S_{b\eta}$ if and only if $(\vartheta_0 - p_b) \in \Upsilon_{b,\eta}{}^{\perp_\eta}$.

Proof: Let ϑ_0 be an, η-approximation to p_b, from (1), we obtain

$$\Leftrightarrow \| \vartheta_0 - p_b \|_{b,\eta} \leq \| \vartheta - p_b \|_{b,\eta} + \eta \text{ for all } \vartheta \in \Upsilon_{b,\eta}$$
$$\Leftrightarrow \| \vartheta_0 - p_b \|_{b,\eta} \leq \| (\vartheta_0 + \alpha\vartheta) - p_b \|_{b,\eta} + \eta \text{ for all } \vartheta \in \Upsilon_{b,\eta}$$
$$\Leftrightarrow \| (\vartheta_0 - p_b) + \alpha\vartheta \|_{b,\eta} + \eta \text{ for all } \vartheta \in \Upsilon_{b,\eta}$$
$$\Leftrightarrow (\vartheta_0 - p_b) \perp_\eta \vartheta \text{ for all } \vartheta \in \Upsilon_{b,\eta}$$
$$\Leftrightarrow (\vartheta_0 - p_b) \in \Upsilon_{b,\eta}{}^{\perp_\eta}.$$

Lemma 2.2: Let $F_{b,\eta}(p_b)$ be a η-proximal set in space $S_{b,\eta}$. Then $\vartheta_0 \in F_{b,\eta}(p_b)$ if and only if $0 \in F_{b,\eta}(p_b - \vartheta_0)$.

Proof: Let $0 \in F_{b,\eta}(p_b - \vartheta_0)$

$$\Leftrightarrow (p_b - \vartheta_0) - 0 \in \Upsilon_{b,\eta}{}^{\perp_\eta}$$
$$\Leftrightarrow (p_b - \vartheta_0) \in \Upsilon_{b,\eta}{}^{\perp_\eta}$$
$$\Leftrightarrow \vartheta_0 \in F_{b,\eta}(p_b).$$

Lemma 2.3: Let $G_{b,\eta}(p_b)$ be η-c-proximal set in space $S_{b,\eta}$ and $\vartheta_0 \in \Upsilon_{b,\eta}{}^{\perp_\eta}$. Then $\vartheta_0 \in \Upsilon_{b,\eta}$ is an η-co-approximation to p_b.

Proof: Let $(p_b - \vartheta_0) \in \Upsilon_{b,\eta}{}^{\perp_\eta}$

$$\Rightarrow (p_b - \vartheta_0) \perp_\eta \Upsilon_{b,\eta}$$
$$\Rightarrow (p_b - \vartheta_0) \perp_\eta \vartheta$$
$$\Rightarrow \| \vartheta \|_{b,\aleph} \leq \| \vartheta - \alpha(p_b - \vartheta_0) \|_{b,} + \eta \text{ for } \vartheta \in \Upsilon_{b,} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta + \alpha\vartheta_0 - \alpha\vartheta_0 \|_{b,\aleph} \leq \| \vartheta - \alpha(p_b - \vartheta_0) \|_{b,} + \eta \text{ for } \vartheta \in \Upsilon_{b,} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta_1 - \alpha\vartheta_0 \|_{b,\aleph} \leq \| \vartheta_1 - \alpha p_b \|_{b,} + \eta \text{ for } \vartheta_1 \in \Upsilon_{b,} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta_1 - \vartheta_0 \|_{b,\eta} \leq \| \vartheta_1 - p_b \|_{b\eta} + \delta \text{ for } \vartheta_1 \in \Upsilon_{b,\eta}$$
$$\Rightarrow \vartheta_0 \text{ is an } \eta\text{-co-approximation to } p_b.$$

Lemma 2.4: Let $\Upsilon_{b,\eta}$ be subspace of bounded space $S_{b,\delta}$ and $p_b \in S_{b,\eta}$. Then for $\vartheta_0 \in \Upsilon_{b,\eta}$, $p_b - \vartheta_0 \perp_\eta \Upsilon_{b,\eta}$ if and only if $\alpha\vartheta_0$ is an η-c-approximation to αp_b.

Proof: Let $\alpha\vartheta_0$ is an η-c-approximation to αp_b,

$$\Rightarrow \| \vartheta - \alpha\vartheta_0 \|_{b,\eta} \leq \| \vartheta - \alpha p_b \|_{b,\eta} + \eta \text{ for } \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta - \alpha\vartheta_0 \|_{b,\eta} \leq \| \vartheta - \alpha\vartheta_0 + \alpha\vartheta_0 - \alpha p_b \|_{b,\delta} + \eta \text{ for } \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta - \alpha\vartheta_0 \|_{b,\eta} \leq \| \vartheta - \alpha\vartheta_0 + \alpha(\vartheta_0 - p_b) \|_{b,\eta} + \eta \text{ for } \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta - \alpha\vartheta_0 \|_{b,\eta} \leq \| \vartheta_1 - \alpha(p_b - \vartheta_0) \|_{b\eta} + \eta \, for \, \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow p_b - \vartheta_0 \perp_\eta \Upsilon_{b,\eta}.$$

Conversely; assume $p_b - \vartheta_0 \perp_\delta \Upsilon_{b,\eta}$

$$\Rightarrow p_b - \vartheta_0 \perp_{,\eta} \vartheta \text{ for } \vartheta \in \Upsilon_{b,\eta}$$
$$\Rightarrow \| \vartheta \|_{b,\eta} \leq \| \vartheta - \alpha(p_b - \vartheta_0) \|_{b,\eta} +, \eta \text{ for } \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta \|_{b,\delta} \leq \| \vartheta + \alpha\vartheta_0 - \alpha p_b \|_{b,\delta} + \delta \text{ for } \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta \|_{b,\eta} \leq \| \vartheta_1 - \alpha p_b \|_{b,,\eta} + \delta \text{ for } \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \| \vartheta_1 - \alpha\vartheta_0 \|_{b,,\eta} \leq \| \vartheta_1 - \alpha p_b \|_{b,\eta} +, \eta \text{ for } \vartheta \in \Upsilon_{b,\eta} \text{ and } \alpha \text{ scalar}$$
$$\Rightarrow \alpha\vartheta_0 \text{ is an, } \eta\text{-c-approximation to } \alpha p_b.$$

Lemma 2.5: Let p_b belong to the space $S_{b,\eta}$ and $\Upsilon_{b,\eta}$ subspace of $S_{b,\eta}$. Then ϑ_0 is an η-c-approximation to p_b if and only if 0 is an η-c-approximation to $p_b - \vartheta_0$.

Proof: Let ϑ_0 be an η-c-approximation to p_b,

$$\Leftrightarrow \| \vartheta - \vartheta_0 \|_{b,\eta} \leq \| \vartheta - p_b \|_{b,\eta} +, \eta \quad \text{for } \vartheta \in \Upsilon_{b,\eta}$$
$$\Leftrightarrow \| \vartheta - \vartheta_0 \|_{b,\eta} \leq \| (\vartheta - \vartheta_0) - (p_b - \vartheta_0) \|_{b,\eta} + \eta \quad \text{for } \vartheta \in \Upsilon_{b,\eta}$$
$$\Leftrightarrow \| \vartheta_1 \|_{b,\eta} \leq \| \vartheta_1 - (p_b - \vartheta_0) \|_{b,\eta} + \eta \text{ for } \vartheta \in \Upsilon_{b,\eta}$$
$$\Leftrightarrow 0 \text{ is an } \eta\text{-c-approximation to } p_b - \vartheta_0.$$

3. Main Results

In this section, we introduced some theorems in which the estimation of the best δ-approximation, η-c-approximation and δ-orthogonal in bounded space $S_{b,\eta}$.

Theorem 3.1: Let Ψ be an isometry mapping on normed space $S_{b,\delta}$ and $G_{b,\eta}(\varphi_b)$ subspace δ-c-proximal of $S_{b,\eta}$. Then the followings are true

i. $\Psi\left(G_{b,\eta}(\varphi_b)\right) \subseteq G_{b,\eta}(\Psi(\varphi_b))$.
ii. If φ_b is Ψ-fixed, then $\Psi\left(G_{b,\eta}(\varphi_b)\right) \subseteq G_{b,\eta}(\varphi_b)$.
iii. If φ_b is Ψ-fixed and $G_{b,\eta}(\varphi_b) = \{\varphi_b\}$, then $\Psi(\varphi_b) = \varphi_b$.
iv. If φ_b is Ψ-fixed and $\{q_b \in H; \Psi(q_b) = q_b\} \cap G_{b,\eta}(\varphi_b) = \varnothing$, then either $G_{b,\eta}(\varphi_b)$ is empty set or $G_{b,\eta}(\varphi_b)$ has more than one element.

Proof: (i) Let $\tau_b \in G_{b,\eta}(\varphi_b) \Rightarrow \Psi(\tau_b) \in \Psi\left(G_{b,\eta}(\varphi_b)\right)$.

We can choose ϕ_b any function contained in $G_{b,\eta}(\varphi_b)$ and $\Psi\left(G_{b,\eta}(\varphi_b)\right) = G_{b,\eta}(\varphi_b)$ implies there is function $\omega_b \in G_{b,\eta}(\varphi_b)$ such that $\Psi(\omega_b) = \phi_b$.

$$\| \Psi(\tau_b) - \phi_b \|_{b,\eta} = \| \Psi(\tau_b) - \Psi(\omega_b) \|_{b,\eta} = \| \tau_b - \omega_b \|_{b,\eta}$$
$$\leq \| \varphi_b - \phi_b \|_{b,\eta} + \eta$$
$$= \| \Psi(\tau_b) - \Psi(\omega_b) \|_{b,\eta} + \eta$$
$$= \| \Psi(\tau_b)) - \phi_b \|_{b,\eta} + \eta \text{ for all } \phi_b \in G_{b,\eta}(\varphi_b)$$
$$\Rightarrow \Psi(\tau_b) \in \Psi\left(G_{b,\eta}(\varphi_b)\right)$$
$$\Rightarrow \tau_b \text{ is } \eta\text{-co-approximation to } \varphi_b.$$

ii. Let $\tau_b \in G_{b,\eta}(\varphi_b)$, form (i) we have

$$\Psi(\tau_b) \in \Psi\left(G_{b,\eta}(\varphi_b)\right) \subseteq G_{b,\eta}\left(\Psi(\varphi_b)\right).$$

iii. From (ii), we have $\Psi\left(G_{b,\eta}\left(\varphi_b\right)\right) \subseteq G_{b,\eta}\left(\Psi\left(\varphi_b\right)\right)$.

Since, $Q_{b,\delta}\left(\varphi_b\right)=\left\{\varphi_b\right\}$.

Let $\varphi_b \in G_{b,\eta}\left(\varphi_b\right) \Rightarrow \Psi\left(\varphi_b\right) \in \Psi\left(G_{b,\eta}\left(\varphi_b\right)\right) \subseteq G_{b,\eta}\left(\Psi\left(\varphi_b\right)\right)=\left\{\varphi_b\right\}$

$$\Rightarrow G_{b,\eta}\left(\varphi_b\right)=\left\{\varphi_b\right\}.$$

iv. From ii, we have $\Psi\left(\varphi_b\right) \in \Psi\left(G_{b,\eta}\left(\varphi_b\right)\right)$, φ_b is Ψ-fixed

$$G_{b,\eta}\left(\varphi_b\right) \cap \left\{\tau_b \in H ; \Psi\left(\tau_b\right)=\tau_b\right\} \neq \varnothing \Rightarrow \Psi\left(\tau_b\right) \neq \tau_b .$$

Thus, if $\Psi\left(\tau_b\right)=\tau_b$, then τ_b is an ,η-co-approximation to φ_b and there is no, η-co-approximation to φ_b.

$$\Rightarrow G_{b,\eta}\left(\varphi_b\right)=\varnothing .$$

If $\Psi\left(\tau_b\right) \neq \tau_b$, then φ_b has at least two -co-approximation.

Note: Its simple to note that similar results are true for $G_{b,\eta}\left(\varphi_b\right)$.

Theorem 3.2: Let $S_{b,\eta}$ be the space of all bounded functions and $\Psi: S_{b,\eta} \rightarrow S_{b,\eta}$ is linear isometry operator. Then, for $\varphi_b, \tau_b \in S_{b,\eta}$

i. $\varphi_b \perp_\eta \tau_b$ if and only if $\Psi\left(\varphi_b\right) \perp_\eta \Psi\left(\tau_b\right)$.

ii. For a subspace $\Upsilon_{b,\eta}$ of the space $S_{b,\eta}$,

$$\Psi\left(\Upsilon_{b,\eta}{}^{\perp_\eta}\right)=\left(\Psi\left(\Upsilon_{b,\eta}\right)\right)_{b,\eta}^{\perp_\eta} .$$

iii. For a subspace $\Upsilon_{b,\eta}$ of the space $S_{b,\eta}$, then

$$\Psi\left(\Upsilon_{\perp_\eta}^{b,\eta}\right)=\left(\Psi\left(\Upsilon_{b,\eta}\right)\right)_{\perp_\eta}^{b,\eta} .$$

Proof: (i) We have for $\varphi_b, \tau_b \in S_{b,\eta}, \varphi_b \perp_\eta \tau_b$

$$\begin{aligned}
&\Leftrightarrow \| \varphi_b \|_{b,\eta} \leq \| \varphi_b-\alpha \tau_b \|_{b,\eta}+\eta, \text{ for any scalar, } \alpha \\
&\Leftrightarrow \| \Psi\left(\varphi_b\right) \|_{b,\eta} \leq \| \Psi(\varphi_b-\alpha \tau_b) \|_{b,\eta}+\eta \\
&=\| \Psi(\varphi_b)-\alpha \Psi(\tau_b) \|_{b,\eta} \\
&\Leftrightarrow \Psi\left(\varphi_b\right) \perp_\eta \Psi\left(\tau_b\right) .
\end{aligned}$$

(ii) Let $\tau_b \in \Upsilon_{b,\eta}{}^{\perp_\eta} \ni \tau_b=\Psi(\varphi_b) ; \varphi_b \in \Upsilon_{b,\eta}{}^{\perp_\eta}$.

So, $\varphi_b \in \Upsilon_{b,\eta}{}^{\perp_\eta} \Rightarrow \varphi_b \perp_\eta \Upsilon_{b,\eta} \Rightarrow \Psi\left(\varphi_b\right) \perp_\eta \Psi(\Upsilon_{b,\eta})$

$$\Rightarrow \tau_b \perp_\eta \Psi(\Upsilon_{b,\eta}) \Rightarrow \tau_b \in (\Psi(\Upsilon_{b,\eta})_{b,\eta}^{\perp_\eta} .$$

Therefore, $\left(\Upsilon_{b,\eta}{}^{\perp_\eta}\right) \subseteq\left(\Psi\left(\Upsilon_{b,\eta}\right)\right)_{b,\eta}^{\perp_\eta}$.

Conversely, $\tau_b \in\left(\Psi\left(\Upsilon_{b,\eta}\right)\right)_{b,\eta}^{\perp_\delta} \Rightarrow \tau_b \perp_\eta \Psi\left(\Upsilon_{b,\eta}\right)$.

Since, Ψ is onto operator $\tau_b=\Psi\left(\varphi_b\right)$ for $\varphi_b \in S_{b,\eta}$

$$\Psi\left(\varphi_b\right) \perp_\eta \Psi\left(\Upsilon_{b,\eta}\right) \Rightarrow \Psi\left(\varphi_b\right) \perp_\eta \Psi\left(\tau_b\right), \text{ for } \tau_b \in \Upsilon_{b,\eta} .$$

Thus, $\| \Psi(\varphi_b) \|_{b,\eta} \leq \| \Psi(\varphi_b) - \alpha\Psi(\tau_b) \|_{b,\eta} + \eta$, for any scalar α

$$\Rightarrow \varphi_b \perp_\eta \tau_b, \text{for } \tau_b \in \Upsilon_{b,\eta}, \varphi_b \perp_\eta \Upsilon_{b,\eta} \Rightarrow \varphi_b \in {\Upsilon_{b,\eta}}^{\perp_\eta}.$$

Now, $\tau_b = \Psi(\varphi_b) \in \Psi\left({\Upsilon_{b,\eta}}^{\perp_\eta} \right)$.

$$\Rightarrow \left(\Psi\left(\Upsilon_{b,\eta} \right) \right)_{b,\eta}^{\perp_\eta} \subseteq \Psi\left({\Upsilon_{b,\eta}}^{\perp_\eta} \right).$$

So, $\left(\Psi\left(\Upsilon_{b,\eta} \right) \right)_{b,\eta}^{\perp_\eta} = \Psi\left({\Upsilon_{b,\eta}}^{\perp_\eta} \right)$.

(iii) Let $\tau_b \in \Psi\left(\Upsilon_{\perp_\eta}^{b,\eta} \right) \Rightarrow q_b = \Psi(\varphi_b) \in \Psi\left(\Upsilon_{\perp_\eta}^{b,\eta} \right)$.

So, $\varphi_b \in \Upsilon_{\perp_\eta}^{b,\eta} \Rightarrow \Upsilon_{b,\eta} \perp_\eta \varphi_b \Rightarrow \Psi\left(\Upsilon_{b,\eta} \right) \perp_\eta \Psi\left(\varphi_b \right)$,

$$\Rightarrow \Psi\left(\Upsilon_{b,\eta} \right) \perp_\eta \tau_b \Rightarrow \tau_b \in \left(\Psi\left(\Upsilon_{b,\eta} \right) \right)_{\perp_\eta}^{b,\eta}$$

$$\Rightarrow \Psi\left(\Upsilon_{\perp_\eta}^{b,\eta} \right) \subseteq \left(\Psi\left(\Upsilon_{b,\eta} \right) \right)_{\perp_\eta}^{b,\eta}.$$

Conversely, $\tau_b \in \left(\Psi\left(\Upsilon_{b,\eta} \right) \right)_{\perp_\eta}^{b,\eta} \Rightarrow \Psi\left(\Upsilon_{b,\eta} \right) \perp_\delta \tau_b$, we have Ψ is onto, $\tau_b = \Psi(\varphi_b)$; $\varphi_b \in S_{b,\eta}$ and $\Psi\left(\Upsilon_{b,\eta} \right) \perp_\eta \Psi(\varphi_b)$

$$\Rightarrow \Psi\left(\tau_b \right) \perp_\delta \Psi(\varphi_b), \tau_b \in \Upsilon_{b,\eta}$$
$$\Rightarrow \| \Psi\left(\tau_b \right) \|_{b,\delta} \leq \| \Psi\left(\tau_b \right) - \alpha\Psi(\varphi_b) \|_{b,\eta} + \eta, \text{for any } \alpha \text{ and } \tau_b \in \Upsilon_{b,\eta}$$
$$\Rightarrow \| \tau_b \|_{b,\delta} \leq \| \tau_b - \alpha\varphi_b \|_{b,\eta} + \delta, \text{for any } \alpha \text{ and } \tau_b \in \Upsilon_{b,\eta}$$
$$\Rightarrow \Upsilon_{b,\eta} \perp_\delta \varphi_b \Rightarrow \varphi_b \in \Upsilon_{\perp_\eta}^{b,\eta}, \text{we have } \tau_b = \Psi(\varphi_b) \in \Psi\left(\Upsilon_{\perp_\eta}^{b,\eta} \right)$$
$$\Rightarrow \left(\Psi\left(\Upsilon_{b,\eta} \right) \right)_{\perp_\eta}^{b,\eta} \subseteq \Psi\left(\Upsilon_{\perp_\eta}^{b,\eta} \right)$$

We obtain, $(\Psi(\Upsilon_{b,\eta}))_{\perp_\eta}^{b,\eta} = \Psi\left(\Upsilon_{\perp_\eta}^{b,\eta} \right)$.

Theorem 3.3: Let $\Upsilon_{b,\eta}$ be a subspace of the space $S_{b,\eta}$ and $\eta > 0$. Then

i. $\Upsilon_{b,\eta}$ is η-aproximal if and only if $S_{b,\delta} = \Upsilon_{b,\eta} + {\Upsilon_{b,\eta}}^{\perp_\eta}$.
ii. $\Upsilon_{b,\eta}$ is η-Chebyshve if and only if $S_{b,\delta} = \Upsilon_{b,\eta} \oplus {\Upsilon_{b,\eta}}^{\perp_\eta}$.
iii. $\Upsilon_{b,\eta}$ is η-semi Chebyshve if and only if for $p_b \in S_{b,\eta}$ has at most one sum decomposition $\Upsilon_{b,\eta} + {\Upsilon_{b,\eta}}^{\perp_\eta}$.

Proof: (i) Let $Y_{b,\eta}$ be an η-aproximal set and $\varphi_b \in S_{b,\eta}$. We have $F_{b,\eta}\left(\varphi_b \right) \neq \varnothing$ implies, there is function $\tau_b \in F_{b,\eta}\left(\varphi_b \right)$ such that $\varphi_b - \tau_b \in {\Upsilon_{b,\eta}}^{\perp_\delta}$ & $\varphi_b = \tau_b + \left(\varphi_b - \tau_b \right) \in \Upsilon_{b,\eta} + {\Upsilon_{b,\eta}}^{\perp_\eta}$, implies, $S_{b,\eta} \subseteq \Upsilon_{b,\eta} + {\Upsilon_{b,\eta}}^{\perp_\delta}$, we have always $\Upsilon_{b,\eta} + {\Upsilon_{b,\eta}}^{\perp_\eta} \subseteq S_{b,\eta}$.

So, $S_{b,\eta} = \Upsilon_{b,\eta} + {\Upsilon_{b,\eta}}^{\perp_\eta}$.

Conversely, let $S_{b,\eta} = \Upsilon_{b,\eta} + \Upsilon_{b,\eta}{}^{\perp_\eta}$, $p_b \in S_{b,\eta}$,

$$\Rightarrow \varphi_b \in \Upsilon_{b,\eta} + \Upsilon_{b,\eta}{}^{\perp_\eta}$$
$$\Rightarrow \varphi_b = \tau_b + (\varphi_b - \tau_b)$$
$$\Rightarrow \varphi_b - \tau_b \in \Upsilon_{b,\eta}{}^{\perp_\eta} \ \& \ \tau_b \in F_{b,\eta}(\varphi_b)$$
$$\Rightarrow \Upsilon_{b,\eta} \text{ is } \eta\text{-aproximal.}$$

(ii) Let $\Upsilon_{b,\eta}$ be an $\perp_\eta$-Chebyshev in $S_{b,\eta}$. Then $\Upsilon_{b,\eta}$ is η-aproximal of $S_{b,\eta}$, from (i) we have $S_{b,\eta} = \Upsilon_{b,\eta} + \Upsilon_{b,\eta}{}^{\perp_\eta}$.

Let $\varphi_b \in S_{b,\eta} \ni \varphi_b = \phi_b + u_b = \omega_b + v_b$;

$$u_b, v_b \in \Upsilon_{b,\eta} \ \& \ \phi_b, \omega_b \in \Upsilon_{b,\eta}{}^{\perp_\eta}$$
$$\Rightarrow \phi_b - \omega_b = u_b - v_b \in \Upsilon_{b,\eta}.$$

Since, $r_b \in \Upsilon_{b,\eta}{}^{\perp_\eta} \Rightarrow r_b - 0 \in \Upsilon_{b,\eta}{}^{\perp_\eta} \Rightarrow 0 \in F_{b,\eta}(\phi_b)$

$$\Rightarrow u_b \in F_{b,\eta}(\phi_b + u_b)$$
$$\Rightarrow u_b \in F_{b,\eta}(\varphi_b).$$

Similarly, we prove that $v_b \in F_{b,\eta}(\varphi_b)$, since $\Upsilon_{b,\eta}$ is an η-Chebyshev,

$$\Rightarrow u_b = v_b$$
$$\Rightarrow \phi_b - \omega_b$$
$$\Rightarrow \varphi_b \in S_{b\eta}, \text{has a unique representation}$$
$$\Rightarrow S_{b,\eta} = \Upsilon_{b,\eta} \oplus \Upsilon_{b,\eta}{}^{\perp_\eta}.$$

Conversely, let $S_{b,\eta} = \Upsilon_{b,\eta} \oplus \Upsilon_{b,\eta}{}^{\perp_\eta}$, we need to prove that $\Upsilon_{b,\eta}$ is an η-Chebyshev.

From (i), $S_{b,\eta} = \Upsilon_{b,\eta} + \Upsilon_{b,\eta}{}^{\perp_\eta} \Leftrightarrow \Upsilon_{b,\eta}$ is an η-proximal.

Let $\varphi_b \in S_{b,\eta}$ has two different η-approximation in $\Upsilon_{b,\eta}$ as u_b, v_b

$$\Rightarrow \varphi_b - u_b = \varphi_b - v_b \in \Upsilon_{b,\eta}{}^{\perp_\eta}$$
$$\Rightarrow p_b = u_b + p_b - v_b, \text{ this is contradiction to our assumption.}$$
$$\Rightarrow \varphi_b \text{ has a unique representation}$$
$$\Rightarrow S_{b,\eta} = \Upsilon_{b,\eta} \oplus \Upsilon_{b,\eta}{}^{\perp_\delta}.$$

(iii) The proof of this case, immediately from (i) & (ii).

4. Conclusion

These concepts η-approximation, η-co-approximation and η-orthogonal of bounded functions in the space $S_{b,\eta}$ have been estimated and isometry operator preserving of this concepts it is also easy to extend our main results to weighed space containing unbounded functions.

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